Strongly clean matrix rings over local rings

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Abstract

An element of a ring $R$ with identity is called strongly clean if it is the sum of an idempotent and a unit that commute, and $R$ is called strongly clean if every element of $R$ is strongly clean. In this paper, we determine when a $2 \times 2$ matrix $A$ over a commutative local ring is strongly clean. Several equivalent criteria are given for such a matrix to be strongly clean. Consequently, we obtain several equivalent conditions for the $2 \times 2$ matrix ring over a commutative local ring to be strongly clean, extending a result of Chen, Yang, and Zhou.

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1. Introduction

Let $R$ be an associative ring with identity. Call $R$ clean if every element of $R$ is the sum of an idempotent and a unit, and call $R$ strongly clean if every element of $R$ is the sum of an idempotent and a unit that commute. Semiperfect rings and unit-regular rings are examples of clean rings, as shown by Camillo and Yu [2], and Camillo and Khurana [1]. For the study of clean rings, we refer to [1,2,5–7,9]. Strongly clean rings were introduced by Nicholson [8], where their connection to Fitting’s Lemma was discussed. Clearly local rings and commutative semiperfect rings are strongly clean.

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In [5], Han and Nicholson showed that a ring $R$ is clean if and only if the matrix ring $M_n(R)$ is clean for every $n \geq 1$. However, the analog for strongly clean rings fails to hold. Examples of non-strongly clean $2 \times 2$ matrices over a commutative local ring can be found in [10,11]. Recently, it was proved by Chen, Yang and Zhou in [4] that for each prime $p$, $M_2(\mathbb{Z}(p))$ is not strongly clean, where $\mathbb{Z}(p)$ is the localization of $\mathbb{Z}$ at the prime ideal generated by $p$. In another recent paper [3], the same authors investigated when a $2 \times 2$ matrix ring $M_2(R)$ over a commutative local ring $R$ is strongly clean, and they obtained a simple criterion for such a matrix ring to be strongly clean. However, their criterion cannot be used to determine whether an individual matrix $A$ in $M_2(R)$ is strongly clean when the matrix ring $M_2(R)$ is not necessarily strongly clean.

In this paper, we determine when a $2 \times 2$ matrix $A$ over a commutative local ring is strongly clean. In Section 2, several equivalent criteria for a $2 \times 2$ matrix $A$ over a commutative local ring to be strongly clean are obtained (Theorem 2.6). In particular, it is shown that such a matrix $A$ is strongly clean if and only if either $A$ is invertible, or $A - I$ is invertible, or $A$ is diagonalizable in $M_2(R)$. Consequently, we obtain several criteria for $M_2(R)$ to be strongly clean, extending the main result (Theorem 8) in [3]. In Section 3, we apply the criteria obtained in Theorem 2.6 to determine when a $2 \times 2$ matrix $A$ over $\mathbb{Z}(p)$ is strongly clean. We show that such a matrix $A$ is strongly clean if and only if either $A$ is invertible, or $A - I$ is invertible, or $(\text{tr} A)^2 - 4 \det A$ is a square of a unit in $\mathbb{Z}(p)$.

Throughout the paper, $U(R)$ and $J(R)$ denote the group of units of $R$ and the Jacobson radical of $R$, respectively. For an element $a$ in a ring $R$, if $a = e + u$ where $e$ is an idempotent and $u$ is a unit such that $eu = ue$, then $a = e + u$ is referred to a strongly clean expression of $a$ in $R$. Recall that two matrices $A$ and $B$ are similar if $A = P^{-1}BP$ for some invertible matrix $P$. A property is called a similarity invariant if it is shared by all similar matrices. For example, $\det A$, $\text{tr} A$ and strongly cleanness are among similarity invariants.

2. Strongly clean matrix rings

This section investigates the question of when a $2 \times 2$ matrix $A$ over a commutative local ring is strongly clean. Our main result is Theorem 2.6, which provides several criteria for such a matrix $A$ to be strongly clean. As a consequence, several necessary and sufficient conditions for a $2 \times 2$ matrix ring over a commutative local ring to be strongly clean are obtained in Theorem 2.8. We start with two useful lemmas.

