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First Steps in Synthetic Computability Theory

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Abstract

Computability theory, which investigates computable functions and computable sets, lies at the foundation of computer science. Its classical presentations usually involve a fair amount of Gödel encodings which sometime obscure ingenious arguments. Consequently, there have been a number of presentations of computability theory that aimed to present the subject in an abstract and conceptually pleasing way. We build on two such approaches, Hyland's effective topos and Richman's formulation in Bishop-style constructive mathematics, and develop basic computability theory, starting from a few simple axioms. Because we want a theory that resembles ordinary mathematics as much as possible, we never speak of Turing machines and Gödel encodings, but rather use familiar concepts from set theory and topology.

Keywords: synthetic computability theory, constructive mathematics.

1 Introduction

Classical presentations of the theory of computable functions and computably enumerable sets [3,9,15,19] start by describing one or more equivalent models of computation, such as Turing machines or schemata for defining partial recursive functions. Descriptions of Turing machines and numerous uses of Gödel encodings give classical computability theory its distinguishing flavor, which one needs to get used to before one can appreciate the ingenious constructions that are unique to computability theory. There are also presentations of computability as an abstract theory that proceeds from basic axioms,

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with as few explicit references to Turing machines and Gödel encodings as possible [13,8,6], where in most cases the axiomatizations take as the primitive notion the computation of a program on a given input, or a similar concept.

A different approach to computability theory is to work in a mathematical universe with computability built in, such as M. Hyland's *effective topos* [10] or P. Mulry's [14] *recursive topos*. In these settings, details about computations are hidden by a level of abstraction, so instead of fiddling with Turing machines and Gödel codes, one uses abstract tools, namely category theory, to achieve the desired results. How this can be done was shown by D.S. Scott, P. Mulry [14], M. Hyland [10], D. McCarty [12], G. Rosolini [18], and others.

We are going to draw on experience from the effective topos and synthetic domain theory, in particular on G. Rosolini's work [18]. However, instead of working explicitly with the topos, which requires a certain amount of knowledge of category theory *in addition* to familiarity with computability theory, we shall follow F. Richman and D. Bridges [17,2], and work within ordinary (constructive) set theory enriched with few simple axioms about sets and sequences of natural numbers. Because one of our axioms contradicts Aristotle's Law of Excluded Middle, the underlying logic and set theory must be *intuitionistic*. Arguably, main-stream mathematicians consider intuitionistic mathematics to be more exotic than computability theory. This opinion is countered by the claim that intuitionistic logic is the natural mathematical foundation for computer science. The present paper supports this view by developing computability theory in an intuitionistic setting.

Our goal is to develop a theory of computability *synthetically:* we work in a mathematical universe in which *all* sets and functions come equipped with intrinsic computability structure. Precisely because computability is omnipresent, we never have to speak about it—there will be no mention of Turing machines, or any other notion of computation. In the synthetic universe, the computable functions are simply all the functions, the computably enumerable sets are all the enumerable sets, etc. So we may just speak about ordinary sets and functions and never worry about which ones are computable. For example, there is no question about what the computable real numbers should be, or how to define computability on a complicated mathematical structure. We just do "ordinary" math—in an extraordinary universe.

You may wonder how exactly such a universe is manufactured. The prime model of our theory is the effective topos. Its existence guarantees that the theory is as consistent as the rest of mathematics. In fact, the specialists will recognize it all as just clever use of the *internal language* of the effective topos. Knowledge of topos theory or the effective topos is not needed to understand synthetic computability, although familiarity with it will certainly help explain some of the axioms and constructions.

The intended audience for synthetic computability is manifold, as there are several communities interested in computability. The constructive mathematicians should have no trouble understanding the matter, since it is written in their language, although the ascetic ones may find the extra axioms unacceptable. The classical computability theorists are a target audience whose approval will be the difficult to win. This is so because considerable effort is required when one first switches from classical to constructive logic, so the payoff needs to be noteworthy. A second reason is that synthetic computability has not been developed far enough to approach current research topics in computability theory. Hopefully, some experts in computability theory will be convinced that synthetic computability is a useful supplemental tool. Computer scientists tend to be more open-minded than mathematicians, so they need not worry us too much.

