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Research Article

Existence of Positive Solutions to a Boundary Value Problem for a Delayed Nonlinear Fractional Differential System

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Though boundary value problems for fractional differential equations have been extensively studied, most of the studies focus on scalar equations and the fractional order between 1 and 2. On the other hand, delay is natural in practical systems. However, not much has been done for fractional differential equations with delays. Therefore, in this paper, we consider a boundary value problem of a general delayed nonlinear fractional system. With the help of some fixed point theorems and the properties of the Green function, we establish several sets of sufficient conditions on the existence of positive solutions. The obtained results extend and include some existing ones and are illustrated with some examples for their feasibility.

1. Introduction

In the past decades, fractional differential equations have been intensively studied. This is due to the rapid development of the theory of fractional differential equations itself and the applications of such construction in various sciences such as physics, mechanics, chemistry, and engineering [1, 2]. For the basic theory of fractional differential equations, we refer the readers to [3–7].

Recently, many researchers have devoted their attention to studying the existence of (positive) solutions of boundary value problems for differential equations with fractional order [8–23]. We mention that the fractional order α involved is generally in $(1,2]$ with the exception that $\alpha \in (2,3]$ in [12, 23] and $\alpha \in (3,4]$ in [8, 17]. Though there have been extensive study on systems of fractional differential equations, not much has been done for boundary value problems for systems of fractional differential equations [18–20].

On the other hand, we know that delay arises naturally in practical systems due to the transmission of signal or the mechanical transmission. Though theory of ordinary differential equations with delays is mature, not much has been done for fractional differential equations with delays [24–31].

As a result, in this paper, we consider the following nonlinear system of fractional order differential equations with delays,

$$\begin{gathered} D^{\alpha_i} u_i(t) + f_i(t, u_1(\tau_{i1}(t)), \ldots, u_N(\tau_{iN}(t))) = 0, \quad t \in (0,1), \\ u_i^{(j)}(0) = 0, \quad j = 0, 1, \ldots, n_i - 2, \quad i = 1, 2, \ldots, N, \\ u_i^{(n_i - 1)}(1) = \eta_i, \quad i = 1, 2, \ldots, N, \end{gathered} \tag{1.1}$$

where D^{α_i} is the standard Riemann-Liouville fractional derivative of order $\alpha_i \in (n_i - 1, n_i]$ for some integer $n_i > 1$, $\eta_i \geq 0$ for $i = 1, \ldots, N$, $0 \leq \tau_{ij}(t) \leq t$ for $i, j = 1, 2, \ldots, N$, and f_i is a nonlinear function from $[0,1] \times \mathbb{R}_+^N$ to $\mathbb{R}_+ = [0, \infty)$. The purpose is to establish sufficient conditions on the existence of positive solutions to (1.1) by using some fixed point theorems and some properties of the Green function. By a positive solution to (1.1) we mean a mapping with positive components on $(0,1)$ such that (1.1) is satisfied. Obviously, (1.1) includes the usual system of fractional differential equations when $\tau_{ij}(t) \equiv t$ for all i and j. Therefore, the obtained results generalize and include some existing ones.

The remaining part of this paper is organized as follows. In Section 2, we introduce some basics of fractional derivative and the fixed point theorems which will be used in Section 3 to establish the existence of positive solutions. To conclude the paper, the feasibility of some of the results is illustrated with concrete examples in Section 4.

2. Preliminaries

We first introduce some basic definitions of fractional derivative for the readers' convenience.

Definition 2.1 (see [3, 32]). The fractional integral of order $\alpha (> 0)$ of a function $f : (0, \infty) \to \mathbb{R}$ is defined as

$$I^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t \frac{f(s)}{(t-s)^{1-\alpha}} ds \tag{2.1}$$

provided that the integral exists on $(0, \infty)$, where $\Gamma(\alpha) = \int_0^{\infty} e^{-t} t^{\alpha - 1} dt$ is the Gamma function.

Note that I^{α} has the semigroup property, that is,

$$I^{\alpha} I^{\beta} = I^{\alpha + \beta} = I^{\beta} I^{\alpha} \quad \text{for } \alpha > 0, \ \beta > 0. \tag{2.2}$$

Definition 2.2 (see [3, 32]). The Riemann-Liouville derivative of order $\alpha(>0)$ of a function $f:(0,\infty)\to\mathbb{R}$ is given by

$$D^{\alpha}f(t)=\frac{1}{\Gamma(n-\alpha)}\frac{d^n}{dt^n}\int_0^t\frac{f(s)}{(t-s)^{\alpha+1-n}}ds \tag{2.3}$$

provided that the right-hand side is pointwise defined on $(0,\infty)$, where $n=\lfloor\alpha\rfloor+1$.

It is well known that if $n-1<\alpha\le n$ then $D^{\alpha}t^{\alpha-k}=0, k=1,2,\ldots,n$. Furthermore, if $y(t)\in L^1[0,T]$ and $\alpha>0$ then $D^{\alpha}I^{\alpha}y(t)=y(t)$ for $t\in(0,T]$.

The following results on fractional integral and fractional derivative will be needed in establishing our main results.

Lemma 2.3 (see [10]). *Let $\alpha>0$. Then solutions to the fractional equation $D^{\alpha}h(t)=0$ can be written as*

$$h(t)=c_1t^{\alpha-1}+c_2t^{\alpha-2}+\cdots+c_nt^{\alpha-n}, \tag{2.4}$$

where $c_i\in\mathbb{R}$, $i=1,2,\ldots,n=\lfloor\alpha\rfloor+1$.