**Lemma 2.1.** [3, Lemma 1] Let $R$ be a commutative ring such that $J(R)$ is prime, $w \in J(R)$ and $u \in U(R)$. The following statements are equivalent:

1. $x^2 - ux = w$ is solvable in $R$.
2. $x^2 - ux = w$ is solvable in $J(R)$.
3. $x^2 - ux = w$ is solvable in $U(R)$.

**Lemma 2.2.** [3, Lemma 4] Let $R$ be a commutative ring. Let $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$, $E = \begin{pmatrix} a & b \\ c & -a \end{pmatrix} \in M_2(R)$ and $U = A - E$. Then $A = E + U$ is a strongly clean expression of $A$ if and only if the following conditions hold:

$$bc = a - a^2,$$

$$a_{21}b = a_{12}c,$$
Proof. We will show that the following statements are equivalent:

\[ t(a - a^2) = a_{12}a_{21}, \]
\[ \text{where } s = a_{11} - a_{22}. \]

**Remark 2.3.** We note that solving \( a - a^2 \) from the first, third and fourth equations in the above lemma yields \( t(a - a^2) = a_{12}a_{21}, \) where \( t = (\text{tr}A)^2 - 4 \text{ det}A. \)

The following proposition is crucial for developing criteria for a \( 2 \times 2 \) matrix over a commutative ring to be strongly clean.

**Proposition 2.4.** Let \( R \) be a commutative ring such that \( J(R) \) is prime. Assume that \( A = \begin{pmatrix} a_{11} & a_{12} \\ w & 0 \end{pmatrix} \in M_2(R), \) where \( a_{11}, a_{12} \in \mathcal{U}(R) \) and \( w \in J(R) \) is such that \( \text{det}(A - I) \notin \mathcal{U}(R). \) Then the following statements are equivalent:

1. \( A \) is strongly clean.
2. The equation \( x^2 - x = \frac{\text{det}A}{(\text{tr}A)^2 - 4 \text{ det}A} \) is solvable in \( R. \)
3. The characteristic equation of \( A, x^2 - (\text{tr}A)x + \text{ det}A = 0 \) is solvable in \( R. \)

**Proof.** We will show that \( (3) \Rightarrow (1) \Rightarrow (2) \Rightarrow (3). \)

\( (3) \Rightarrow (1). \) Assume that the characteristic equation of \( A, \)
\[ x^2 - (\text{tr}A)x + \text{det}A = 0 \] is solvable in \( R. \) Since \( \text{tr}A = a_{11} \in \mathcal{U} \) and \( \text{det}A = -a_{12}w \in J(R), \) by Lemma 2.1 we can find two solutions \( \lambda_1 \) and \( \lambda_2 \) to (*) such that \( \lambda_1 \in J(R) \) and \( \lambda_2 = \text{tr}A - \lambda_1 = a_{11} - \lambda_1 \in \mathcal{U}(R). \) We now show that there exist two eigenvectors \( X_1 \) and \( X_2 \) of \( A \) such that \( P = (X_1, X_2) \in M_2(R) \) is invertible and \( P^{-1}AP = \begin{pmatrix} \lambda_1 & 0 \\ \lambda_1^1 - \lambda_1 & a_{11} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} \lambda_1^1 - \lambda_1 & 0 \\ 0 & a_{11} \end{pmatrix} \) is strongly clean. Thus \( A \) is strongly clean.

Let \( X_1 = \begin{pmatrix} a_{12} \\ -(a_{11} - \lambda_1) \end{pmatrix} \) and \( X_2 = \begin{pmatrix} a_{11} \lambda_1 \\ w \end{pmatrix}. \) Then it is straightforward to check \( AX_1 = \lambda_1 X_1 \) and \( AX_2 = \lambda_2 X_2 = (a_{11} - \lambda_1)X_2. \) Now \( P = (X_1, X_2) \) is invertible because \( \text{det}P = (a_{11} - \lambda_1)^2 + a_{12}w \in \mathcal{U}(R). \) Therefore, \( P^{-1}AP = \begin{pmatrix} \lambda_1 & 0 \\ 0 & a_{11} - \lambda_1 \end{pmatrix} \) as desired.