When an old subject is reformulated in a new way, as is the case here, success may be claimed to a lesser degree if the new formulation leads to a more elegant account, and to a larger degree when it leads to new results. The readers will be the judges of the first criterion, while it is too early to say much about the second one.

The rest of the paper is organized as follows. Section 2 reviews a few points about constructive mathematics. Section 3 sets up the basic theory of enumerable sets and semidecidable truth values. Section 4 develops the beginnings of synthetic computability theory. A selection of standard theorems in computability is proved, among others: Single Value Theorem, Enumerability Theorem for partial recursive functions, non-existence of a computable enumeration of total recursive functions, Projection Theorem, Post's Theorem, existence of computably enumerable non-recursive set, existence of inseparable, creative, simple and immune sets, Recursion Theorem, Berger's Branching Lemma, Rice-Shapiro, and Myhill-Shepherdson Theorems. Some of these are generalized, and the formulation of Recursion Theorem is new.

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2 Constructive Mathematics

The foundation of synthetic computability is constructive mathematics. This is not a philosophical decision or a matter of opinion. Aristotle's Law of Excluded Middle contradicts one of the axioms of synthetic computability, which gives us no choice but to abandon classical logic.

We work in the internal language of a topos, or Bishop-style constructive mathematics. Our logic is intuitionistic, all quantifiers are bounded by sets, and the members A_i of a family of sets $\{A_i \mid i \in I\}$ must all be contained in a previously constructed set A. An example of a formal system that suits our needs is *Zermelo set theory*, as presented by P. Taylor [21, §2.2]. We do not employ impredicativity in an essential way, and most results could be done in a weaker system based on first-order logic.

Occasionally, we shall speak of the "computational interpretation" of this or that concept, and refer to computations and Gödel codes. Such excursions into classical computability theory are only meant to motivate and clarify what we are doing. The theory proper really does speak *only* about ordinary sets and functions, and does not rely in any way on classical computability theory.

We presuppose the existence of the set of natural numbers $\mathbb{N}$, which satisfies the usual Peano axioms. We let $\mathsf{1} = \{\star\}$ be the singleton set containing precisely one element $\star$. We define $\mathbb{N}^+$ to be the set of monotone binary sequences,

$$\mathbb{N}^+ = \left\{f \in 2^{\mathbb{N}} \mid \forall\, n \in \mathbb{N} \,.\, (f(n) = 1 \implies f(n+1) = 1)\right\} \;.$$

There is an injection $i : \mathbb{N} \to \mathbb{N}^+$ which maps n to $i(n) = \lambda k : \mathbb{N} \,.\, (n \leq k)$. We identify $i(n)$ with n and view $\mathbb{N}$ as a subset of $\mathbb{N}^+$. The elements $i(n)$ are the *finite* points of $\mathbb{N}^+$. The constantly zero sequence $\lambda k : \mathbb{N} \,.\, 0$ is an element of $\mathbb{N}^+$ but not of $\mathbb{N}$. We call it *infinity* and denote it by ∞. A picture of $\mathbb{N}^+$ is that of a one-point compactification of $\mathbb{N}$. The set $\mathbb{N}^+$ is ordered by the "less than" relation $<$, defined as

$$f < g \iff \exists\, k \in \mathbb{N} \,.\, (f(k) = 1 \wedge g(k) = 0) \;.$$

It is irreflexive, asymmetric, transitive and linear, in the sense that $f < g$ implies $h < f$ or $g < h$, for all $f, g, h \in \mathbb{N}^+$. When restricted to $\mathbb{N}$, $<$ is the usual "less than" relation. For every $n \in \mathbb{N}$ we have $n < \infty$, as expected. If $n \in \mathbb{N}$ and $g \in \mathbb{N}^+$, the proposition $i(n) < g$ is decidable because it is equivalent to $g(n) = 0$. Similarly, $\leq$ on $\mathbb{N}^+$ is defined by

$$f \leq g \iff \neg(g < f) \iff \forall\, k \in \mathbb{N} \,.\, (f(k) = 0 \implies g(k) = 0) \;.$$

The set $\mathbb{N}^+$ is the *generic converging sequence* and will be useful in proving the Rice-Shapiro and Myhill-Shepherdson theorems.