Lemma 2.4 (see [10]). *Let $\alpha>0$. Then*

$$I^{\alpha}D^{\alpha}h(t)=h(t)+c_1t^{\alpha-1}+c_2t^{\alpha-2}+\cdots+c_nt^{\alpha-n} \tag{2.5}$$

for some $c_i\in\mathbb{R}$, $i=1,2,\ldots,n=\lfloor\alpha\rfloor+1$.

Now, we cite the fixed point theorems to be used in Section 3.

Lemma 2.5 (the Banach contraction mapping theorem [33]). *Let M be a complete metric space and let $T:M\to M$ be a contraction mapping. Then T has a unique fixed point.*

Lemma 2.6 (see [16, 34]). *Let C be a closed and convex subset of a Banach space X. Assume that U is a relatively open subset of C with $0\in U$ and $T:\overline{U}\to C$ is completely continuous. Then at least one of the following two properties holds:*

(i) *T has a fixed point in $\overline{U}$;*

(ii) *there exists $u\in\partial U$ and $\lambda\in(0,1)$ with $u=\lambda Tu$.*

Lemma 2.7 (the Krasnosel'skii fixed point theorem [33, 35]). *Let P be a cone in a Banach space X. Assume that Ω_1 and Ω_2 are open subsets of X with $0\in\Omega_1$ and $\overline{\Omega}_1\subseteq\Omega_2$. Suppose that $T:P\bigcap(\overline{\Omega}_2\setminus\Omega_1)\to P$ is a completely continuous operator such that either*

(i) *$\|Tu\|\le\|u\|$ for $u\in P\bigcap\partial\Omega_1$ and $\|Tu\|\ge\|u\|$ for $u\in P\bigcap\partial\Omega_2$*

or

(ii) *$\|Tu\|\ge\|u\|$ for $u\in P\bigcap\partial\Omega_1$ and $\|Tu\|\le\|u\|$ for $u\in P\bigcap\partial\Omega_2$.*

Then T has a fixed point in $\overline{\Omega}_2\setminus\Omega_1$.

3. Existence of Positive Solutions

Throughout this paper, we let $E = C([0,1], \mathbb{R}^N)$. Then $(E, \|\cdot\|_E)$ is a Banach space, where

$$\|u\|_E = \max_{1\le i\le N} \max_{0\le t\le 1} |u_i(t)| \quad \text{for } u = (u_1, \ldots, u_N)^T \in E. \tag{3.1}$$

In this section, we always assume that $f = (f_1, \ldots, f_N)^T \in C([0,1] \times \mathbb{R}_+^N, \mathbb{R}_+^N)$.

Lemma 3.1. *System* (1.1) *is equivalent to the following system of integral equations:*

$$\begin{aligned} u_i(t) = &\int_0^1 G_i(t,s) f_i(s, u_1(\tau_{i1}(s)), \ldots, u_N(\tau_{iN}(s)))ds \\ &+ \frac{\eta_i t^{\alpha_i - 1}}{(\alpha_i - 1)\cdots(\alpha_i - n_i + 1)}, \quad i = 1, 2, \ldots, N, \end{aligned} \tag{3.2}$$

where

$$G_i(t,s) = \begin{cases} \dfrac{t^{\alpha_i-1}(1-s)^{\alpha_i-n_i} - (t-s)^{\alpha_i-1}}{\Gamma(\alpha_i)}, & 0 \le s \le t \le 1, \\ \dfrac{t^{\alpha_i-1}(1-s)^{\alpha_i-n_i}}{\Gamma(\alpha_i)}, & 0 \le t \le s \le 1. \end{cases} \tag{3.3}$$

Proof. It is easy to see that if $(u_1, u_2, \ldots, u_N)^T$ satisfies (3.2) then it also satisfies (3.2). So, assume that $(u_1, u_2, \ldots, u_N)^T$ is a solution to (1.1). Integrating both sides of the first equation of (1.1) of order α_i with respect to t gives us

$$\begin{aligned} u_i(t) = &-\frac{1}{\Gamma(\alpha_i)} \int_0^t (t-s)^{\alpha_i-1} f_i(s, u_1(\tau_{i1}(s)), \ldots, u_N(\tau_{iN}(s)))ds \\ &+ c_{1i} t^{\alpha_i-1} + c_{2i} t^{\alpha_i-2} + \cdots + c_{n,i} t^{\alpha_i-n_i} \end{aligned} \tag{3.4}$$

for $0 \le t \le 1$, $i = 1, 2, \ldots, N$. It follows that

$$\begin{aligned} u_i'(t) = &-\frac{\alpha_i - 1}{\Gamma(\alpha_i)} \int_0^t (t-s)^{\alpha_i-2} f_i(s, u_1(\tau_{i1}(s)), \ldots, u_N(\tau_{iN}(s)))ds \\ &+ (\alpha_i - 1)c_{1i} t^{\alpha_i-2} + (\alpha_i - 2)c_{2i} t^{\alpha_i-3} + \cdots + (\alpha_i - n_i + 1)c_{n-1,i} t^{\alpha_i-n_i} \end{aligned} \tag{3.5}$$

for $0 \le t \le 1$, $i = 1, 2, \ldots, N$. This, combined with the boundary conditions in (1.1), yields

$$c_{n_i-1,i} = 0, \quad i = 1, 2, \ldots, N. \tag{3.6}$$

Similarly, one can obtain

$$c_{n_i-2,i} = c_{n_i-3,i} = \cdots = c_{2,i} = 0, \tag{3.7}$$

$$\begin{aligned} u_i^{(n_i-1)}(t) = &-\frac{(\alpha_i-1)\cdots(\alpha_i-n_i+1)}{\Gamma(\alpha_i)}\int_0^t (t-s)^{\alpha_i-n_i} f_i(s,u_1(\tau_{i1}(s)),\ldots,u_N(\tau_{iN}(s)))ds \\ &+ (\alpha_i-1)\cdots(\alpha_i-n_i+1)c_{1i}t^{\alpha_i-n_i}, \end{aligned} \tag{3.8}$$