\( (1) \Rightarrow (2). \) Assume that \( A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} + \begin{pmatrix} a_{11} - a \\ a_{12} - b \\ w - c \\ -d \end{pmatrix} = E + U \) is such that \( E^2 = E, EU = UE \) and \( \text{det}U \in \mathcal{U}(R). \)

Comparing the \((1, 1)\) and \((1, 2)\)-entries of \( EU = UE \) yields \( bw = a_{12}c \) and \( a_{11}b = (a - d)a_{12}. \) Thus
\[ c = \frac{bw}{a_{12}} \in J(R) \tag{2.1} \]
and
\[ b = \frac{(a - d)a_{12}}{a_{11}}. \tag{2.2} \]
Since \( \det U = (a_{11} - a)(-d) - (a_{12} - b)(w - c) \in \mathcal{U}(R) \) and \( (a_{12} - b)(w - c) \in J(R) \) by (2.1), we conclude that

\[
a_{11} - a \in \mathcal{U}(R) \quad \text{and} \quad d \in \mathcal{U}(R).
\]

Comparing the (1, 1) and (2, 2)-entries of \( E^2 = E \) yields that \( a - a^2 = bc \) and \( d - d^2 = bc \). This gives

\[
(a - d)(1 - a - d) = 0.
\]

Since \( a(1 - a) = bc \in J(R) \) and \( J(R) \) is prime, either \( a \in J(R) \) or \( 1 - a \in J(R) \). The latter together with \( d \in \mathcal{U}(R) \) imply that \( 1 - a - d \in \mathcal{U}(R) \) and thus \( a - d = 0 \) by (2.4). It follows from (2.1) and (2.2) that \( b = c = 0 \), so \( E = al \) with \( a^2 = a \). Since \( J(R) \) is prime, we must have either \( a = 0 \) or \( a = 1 \). This implies that either \( A = U \) or \( A - I = U \) is invertible, contradicting the assumption. So we must have \( a \in J(R) \), and thus \( a - d \) is a unit. By (2.4), \( d = 1 - a \). It now follows immediately from Lemma 2.2 that \( a - a^2 = bc \), \( sb = a_{12}(2a - 1) \) and \( sc = a_{21}(2a - 1) \). Solving \( a^2 - a \) from these equations yields that \( a^2 - a = \frac{\det A}{(\text{tr} A)^2 - 4 \det A} \) (see Remark 2.3). Therefore, \( x^2 - x = \frac{\det A}{(\text{tr} A)^2 - 4 \det A} \) has a solution \( a \) in \( R \).

(2) \( \Rightarrow \) (3). Assume that the equation \( x^2 - x = \frac{\det A}{(\text{tr} A)^2 - 4 \det A} \) is solvable in \( R \). We first construct a matrix \( B \) such that its characteristic equation is the above one. Clearly, if \( B = \left( \begin{array}{cc} 1 & 0 \\ \frac{1}{(\text{tr} A)^2 - 4 \det A} & 1 \end{array} \right) \), then \( B \) is a desired matrix. Moreover, \( \det(B - I) = \det B = -\frac{\det A}{(\text{tr} A)^2 - 4 \det A} \in J(R) \), so \( B \) satisfies all assumptions. Since the characteristic equation of \( B \) is solvable in \( R \), by what we just proved (the implication of (3) \( \Rightarrow \) (1)) \( B \) is strongly clean. Using the implication of (1) \( \Rightarrow \) (2) on \( B \), we conclude that the equation \( y^2 - y = \frac{\det B}{(\text{tr} B)^2 - 4 \det B} \) is solvable in \( R \). Since \( \frac{\det B}{(\text{tr} B)^2 - 4 \det B} = \frac{-\det A}{(\text{tr} A)^2 - 4 \det A}/(1 - 4(-\det A/(\text{tr} A)^2 - 4 \det A)) = \frac{-\det A}{(\text{tr} A)^2 - 4 \det A} \), the above equation reduces to \( y^2 - y = \frac{-\det A}{(\text{tr} A)^2} \). Replacing \( (\text{tr} A)y \) by \( x \) yields that \( x^2 - (\text{tr} A)x + \det A = 0 \) is solvable in \( R \). This completes the proof. □

**Remark 2.5.**

(1) If condition (2) in Proposition 2.4 is satisfied, then the equation \( x^2 - x = \frac{\det A}{(\text{tr} A)^2 - 4 \det A} \) has a solution \( x_0 \in J(R) \) by Lemma 2.1. Let \( a = x_0, b = a_{12}/a_{11}(2a - 1), c = a_{21}/a_{11}(2a - 1), d = 1 - a, \) and let \( E = \left( \begin{array}{cc} a & b \\ c & 1-a \end{array} \right) \in M_2(R) \) and \( U = A - E \). It is straightforward to check that all five conditions in Lemma 2.2 are satisfied. Therefore, \( A = E + U \) is a strongly clean expression of \( A \).