2.1 Truth values

We identify a truth value p with its extension $\{u \in \mathsf{1} \mid p\}$, which is a subset of $\mathsf{1}$. This way the set of truth values is simply the powerset of $\mathsf{1}$, $\Omega = \mathcal{P}\mathsf{1}$. Falsehood $\bot$ and truth $\top$ are elements of Ω, represented by $\emptyset$ and $\mathsf{1}$, respectively. Classically, $\bot$ and $\top$ are the only elements of Ω. Intuitionistically, this is not the case, but neither is it the case that there is a truth value that differs from both $\bot$ and $\top$.

There are various interesting subsets of Ω. For example, we may consider the set of *decidable* truth values, which are those that satisfy the Law of Excluded Middle,

$$\mathsf{2} = \{p \in \Omega \mid p \lor \neg p\} .$$

Certainly $\bot$ and $\top$ are decidable truth values. Now if $p \lor \neg p$ then $p = \top$ or $p = \bot$, so we see that $\mathsf{2}$ does not contain any other elements. We have given it a good name, also because $\mathsf{2} = \mathsf{1} + \mathsf{1}$. When $\bot$ and $\top$ are seen as elements of $\mathsf{2}$ we prefer to denote them by 0 and 1, respectively. The set $\mathsf{2}$ is a Boolean algebra whose operations $\land$ and $\lor$ are inherited from Ω.

Another distinguished subset of Ω, which is related to classical logic, is the set of *classical* truth values:

$$\Omega_{\neg\neg} = \{p \in \Omega \mid \neg\neg p \implies p\} .$$

The elements of $\Omega_{\neg\neg}$ are those truth values whose truth may be established by reductio ad absurdum: if $\neg p$ implies falsehood then p holds. Because reductio ad absurdum implies classical logic we call the elements of $\Omega_{\neg\neg}$ "classical". This is not standard terminology, but we really do want to avoid the standard awkward phrase "$\neg\neg$-stable truth values", which expresses the fact that $\neg\neg$ is a closure operator on Ω and that $\Omega_{\neg\neg}$ is the set of its fixed points. While Ω is only a complete Heyting algebra, $\Omega_{\neg\neg}$ is also a *complete Boolean algebra.*

The decidable truth values $\bot$ and $\top$ are classical, thus $\mathsf{2} \subseteq \Omega_{\neg\neg} \subseteq \Omega$. In 3.3 we shall define a third subset of Ω which plays a central role in synthetic computability.

A predicate $P \subseteq A$ may be equivalently expressed by a *characteristic function* $\chi_P : A \to \Omega$. Too see this, notice that $\mathcal{P}A \cong \mathcal{P}(A \times \mathsf{1}) \cong (\mathcal{P}\mathsf{1})^A = \Omega^A$. Propositional functions which map into a distinguished subset of Ω, such as $\mathsf{2}$ or $\Omega_{\neg\neg}$, determine special kinds of predicates. For example, a function $p : A \to \mathsf{2}$ represents a subset $S = \{x \in A \mid p(x) = \top\}$ which satisfies $\forall\, x \in A\,.\,(x \in S \lor x \notin S)$. Such predicates and subsets are called *decidable*. Computationally we may view a decidable predicate as one for which there exists a computable decision procedure. A function $p : A \to \Omega_{\neg\neg}$ represents a *classical* predicate or subset $S \subseteq A$. Such a subset satisfies

$\forall x \in A . (\neg\neg(x \in S) \implies x \in S)$.

When equality on a set has a special property, we usually give it a name. Thus a set A is *decidable* if equality on A is a decidable predicate:

$$\forall x, y \in A . (x = y \vee x \neq y) \ .$$

The following sets are decidable: natural numbers, a subset of a decidable set, the cartesian product and the sum of decidable sets.

Similarly, a set A is *classical* if equality on A is a classical predicate:

$$\forall x, y \in A . (\neg(x \neq y) \implies x = y) \ .$$

Every decidable set is also classical, but the converse need not hold. A subset of a classical set is classical.