$i = 1,2,\ldots,N$. Then it follows from (3.8) and the boundary condition $u_i^{(n_i-1)}(1) = \eta_i$ that

$$c_{1,i} = \frac{\eta_i}{(\alpha_i-1)\cdots(\alpha_i-n_i+1)} + \frac{1}{\Gamma(\alpha_i)}\int_0^1 (1-s)^{\alpha_i-n_i} f_i(s,u_1(\tau_{i1}(s)),\ldots,u_N(\tau_{iN}(s)))ds. \tag{3.9}$$

Therefore, for $i = 1,2,\ldots,N$,

$$\begin{aligned} u_i(t) = &-\frac{1}{\Gamma(\alpha_i)}\int_0^t (t-s)^{\alpha_i-1} f_i(s,u_1(\tau_{i1}(s)),\ldots,u_N(\tau_{iN}(s)))ds + \frac{\eta_i t^{\alpha_i-1}}{(\alpha_i-1)\cdots(\alpha_i-n_i+1)} \\ &+ \frac{t^{\alpha_i-1}}{\Gamma(\alpha_i)}\int_0^1 (1-s)^{\alpha_i-n_i} f_i(s,u_1(\tau_{i1}(s)),\ldots,u_N(\tau_{iN}(s)))ds \\ = &\frac{1}{\Gamma(\alpha_i)}\int_0^t \left(t^{\alpha_i-1}(1-s)^{\alpha_i-n_i} - (t-s)^{\alpha_i-1}\right) f_i(s,u_1(\tau_{i1}(s)),\ldots,u_N(\tau_{iN}(s)))ds \\ &+ \frac{1}{\Gamma(\alpha_i)}\int_t^1 t^{\alpha_i-1}(1-s)^{\alpha_i-n_i} f_i(s,u_1(\tau_{i1}(s)),\ldots,u_N(\tau_{iN}(s)))ds + \frac{\eta_i t^{\alpha_i-1}}{(\alpha_i-1)\cdots(\alpha_i-n_i+1)} \\ = &\int_0^1 G_i(t,s) f_i(s,u_1(\tau_{i1}(s)),\ldots,u_N(\tau_{iN}(s)))ds + \frac{\eta_i t^{\alpha_i-1}}{(\alpha_i-1)\cdots(\alpha_i-n_i+1)}. \end{aligned} \tag{3.10}$$

This completes the proof. □

The following two results give some properties of the Green functions $G_i(t,s)$.

Lemma 3.2. *For $i = 1,2,\ldots,N$, $G_i(t,s)$ is continuous on $[0,1]\times[0,1]$ and $G_i(t,s) > 0$ for $(t,s) \in (0,1)\times(0,1)$.*

Proof. Obviously, $G_i(t,s)$ is continuous on $[0,1]\times[0,1]$. It remains to show that $G_i(t,s) > 0$ for $(t,s) \in (0,1)\times(0,1)$. It is easy to see that $G_i(t,s) > 0$ for $0 < t \le s < 1$. We only need to show that $G_i(t,s) > 0$ for $0 < s \le t < 1$. For $0 < s \le t \le 1$, let

$$g_i(t,s) = t^{\alpha_i-1}(1-s)^{\alpha_i-n_i} - (t-s)^{\alpha_i-1}, \tag{3.11}$$

$$h_i(t,s) = (1-s)^{\alpha_i-n_i} - \left(1-\frac{s}{t}\right)^{\alpha_i-1}. \tag{3.12}$$

Then

$$g_i(t,s) = t^{\alpha_i-1} h_i(t,s), \quad 0 < s \le t < 1. \tag{3.13}$$

Note that $h_i(s,s) > 0$ and $(\partial h_i/\partial t)(t,s) = -(\alpha_i - 1)(1 - s/t)^{\alpha_i-2} s t^{-2} < 0$ for $0 < s \le t < 1$. It follows that $h_i(t,s) > 0$ and hence $g_i(t,s) > 0$ for $0 < s \le t < 1$.

Therefore, $G_i(t,s) > 0$ for $0 < s \le t < 1$ and the proof is complete. □

Lemma 3.3. *(i) If $n_i = 2$, then $G_i(t,s) \le G_i(s,s)$ for $(t,s) \in (0,1) \times (0,1)$.*

(ii) If $n_i > 2$, then $G_i(t,s) < G_i(1,s)$ for $(t,s) \in (0,1) \times (0,1)$.

Proof.

(i) Obviously, $G_i(t,s) \le G_i(s,s)$ for $0 < t \le s < 1$. Now, for $0 < s \le t < 1$, we have

$$\frac{\partial g_i(t,s)}{\partial t} = (\alpha_i - 1) t^{\alpha_i-2} \left[(1-s)^{\alpha_i-2} - \left(1 - \frac{s}{t}\right)^{\alpha_i-2} \right] \le 0, \tag{3.14}$$

where g_i is the function defined by (3.11). It follows that $G_i(t,s) \le G_i(s,s)$ for $0 < s \le t < 1$. In summary, we have proved (i).