(2) In view of the proof of Proposition 2.4, we conclude that matrix \( A = \left( \begin{array}{cc} a_{11} & a_{12} \\ w & 0 \end{array} \right) \) is strongly clean if and only if it is diagonalizable in \( M_2(R) \).

We are now ready to provide criteria in terms of similarity invariants for a \( 2 \times 2 \) matrix over a commutative local ring to be strongly clean.

**Theorem 2.6.** Let \( R \) be a commutative local ring and let \( A \in M_2(R) \) be such that \( \det A \in J(R) \) and \( \det(A - I) \in J(R) \). Then the following statements are equivalent:
Remark 2.7.

Case I. Just proved we may assume that $A$ is diagonalizable in $M_2(R)$.

Proof. Let $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \in M_2(R)$ be such that $\det A \in J(R)$ and $\det(A - I) \in J(R)$. Then $\text{tr} A = 1 + \det A - \det(A - I) \in U(R)$. We first show that $A$ is similar to the matrix $\begin{pmatrix} \frac{\text{tr} A}{\det A} & b_{12} \\ \det A \end{pmatrix}$ for some unit $b_{12} \in U(R)$ by using case-by-case analysis. Since $R$ is a local ring, either $a_{12} \in U(R)$ or $a_{12} \in J(R)$.

Case I. $a_{12} \in U(R)$. Let $P = \begin{pmatrix} 0 & 1 \\ a_{12} & 0 \end{pmatrix}$. Then
\[
P^{-1}AP = \begin{pmatrix} 1 & 0 \\ -\frac{a_{12}}{a_{22}} & 1 \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ a_{12} & 1 \end{pmatrix} = \begin{pmatrix} \text{tr} A & a_{12} \\ -\frac{\det A}{a_{12}} & 0 \end{pmatrix},
\]
as desired.

Case II. $a_{12} \in J(R)$. Assume that $a_{21} \in U(R)$. Since $A$ is similar to $\begin{pmatrix} a_{22} & a_{21} \\ a_{12} & a_{11} \end{pmatrix}$, we are back to Case I.

Next we assume that both $a_{12}$ and $a_{21}$ are in $J(R)$. Since $\det A = a_{11}a_{22} - a_{12}a_{21} \in J(R)$, we have $a_{11}a_{22} \in J(R)$. This together with $\text{tr} A = a_{11} + a_{22} \in U(R)$ imply that either $a_{11} \in U(R)$ and $a_{22} \in J(R)$, or $a_{11} \in J(R)$ and $a_{22} \in U(R)$. Since $\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ is similar to $\begin{pmatrix} a_{22} & a_{21} \\ a_{12} & a_{11} \end{pmatrix}$, without loss of generality we may always assume that $a_{11} \in U(R)$ and $a_{12}, a_{21}, a_{22} \in J(R)$. Let $P = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. Then
\[
P^{-1}AP = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} b_{12} \\ a_{21} + a_{22} \end{pmatrix}
\]
where $b_{12} = a_{11} + a_{12} - a_{21} - a_{22} \in U(R)$. Again we are back to Case I, and therefore, $A$ is similar to $\begin{pmatrix} \frac{\text{tr} A}{\det A} & b_{12} \\ \det A \end{pmatrix}$ for some $b_{12} \in U(R)$.