(ii) Again, one can easily see that $G_i(t,s) < G_i(1,s)$ for $0 < t \le s < 1$. When $0 < s \le t \le 1$, we have in this case that

$$\begin{aligned} \frac{\partial g_i(t,s)}{\partial t} &= (\alpha_i - 1) t^{\alpha_i-2} \left[(1-s)^{\alpha_i-n_i} - \left(1 - \frac{s}{t}\right)^{\alpha_i-2} \right] \\ &\ge (\alpha_i - 1) t^{\alpha_i-2} \left[(1-s)^{\alpha_i-n_i} - (1-s)^{\alpha_i-2} \right] \\ &> 0, \end{aligned} \tag{3.15}$$

which implies that $G_i(t,s) \le G_i(1,s)$ for $0 < s \le t < 1$. To summarize, we have proved (ii) and this completes the proof. □

Now, we are ready to present the main results.

Theorem 3.4. *Suppose that there exist functions $\lambda_{ij}(t) \in C([0,1], \mathbb{R}_+)$, $i, j = 1, 2, \ldots, N$, such that*

$$\left| f_i(t, u_1, \ldots, u_N) - f_i(t, v_1, \ldots, v_N) \right| \le \sum_{j=1}^{N} \lambda_{ij}(t) \left| u_j - v_j \right| \tag{3.16}$$

for $t \in [0,1]$, $i = 1,2,\ldots,N$. If

$$\max_{1 \le i \le N,\, n_i > 2} \int_0^1 G_i(1,s) \left(\sum_{j=1}^{N} \lambda_{ij}(s) \right) ds < 1, \tag{3.17}$$

$$\max_{1\le i\le N,\, n_i=2}\int_0^1 G_i(s,s)\left(\sum_{j=1}^{N}\lambda_{ij}(s)\right)ds<1, \tag{3.18}$$

then (1.1) *has a unique positive solution.*

Proof. Let

$$\Omega=\{u\in E\mid u_i(t)\ge 0 \text{ for } t\in[0,1],\ i=1,2,\dots,N\}. \tag{3.19}$$

It is easy to see that Ω is a complete metric space. Define an operator T on Ω by

$$Tu(t)=\int_0^1 G(t,s)g(s)ds+\operatorname{diag}\left(\dots,\frac{\eta_i t^{\alpha_i-1}}{(\alpha_i-1)\cdots(\alpha_i-n_i+1)},\dots,\right), \tag{3.20}$$

where $G(t,s)=\operatorname{diag}(G_1(t,s),G_2(t,s),\dots,G_N(t,s))$ and

$$g(t)=\begin{pmatrix} f_1(t,u_1(\tau_{11}(t)),u_2(\tau_{12}(t)),\dots,u_N(\tau_{1N}(t)))\\ f_2(t,u_1(\tau_{21}(t)),u_2(\tau_{22}(t)),\dots,u_N(\tau_{2N}(t)))\\ \vdots\\ f_N(t,u_1(\tau_{N1}(t)),u_2(\tau(_{N2}(t)),\dots,u_N(\tau_{NN}(t)))\end{pmatrix}. \tag{3.21}$$

Because of the continuity of G and f, it follows easily from Lemma 3.2 that T maps Ω into itself. To finish the proof, we only need to show that T is a contraction. Indeed, for $u,v\in\Omega$, by (3.16) we have

$$\begin{aligned}
&|(Tu(t))_i-(Tv(t))_i|\\
&\quad=\left|\int_0^1 G_i(t,s)\big(f_i(s,u_1(\tau_{i1}(s)),\dots,u_N(\tau_{iN}(s)))-f_i(s,v_1(\tau_{i1}(s)),\dots,v_N(\tau_{iN}(s)))\big)ds\right|\\
&\quad\le\int_0^1 G_i(t,s)\big|f_i(s,u_1(\tau_{i1}(s)),\dots,u_N(\tau_{iN}(s)))-f_i(s,v_1(\tau_{i1}(s)),\dots,v_N(\tau_{iN}(s)))\big|ds\\
&\quad\le\int_0^1 G_i(t,s)\left(\sum_{j=1}^{N}\lambda_{ij}(s)\big|u_j(\tau_{ij}(s))-v_j(\tau_{ij}(s))\big|\right)ds.
\end{aligned} \tag{3.22}$$

This, combined with Lemma 3.3 and (3.17) and (3.18), immediately implies that $T:\Omega\to\Omega$ is a contraction. Therefore, the proof is complete with the help of Lemmas 3.1 and 2.5. □

The following result can be proved in the same spirit as that for Theorem 3.4.

Theorem 3.5. *For $i = 1, 2, \ldots, N$, suppose that there exist nonnegative function $\lambda_i(t)$ and nonnegative constants $q_i1, q_i2, \ldots, q_iN$ such that $\sum_{j=1}^{N} q_{ij} = 1$ and*

$$\left|f_i(t, u_1, \ldots, u_N) - f_i(t, v_1, \ldots, v_N)\right| \le \lambda_i(t) \prod_{j=1}^{N} \left|u_j - v_j\right|^{q_{ij}} \tag{3.23}$$

for $t \in [0,1]$, $(u_1, u_2, \ldots, u_N)^T, (v_1, v_2, \ldots, v_N)^T \in \mathbb{R}_+^N$. If

$$\max_{1 \le i \le N,\, n_i > 2} \int_0^1 G_i(1,s)\lambda_i(s)ds < 1, \qquad \max_{1 \le i \le N,\, n_i = 2} \int_0^1 G_i(s,s)\lambda_i(s)ds < 1, \tag{3.24}$$

then (1.1) *has a unique positive solution.*

Theorem 3.6. *For $i = 1, 2, \ldots, N$, suppose that there exist nonnegative real-valued functions $m_i, n_{i1}, \ldots, n_{iN} \in L[0,1]$ such that*

$$f_i(t, u_1, \ldots, u_N) \le m_i(t) + \sum_{j=1}^{N} n_{ij}(t)u_j \tag{3.25}$$

for almost every $t \in [0,1]$ and all $(u_1, u_2, \ldots, u_N)^T \in \mathbb{R}_+^N$. If

$$\begin{gathered} \max_{1 \le i \le N,\, n_i > 2} \left\{ \int_0^1 G_i(1,s) \left(\sum_{j=1}^{N} n_{ij}(s) \right) ds \right\} < 1, \\ \max_{1 \le i \le N,\, n_i = 2} \left\{ \int_0^1 G_i(s,s) \left(\sum_{j=1}^{N} n_{ij}(s) \right) ds \right\} < 1, \end{gathered} \tag{3.26}$$

then (1.1) *has at least one positive solution.*

Proof. Let Ω and $T : \Omega \to \Omega$ be defined by (3.19) and (3.20), respectively. We first show that T is completely continuous through the following three steps.

Step 1. Show that $T : \Omega \to \Omega$ is continuous. Let $\{u^k(t)\}$ be a sequence in Ω such that $u^k(t) \to u(t) \in \Omega$. Then $\Omega_0 = [0,1] \times \{u(t) \mid u^k(t), u(t) \in \Omega, t \in [0,1], k \ge 1\}$ is bounded in $[0,1] \times \mathbb{R}_+^N$. Since f is continuous, it is uniformly continuous on any compact set. In particular, for any $\varepsilon > 0$, there exists a positive integer K_0 such that

$$\begin{aligned} &\left| f_i\Big(t, u_1^k(\tau_{i1}(t)), \ldots, u_N^k(\tau_{iN}(t))\Big) - f_i(t, u_1(\tau_{i1}(t)), \ldots, u_N(\tau_{iN}(t))) \right| \\ &\quad < \frac{\varepsilon}{\max_{1 \le i \le N} \max_{t \in [0,1]} \int_0^1 G_i(t,s)ds} \end{aligned} \tag{3.27}$$

for $t \in [0,1]$ and $k \geq K_0, i = 1,2,\ldots,N$. Then, for $k \geq K_0$, we have

$$\begin{aligned}
&\left|\left(Tu^k(t)\right)_i - (Tu(t))_i\right| \\
&= \left|\int_0^1 G_i(t,s) f_i\left(s, u_1^k(\tau_{i1}(s)), \ldots, u_N^k(\tau_{iN}(s))\right)\right. \\
&\qquad \left. - f_i(s, u_1(\tau_{i1}(s)), \ldots, u_N(\tau_{iN}(s)))\right| ds \\
&\leq \int_0^1 G_i(t,s) \left| f_i\left(s, u_1^k(\tau_{i1}(s)), \ldots, u_N^k(\tau_{iN}(s))\right)\right. \\
&\qquad\qquad \left. - f_i(s, u_1(\tau_{i1}(s)), \ldots, u_N(\tau_{iN}(s)))\right| ds \\
&< \frac{\varepsilon}{\max_{1\leq i\leq N} \max_{t\in[0,1]} \int_0^1 G_i(t,s)ds} \int_0^1 G_i(t,s)ds \leq \varepsilon
\end{aligned} \tag{3.28}$$

for $k \geq K_0$ and $t \in [0,1], i = 1,2,\ldots,N$. Therefore,

$$\left\|Tu^k(t) - Tu(t)\right\| < \varepsilon \quad \text{for } k \geq K_0, \tag{3.29}$$

which implies that T is continuous.

Step 2. Show that T maps bounded sets of Ω into bounded sets. Let A be a bounded subset of Ω. Then $[0,1] \times \{u(t) \mid t \in [0,1], u \in A\} \subseteq [0,1] \times \mathbb{R}_+^N$ is bounded. Since f is continuous, there exists an $M > 0$ such that

$$f_i(t, u_1(\tau_{i1}(t)), \ldots, u_N(\tau_{iN}(t))) \leq M \quad \text{for } u \in A,\ t \in [0,1],\ 1 \leq i \leq N. \tag{3.30}$$

It follows that, for $u \in A, t \in [0,1]$ and $1 \leq i \leq N$,

$$\begin{aligned}
(Tu(t))_i &= \int_0^1 G_i(t,s) f_i(s, u_1(\tau_{i1}(s)), \ldots, u_N(\tau_{iN}(s)))ds + \frac{\eta_i}{(\alpha_i - 1)\cdots(\alpha_i - n_i + 1)} \\
&\leq M \int_0^1 G_i(t,s)ds + \frac{\eta_i}{(\alpha_i - 1)\cdots(\alpha_i - n_i + 1)} \\
&\leq \max_{1\leq i\leq N}\left[M \max_{t\in[0,1]} \int_0^1 G_i(t,s)ds + \frac{\eta_i}{(\alpha_i - 1)\cdots(\alpha_i - n_i + 1)}\right].
\end{aligned} \tag{3.31}$$

Immediately, we can easily see that TA is a bounded subset of Ω.