Next we assume that both $a_{12}$ and $a_{21}$ are in $J(R)$. Since $\det A = a_{11}a_{22} - a_{12}a_{21} \in J(R)$, we have $a_{11}a_{22} \in J(R)$. This together with $\text{tr} A = a_{11} + a_{22} \in U(R)$ imply that either $a_{11} \in U(R)$ and $a_{22} \in J(R)$, or $a_{11} \in J(R)$ and $a_{22} \in U(R)$. Since $\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ is similar to $\begin{pmatrix} a_{22} & a_{21} \\ a_{12} & a_{11} \end{pmatrix}$, without loss of generality we may always assume that $a_{11} \in U(R)$ and $a_{12}, a_{21}, a_{22} \in J(R)$. Let $P = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. Then
\[
P^{-1}AP = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} b_{12} \\ a_{21} + a_{22} \end{pmatrix}
\]
where $b_{12} = a_{11} + a_{12} - a_{21} - a_{22} \in U(R)$. Again we are back to Case I, and therefore, $A$ is similar to $\begin{pmatrix} \frac{\text{tr} A}{\det A} & b_{12} \\ \det A \end{pmatrix}$ for some $b_{12} \in U(R)$.

Since $\det A, \det(A - I), \text{tr} A$ and strongly cleanness are similarity invariants, by what we just proved we may assume that $A = \begin{pmatrix} \frac{\text{tr} A}{\det A} & b_{12} \\ \det A \end{pmatrix}$ for some $b_{12} \in U(R)$. It now follows immediately from Proposition 2.4 and Remark 2.5 that (1) $\iff$ (2) $\iff$ (3) $\iff$ (4). Note that in (5) $\det B = \det(B - I) = \frac{-\det A}{(\text{tr} A)^2 - 4\det A} \in J(R)$ and the characteristic equation of $B$ is $x^2 - x - \frac{\det A}{(\text{tr} A)^2 - 4\det A} = 0$. We conclude that (2) $\iff$ (5) follows from (1) $\iff$ (3).

An element $r \in R$ (a matrix $A \in M_n(R)$) is called a trivial strongly clean element (matrix) if either $r \in U(R)$ or $r - 1 \in U(R)$ (either $\det A \in U(R)$ or $\det(A - I) \in U(R)$).

Remark 2.7.

(1) Let $R$ be a commutative local ring. Assume that $A \in M_2(R)$ is not a trivial strongly clean matrix. Then by Theorem 2.6, $A$ is strongly clean if and only if $A$ is diagonalizable in $M_2(R)$.

(2) We note that not all strongly clean matrices are necessarily diagonalizable. For example, matrices $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ and $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ are clearly strongly clean, but none of them are diagonalizable in $M_2(F)$ where $F$ is a field.
As a consequence of Theorem 2.6, we obtain the following criteria for the $2 \times 2$ matrix ring $M_2(R)$ over a commutative local ring $R$ to be strongly clean, extending the main result [3, Theorem 8] of Chen, Yang and Zhou.

**Theorem 2.8.** Let $R$ be a commutative local ring. Then the following statements are equivalent:

1. $M_2(R)$ is strongly clean.
2. For every $A \in M_2(R)$ with $\det A \in J(R)$ and $\det(A-I) \in J(R)$, the characteristic equation of $A$, $x^2 - (\text{tr} A)x + \det A = 0$ is solvable in $R$.
3. For every $A \in M_2(R)$ with $\det A \in J(R)$ and $\det(A-I) \in J(R)$, the equation $x^2 - x - \frac{\det A}{(\text{tr} A)^2 - 4 \det A} = 0$ is solvable in $R$.
4. For every $A \in M_2(R)$ with $\det A \in J(R)$ and $\det(A-I) \in J(R)$ is diagonalizable in $M_2(R)$.
5. For every $w \in J(R)$, the matrix $B = \begin{pmatrix} 1 & w \\ 0 & 1 \end{pmatrix}$ is strongly clean.
6. For every $w \in J(R)$, the equation $x^2 - x + w = 0$ is solvable in $R$.
7. For every $w \in J(R)$ and $u \in U(R)$, the equation $x^2 - ux + w = 0$ is solvable in $R$.

**Proof.** For every $A \in M_2(R)$, if $\det A \in U(R)$ or $\det(A-I) \in U(R)$, then $A$ is strongly clean. So we may assume that neither $\det A$ nor $\det(A-I)$ is a unit. Since $R$ is local, we must have $\det A \in J(R)$ and $\det(A-I) \in J(R)$, and thus $\text{tr} A \in U(R)$. Now Theorem 2.6 implies that (1) $\iff$ (2) $\iff$ (3) $\iff$ (4) $\iff$ (5).