Step 3. Show that T maps bounded sets of Ω into equicontinuous sets. Let B be a bounded subset of Ω. Similarly as in Step 2, there exists $L > 0$ such that

$$f_i(t, u_1(\tau_{i1}(t)), \dots, u_N(\tau_{iN}(t))) \leq L \quad \text{for } u \in B,\ t \in [0,1],\ 1 \leq i \leq N. \tag{3.32}$$

Then, for any $u \in B$ and $t_1, t_2 \in [0,1]$ and $1 \leq i \leq N$,

$$\begin{aligned}
|(Tu(t_2))_i - (Tu(t_1))_i| &= \Bigg| \frac{\eta_i \left(t_2^{\alpha_i - 1} - t_1^{\alpha_i - 1}\right)}{(\alpha_i - 1)\cdots(\alpha_i - n_i + 1)} \\
&\quad + \int_0^1 (G_i(t_2, s) - G_i(t_1, s)) f_i(s, u_1(\tau_{i1}(s)), \dots, u_N(\tau_{iN}(s))) ds \Bigg| \\
&\leq \frac{\eta_i \left|t_2^{\alpha_i - 1} - t_1^{\alpha_i - 1}\right|}{(\alpha_i - 1)\cdots(\alpha_i - n_i + 1)} + \int_0^1 |G_i(t_2, s) - G_i(t_1, s)| L ds \\
&\leq \frac{\eta_i \left|t_2^{\alpha_i - 1} - t_1^{\alpha_i - 1}\right|}{(\alpha_i - 1)\cdots(\alpha_i - n_i + 1)} + \max_{0 \leq s \leq 1} |G_i(t_2, s) - G_i(t_1, s) L.
\end{aligned} \tag{3.33}$$

Now the equicontituity of T on B follows easily from the fact that G_i is continuous and hence uniformly continuous on $[0,1] \times [0,1]$.

Now we have shown that T is completely continuous. To apply Lemma 2.6, let

$$\begin{aligned}
\mu &= \frac{\max_{1 \leq i \leq N, n_i > 2} \left\{ \int_0^1 G_i(1, s) m_i(s) ds + \eta_i / (\alpha_i - 1)\cdots(\alpha_i - n_i + 1) \right\}}{1 - \max_{1 \leq i \leq N, n_i > 2} \left\{ \int_0^1 G_i(1, s) \left(\sum_{j=1}^N n_{ij}(s) \right) ds \right\}}, \\
\nu &= \frac{\max_{1 \leq i \leq N, n_i = 2} \left\{ \int_0^1 G_i(s, s) m_i(s) ds + \eta_i / (\alpha_i - 1)\cdots(\alpha_i - n_i + 1) \right\}}{1 - \max_{1 \leq i \leq N, n_i = 2} \left\{ \int_0^1 G_i(s, s) \left(\sum_{j=1}^N n_{ij}(s) \right) ds \right\}}.
\end{aligned} \tag{3.34}$$

Fix $r > \max\{\mu, \nu\}$ and define

$$U = \{u \in \Omega \| u \|_E < r\}. \tag{3.35}$$

We claim that there is no $u \in \partial U$ such that $u = \lambda T u$ for some $\lambda \in (0,1)$. Otherwise, assume that there exist $\lambda \in (0,1)$ and $u \in \partial U$ such that $u = \lambda T u$. Then

$$\begin{aligned}
|u_i(t)| &= |\lambda(Tu(t))_i| \le |(Tu(t))_i| \\
&\le \int_0^1 G_i(t,s) f_i(s, u_1(\tau_{i1}(t)), u_2(\tau_{i2}(t)), \ldots, u_N(\tau_{iN}(t))) ds + \frac{\eta_i t^{\alpha_i - 1}}{(\alpha_i - 1)\cdots(\alpha_i - n_i + 1)} \\
&\le \int_0^1 G_i(t,s) \left(m_i(s) + \sum_{j=1}^N n_{ij}(s) u_j(\tau_{ij}(s)) \right) ds + \frac{\eta_i}{(\alpha_i - 1)\cdots(\alpha_i - n_i + 1)} \\
&\le \int_0^1 G_i(t,s) m_i(s) ds + r \int_0^1 G_i(t,s) \sum_{j=1}^N n_{ij}(s) ds + \frac{\eta_i}{(\alpha_i - 1)\cdots(\alpha_i - n_i + 1)}.
\end{aligned} \tag{3.36}$$

If $n_i = 2$, then

$$\begin{aligned}
|u_i(t)| &< \int_0^1 G_i(s,s) m_i(s) ds + \frac{\eta_i}{(\alpha_i - 1)\cdots(\alpha_i - n_i + 1)} + r \int_0^1 G_i(s,s) \sum_{j=1}^N n_{ij}(s) ds \\
&\le \nu \left(1 - \max_{1 \le i \le N,\, n_i = 2} \left\{ \int_0^1 G_i(s,s) \left(\sum_{j=1}^N n_{ij}(s) \right) ds \right\} \right) + r \int_0^1 G_i(s,s) \sum_{j=1}^N n_{ij}(s) ds \\
&< r \left(1 - \max_{1 \le i \le N,\, n_i = 2} \left\{ \int_0^1 G_i(s,s) \left(\sum_{j=1}^N n_{ij}(s) \right) ds \right\} \right) + r \int_0^1 G_i(s,s) \sum_{j=1}^N n_{ij}(s) ds \le r.
\end{aligned} \tag{3.37}$$

Similarly, we can have $|u_i(t)| < r$ if $n_i > 2$. To summarize, $\|u\| < r$, a contradiction to $u \in \partial U$. This proves the claim. Applying Lemma 2.6, we know that T has a fixed point in $\overline{U}$, which is a positive solution to (1.1) by Lemma 3.1. Therefore, the proof is complete. □

As a consequence of Theorem 3.6, we have the following.

Corollary 3.7. *If all f_i, $i = 1, 2, \ldots, N$, are bounded, then* (1.1) *has at least one positive solution.*

To state the last result of this section, we introduce

$$\begin{aligned}
M_1 &= \frac{1}{\max_{1 \le i \le N} \int_0^1 G_i(1,s) ds}, \\
N &= \max\left\{ \max_{1 \le i \le N,\, n_i > 2} \int_0^1 G_i(1,s) ds, \quad \max_{1 \le i \le N,\, n_i = 2} \int_0^1 G_i(s,s) ds \right\}.
\end{aligned} \tag{3.38}$$

Theorem 3.8. *Suppose that there exist $M_2 \in (0, 1/N)$ and positive constants $0 < r_1 < r_2$ with $r_2 \ge \max_{1 \le i \le N}\{\eta_i/(\alpha_i - 1)\cdots(\alpha_i - n_i + 1)\}/(1 - M_2 N)$ such that*

(i) $f_i(t,u_1,\ldots,u_N) \le M_2 r_2$ *for* $(t,u_1,\ldots,u_N) \in [0,1] \times B_{r_2}, i = 1,2,\ldots,N$

and

(ii) $f_i(t,u_1,\ldots,u_N) \ge M_1 r_1$, *for* $(t,u_1,\ldots,u_N) \in [0,1] \times B_{r_1}, i = 1,2,\ldots,N$,

where $B_{r_i} = \{u = (u_1,\ldots,u_N)^T \in \mathbb{R}_+^N \mid \max_{1\le i\le N} u_i \le r_i\}, i = 1,2$. *Then* (1.1) *has at least a positive solution.*

Proof. Let Ω be defined by (3.19) and $\Omega_i = \{u \in E \mid \|u\| < r_i\}$, $i = 1, 2$. Obviously, Ω is a cone in E. From the proof of Theorem 3.6, we know that the operator T defined by (3.20) is completely continuous on Ω. For any $u \in \Omega \cap \partial\Omega_1$, it follows from Lemma 3.3 and condition (ii) that

$$\begin{aligned}
\|Tu\|_E &= \max_{1\le i\le N}\max_{0\le t\le 1}(Tu(t))_i \ge \max_{1\le i\le N}(Tu(1))_i \\
&= \max_{1\le i\le N}\left\{\int_0^1 G_i(1,s) f_i(s,u_1(\tau_{i1}(s)),u_2(\tau_{i2}(s)),\ldots,u_N(\tau_{iN}(s)))ds \right.\\
&\qquad \left. + \frac{\eta_i}{(\alpha_i-1)\cdots(\alpha_i-n_i+1)}\right\} \\
&\ge \max_{1\le i\le N}\left\{\int_0^1 G_i(1,s)M_1 r_1 + \frac{\eta_i}{(\alpha_i-1)\cdots(\alpha_i-n_i+1)}\right\} \ge r_1 = \|u\|_E,
\end{aligned} \tag{3.39}$$

that is,

$$\|Tu\|_E \ge \|u\|_E \quad \text{for } u \in \Omega \cap \partial\Omega_1. \tag{3.40}$$

On the other hand, for any $u \in \Omega \cap \partial\Omega_2$, it follows from Lemma 3.3 and condition (i) that, for $t \in [0,1]$,

$$\begin{aligned}
(Tu(t))_i &\le \int_0^1 G_i(1,s)M_2 r_2 ds + \frac{\eta_i}{(\alpha_i-1)\cdots(\alpha_i-n_i+1)} \\
&\le M_2 r_2 \max_{1\le i\le N,\, n_i>2}\int_0^1 G_i(1,s)ds + \max_{1\le i\le N}\frac{\eta_i}{(\alpha_i-1)\cdots(\alpha_i-n_i+1)} \\
&\le M_2 r_2 \max_{1\le i\le N,\, n_i>2}\int_0^1 G_i(1,s)ds + (1-M_2 N)r_2 \le r_2 = \|u\|_E
\end{aligned} \tag{3.41}$$

if $n_i > 2$, whereas

$$\begin{aligned}
(Tu(t))_i &\le \int_0^1 G_i(s,s)M_2 \mathrm{r}_2 ds + \frac{\eta_i}{(\alpha_i-1)\cdots(\alpha_i-n_i+1)} \\
&\le M_2 r_2 \max_{1\le i\le N,\, n_i=2}\int_0^1 G_i(s,s)ds + \max_{1\le i\le N}\frac{\eta_i}{(\alpha_i-1)\cdots(\alpha_i-n_i+1)} \le r_2 = \|u\|_E
\end{aligned} \tag{3.42}$$

if $n_i = 2$. In summary,

$$\|Tu\| \le \|u\|_E \quad \text{for } u \in \Omega \cap \partial\Omega_2. \tag{3.43}$$

Therefore, we have verified condition (ii) of Lemma 2.7. It follows that T has a fixed point in $\Omega \cap (\overline{\Omega_2} \setminus \Omega_1)$, which is a positive solution to (1.1). This completes the proof. □

4. Examples

In this section, we demonstrate the feasibility of some of the results obtained in Section 3.

Example 4.1. Consider

$$\begin{gathered}
D^{5/2}x_1(t) + \frac{e^{-t}(x_1(t/2) + x_2(\sin t))}{(9+e^t)(1 + x_1(t/2) + x_2(\sin t))} = 0, \quad t \in (0,1), \\
D^{5/2}x_2(t) + \frac{t^2(x_1(t^2) + x_2(\sin t))}{10(1 + (x_1(t^2) + x_2(\sin t))} = 0, \quad t \in (0,1), \\
x_1(0) = x_2(0) = x_1'(0) = x_2'(0) = 0, x_1''(1) = x_2''(1) = \frac{1}{2}.
\end{gathered} \tag{4.1}$$