Next we show that (5) $\implies$ (6) $\implies$ (7) $\implies$ (2).

(5) $\implies$ (6). For every $w \in J(R)$, $\det B = \det(B-I) = w \in J(R)$. Since $B$ is strongly clean, it follows immediately from Theorem 2.6 that the characteristic equation of $B$, $x^2 - x + w = 0$ is solvable in $R$.

(6) $\implies$ (7). For every $u \in U(R)$ and $w \in J(R)$, consider the equation $y^2 - y + \frac{w}{u^2} = 0$. Since $\frac{w}{u^2} \in J(R)$, by the assumption the above equation has a solution $y_0$ in $R$. Now it is easy to check that $x_0 = u y_0$ is a solution to $x^2 - ux + w = 0$.

(7) $\implies$ (2) is obvious (by setting $u = \text{tr} A$ and $w = \det A$).

This completes the proof. $\square$

We note that the equivalency of (6) and (7) in the above theorem answers a question raised in [3], and it also allows us to restate the main result (Theorem 8) of [3] in the following simplified form.

**Corollary 2.9.** Let $R$ be a commutative local ring. Then the $2 \times 2$ matrix ring $M_2(R)$ over $R$ is strongly clean if and only if for every $w \in J(R)$, the equation $x^2 - x + w = 0$ is solvable in $R$.

3. The $2 \times 2$ matrix ring $M_2(\mathbb{Z}_p)$

It was shown in [4] that for any prime $p$, $M_2(\mathbb{Z}_p)$ is not strongly clean. In this section, we apply Theorem 2.6 to determine when a matrix $A \in M_2(\mathbb{Z}_p)$ is strongly clean. The following results slightly improve several results from Section 2 of [4].

**Proposition 3.1.** Let $R$ be a commutative local ring. If $A \in M_2(R)$ is strongly clean, then exactly one of the following holds:
(1) Either $A$ or $A - I$ is invertible.
(2) $\det A \in J(R)$, $\det(A - I) \in J(R)$ and $(\text{tr } A)^2 - 4 \det A = u^2$ for some $u \in \mathcal{U}(R)$.

**Proof.** Let $A$ be a strongly clean matrix. We may assume that $\det A \in J(R)$ and $\det(A - I) \in J(R)$. Thus $\text{tr } A \in \mathcal{U}(R)$ and so $(\text{tr } A)^2 - 4 \det A \in \mathcal{U}(R)$. By Theorem 2.6, there exists $x_0 \in R$ such that $x_0^2 - (\text{tr } A)x_0 + \det A = 0$, or $(2x_0)^2 - 2(\text{tr } A)(2x_0) + 4 \det A = 0$. This gives $(\text{tr } A)^2 - 4 \det A = (2x_0 - \text{tr } A)^2 = u^2$. Since $(\text{tr } A)^2 - 4 \det A$ is a unit, $u$ must be a unit. \( \square \)

**Theorem 3.2.** Let $R$ be a commutative local ring such that either $J(R) = 2R$ and $R$ is a domain, or $2 \in \mathcal{U}(R)$. Then $A \in M_2(R)$ is strongly clean if and only if exactly one of the following holds:

(1) Either $A$ or $A - I$ is invertible.
(2) $\det A \in J(R)$, $\det(A - I) \in J(R)$ and $(\text{tr } A)^2 - 4 \det A = u^2$ for some $u \in \mathcal{U}(R)$.

**Proof.** The necessity follows from Proposition 3.1.

We now show the sufficiency. Let $A \in M_2(R)$. As before, we may assume that $\det A \in J(R)$, $\det(A - I) \in J(R)$ and $(\text{tr } A)^2 - 4 \det A = u^2$ for some $u \in \mathcal{U}(R)$.

**Case I.** $2 \in \mathcal{U}(R)$. Let $x_0 = \frac{\text{tr } A + u}{2}$. Then $x_0$ is a solution to the characteristic equation of $A$, i.e. $(x_0^2 - (\text{tr } A)x_0 + \det A = 0$. By Theorem 2.6 $A$ is strongly clean.

**Case II.** $J(R) = 2R$ and $R$ is a domain. Since $(\text{tr } A + u)^2 = 2(\text{tr } A)^2 + 2(\text{tr } A)u - 4 \det A \in J(R)$ and $R$ is local, we have $\text{tr } A + u \in J(R)$. Let $\text{tr } A + u = 2r$. We now show that $r$ is a solution to the equation $(x)^2 - (\text{tr } A)x + \det A = 0$. Again $A$ is strongly clean by Theorem 2.6. Note that $4(r^2 - (\text{tr } A)r + \det A) = (2r - \text{tr } A)^2 + 4 \det A - (\text{tr } A)^2 = u^2 - u^2 = 0$. Since $R$ is a domain, we conclude $r^2 - (\text{tr } A)r + \det A = 0$ as desired. \( \square \)

In particular, when $R = \mathbb{Z}_{(p)}$, the assumptions in the above theorem are satisfied, so we obtain the following result.

**Corollary 3.3.** Let $R = \mathbb{Z}_{(p)}$. Then a matrix $A \in M_2(R)$ is strongly clean if and only if one of the following holds:

(1) Either $A$ or $A - I$ is invertible.
(2) $\det A \in J(R)$, $\det(A - I) \in J(R)$ and $(\text{tr } A)^2 - 4 \det A = u^2$ for some $u \in \mathcal{U}(R)$.

We conclude this paper by stating an alternative criterion for a matrix $A \in M_2(\mathbb{Z}_{(2)})$ to be strongly clean.

**Proposition 3.4.** Let $R = \mathbb{Z}_{(2)}$. Then a matrix $A \in M_2(R)$ is strongly clean if and only if one of the following holds:

(1) Either $A$ or $A - I$ is invertible.
(2) $\det A \in J(R)$, $\det(A - I) \in J(R)$ and $w = \frac{\det A}{(\text{tr } A)^2 - 4 \det A} = \frac{2n}{m}$, where $(n, m) = 1$, $m = (2l + 1)^2$ and $2n = s(s + 1) - l(l + 1)$ for some integers $l, s$. 
Proof. As before, we need only consider that $A \in M_2(\mathbb{Z}_2)$ such that $\det A \in J(\mathbb{Z}_2)$ and $\det(A - I) \in J(\mathbb{Z}_2)$. By Theorem 2.6, $A$ is strongly clean if and only if the matrix $\left( \begin{array}{cc} det A & 1 \\ \frac{1}{(tr A)^2 - 4 det A} & 0 \end{array} \right)$ is strongly clean. Without loss of generality, we may always assume that $A = \left( \begin{array}{cc} 1 & w \\ 0 & 0 \end{array} \right)$ where $w = \frac{2n}{m} \in J(R)$, $m$ is odd and $(n, m) = 1$.

For the sufficiency, we compute $(tr A)^2 - 4 det A = 1 + \frac{8n}{m} = \frac{(2l+1)^2 + 4(s(s+1) - l(l+1))}{(2l+1)^2} = \frac{(2s+1)^2}{2l+1} = u^2$ where $u \in U(\mathbb{Z}_2)$. So $A$ is strongly clean by Corollary 3.3.

Conversely, if $A$ is strongly clean, then by Corollary 3.3 $(tr A)^2 - 4 det A = u^2 = \frac{(2s+1)^2}{2l+1}$, where $(2s + 1, 2l + 1) = 1$. On the other hand, $(tr A)^2 - 4 det A = 1 + 4w = 1 + \frac{8n}{m}$. Thus $\frac{m+8n}{m} = \frac{(2s+1)^2}{(2l+1)^2}$. This implies that $m = (2l + 1)^2$ and $8n = (2s + 1)^2 - (2l + 1)^2$. Therefore, $2n = s(s + 1) - l(l + 1)$ as desired. \[ \square \]

In view of Proposition 3.4, one can easily check in $M_2(\mathbb{Z}_2)$, $\left( \begin{array}{cc} 1 & 1 \\ \frac{1}{2} & 0 \end{array} \right)$ is strongly clean, but $\left( \begin{array}{cc} 1 & 1 \\ \frac{1}{2} & 0 \end{array} \right)$ is not.

References