Here

$$\begin{gathered}
n_1 = n_2 = 3, \qquad \alpha_1 = \alpha_2 = \frac{5}{2}, \qquad \eta_1 = \eta_2 = \frac{1}{2}, \\
\tau_{11}(t) = \frac{t}{2}, \qquad \tau_{12}(t) = \tau_{22}(t) = \sin t, \qquad \tau_{21}(t) = t^2, \\
f_1(t, x_1, x_2) = \frac{e^{-t}(x_1 + x_2)}{(9+e^t)(1 + x_1 + x_2)}, \qquad f_2(t, x_1, x_2) = \frac{t^2(x_1 + x_2)}{10(1 + x_1 + x_2)}.
\end{gathered} \tag{4.2}$$

One can easily see that (3.16) is satisfied with

$$\lambda_{11}(t) = \lambda_{12}(t) = \frac{e^{-t}}{9+e^t}, \qquad \lambda_{21}(t) = \lambda_{22}(t) = \frac{t^2}{10}. \tag{4.3}$$

Moreover,

$$G_1(1,s) = G_2(1,s)\frac{(1-s)^{-1/2} - (1-s)^{3/2}}{\Gamma(5/2)}, \quad 0 \le s \le 1 \tag{4.4}$$

and hence

$$\max_{1\le i\le 2}\int_0^1 G_i(1,s)\left(\sum_{j=1}^{2}\lambda_{ij}(s)\right)ds \le \int_0^1 \frac{(1-s)^{-1/2}-(1-s)^{3/2}}{\Gamma(5/2)}\max_{0\le s\le 1}\left\{\frac{2s^2}{10},\frac{2}{e^s(9+e^s)}\right\}ds$$
$$\le \frac{1}{5}\int_0^1 \frac{(1-s)^{-1/2}-(1-s)^{3/2}}{\Gamma(5/2)}ds \tag{4.5}$$
$$= \frac{2-(2/5)}{5\cdot(3/4)\sqrt{\pi}} = \frac{32}{75\sqrt{\pi}} < 1.$$

It follows from Theorem 3.4 that (4.1) has a unique positive solution on $[0,1]$.

Example 4.2. Consider

$$D^{5/2}x_1(t) + \frac{tx_1(t)}{20} + \frac{x_2(t)}{20} + \frac{t}{10} + \frac{1}{10} = 0, \quad t\in(0,1),$$
$$D^{3/2}x_2(t) + \frac{x_1(1)}{20} + \frac{tx_2(t)}{20} + \frac{t^2}{10} + \frac{1}{10} = 0, \quad t\in(0,1), \tag{4.6}$$
$$x_1(0) = x_2(0) = x_1'(0) = 0, x_1'',(1) = x_2''(1) = \frac{1}{2}.$$

Here

$$n_1 = 3, \qquad n_2 = 2, \qquad \alpha_1 = \frac{5}{2}, \qquad \alpha_2 = \frac{3}{2}, \qquad \eta_1 = \eta_2 = \frac{1}{2},$$
$$f_i(t,x_1,x_2) = m_i(t) + \sum_{j=1}^{2} n_{ij}(t)x_j, \quad i = 1,2, \tag{4.7}$$

where

$$m_1(t) = \frac{t}{10} + \frac{1}{10}, \qquad m_2(t) = \frac{t^2}{10} + \frac{1}{10},$$
$$n_{11}(t) = n_{22}(t) = \frac{t}{20}, \qquad n_{12}(t) = n_{21}(t) = \frac{1}{20}. \tag{4.8}$$

Hence, f_1 and f_2 satisfy (3.25). Moreover, simple calculations give us

$$\int_0^1 G_1(1,s)ds = \frac{32}{15\sqrt{\pi}}, \qquad \int_0^1 G_2(1,s)ds = \frac{8}{3\sqrt{\pi}},$$
$$\int_0^1 G_1(s,s)ds = \frac{\sqrt{\pi}}{2}, \qquad \int_0^1 G_2(s,s)ds = \sqrt{\pi}. \tag{4.9}$$

Then $M_1 = 3\sqrt{\pi}/8$ and

$$N = \max\left\{\int_0^1 G_1(1,s)ds, \int_0^1 G_2(s,s)ds\right\} = \sqrt{\pi}. \tag{4.10}$$

Choose $M_2 = \sqrt{\pi}/10 \in (0, 1/N) = (0, 1/\sqrt{\pi})$, $r_1 = \sqrt{\pi}/12$ and

$$r_2 = \max\left\{\frac{1/2}{(5/2-1)(5/2-3+1)}, \frac{1/2}{(3/2-2+1)}\right\} \Big/ \left(1 - \frac{\sqrt{\pi}}{10}\sqrt{\pi}\right) = \frac{10}{10-\pi}. \tag{4.11}$$

Then, for $\|(x_1, x_2)^T\| \le r_2$ and $t \in [0,1]$, we have

$$\begin{aligned} f_1(t, x_1, x_2) &= \frac{tx_1}{20} + \frac{x_2}{20} + \frac{t}{10} + \frac{1}{10} \le \frac{r_2}{10} + \frac{1}{10} = \frac{1}{10-\pi} + \frac{1}{10} \\ &\le 0.24581 < 0.2584 < M_2 r_2 \\ f_2(t, x_1, x_2) &= \frac{x_1}{20} + \frac{tx_2}{20} + \frac{t^2}{10} + \frac{1}{10} < M_2 r_2; \end{aligned} \tag{4.12}$$

for $\|(x_1, x_2)^T\| \le r_1$ and $t \in [0,1]$, we have

$$f_1(t, x_1, x_2), f_2(t, x_1, x_2) \ge \frac{1}{10} > \frac{\pi}{32} = M_1 r_1. \tag{4.13}$$

By now we have verified all the assumptions of Theorem 3.8. Therefore, (4.6) has at least one positive solution $x = (x_1, x_2)^T$ satisfying $\sqrt{\pi}/12 \le \|x\| \le 10/(10-\pi)$.

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