Beware of a possible terminological confusion: a "decidable subset" is a subset whose membership predicate is decidable, whereas a "decidable set" is one whose equality is decidable. Thus it is easy to muddle matters when we speak about a subset with decidable equality. The same caveat stands for "classical subsets".

2.2 Axiom of Choice

In constructive mathematics the axiom of choice is handled with some care. We say that *choice holds for sets A and B*, written $\mathrm{AC}(A, B)$, when every total relation between A and B has a choice function:

$$\forall R \subseteq A \times B . \big((\forall x \in A . \exists y \in B . R(x, y)) \\ \implies \exists f \in B^A . \forall x \in A . R(x, f(x))\big) \ .$$

A set A is *projective* when $\mathrm{AC}(A, B)$ holds for every set B. In classical set theory the axiom of choice states that all sets are projective. We are going to be much more restrictive because of the following computational explanation: a set A is projective when every element of A has a canonical Gödel code. Thus we would expect $\mathbb{N}$ to be projective, since in the standard Gödel coding of $\mathbb{N}$ each number is represented canonically just by itself, but we would not expect $\mathbb{N}^{\mathbb{N}}$ to be projective, because we cannot effectively choose a canonical Gödel code for each total recursive function (do you know why?).

In Bishop's constructive mathematics, the natural numbers are indeed presumed to be projective. This is known as Number Choice.

Axiom 2.1 (Number Choice) *The set of natural numbers $\mathbb{N}$ is projective.*

In fact, Bishop's mathematics adopts Dependent Choice, which is the following strengthening of Number Choice.

Axiom 2.2 (Dependent Choice) *If R is a total relation on A and $x \in A$ then there exists $f : \mathbb{N} \to A$ such that $f(0) = x$ and $R(f(n), f(n+1))$ for all $n \in \mathbb{N}$.*

The computational justification for Dependent Choice goes like this. To say that R is effectively total means that from a Gödel code m for $y \in A$ we can compute a Gödel code n for some $z \in A$ such that $R(y, z)$. Now if k is a Gödel code for x, start by defining $f(0) = x$ and $g(0) = k$. Then if $f(i) = y$ is already defined and $g(i) = m$ is a Gödel code for y, compute a Gödel code n of some z such that $R(y, z)$, then define $f(i+1) = z$ and $g(i+1) = n$. This gives us the desired choice function f.

We observe that Number Choice and Dependent Choice are compatible with classical mathematics.

3 Enumerable and Semidecidable Sets

We now embark properly on the subject of computability theory. Unlike in most classical treatments of computability theory, we do not start with partial recursive functions but rather with computably enumerable sets. This is so because in our settings the computably enumerable sets are simply the enumerable sets, while some effort is needed to introduce the partial recursive functions.

3.1 Finite lists and finite sets

We quickly review the basic constructive theory of finite sequences and finite sets. Computationally speaking, a set is finite if we can compute a finite sequence of Gödel codes of its elements, where some elements may be represented more than once.

For a natural number $n \in \mathbb{N}$, define the set $\{1, \ldots, n\} = \{k \in \mathbb{N} \mid k \leq n\}$. A *(finite) list*, or a *finite sequence* of elements in A is a map $\ell : \{1, \ldots, n\} \to A$ for some $n \in \mathbb{N}$. The number n is called the *length* of ℓ and is denoted by $|\ell|$. We write a list ℓ by $[\ell(1), \ldots, \ell(n)]$. Given $x \in A$ we may form a new list $x{::}\ell$ whose first element is x followed by elements of ℓ. Every list is either empty or of the form $x{::}\ell$ for unique x and ℓ. The set of all finite sequences of elements of A is denoted by $\mathsf{Seq}\,A$.

A set A is *finite* if there exists $n \in \mathbb{N}$ and a surjection $e : \{1, \ldots, n\} \to A$, called a *listing* of A. The collection of all finite subsets of A is denoted by $\mathcal{P}_{\mathrm{fin}}A$. There is a quotient map $\mathsf{Seq}\,A \to \mathcal{P}_{\mathrm{fin}}A$ which assigns to a list ℓ its image $\mathsf{im}(\ell) = \{x \in A \mid \exists\, k \in \{1, \ldots, |\ell|\} \,.\, x = \ell(k)\}$. A set is *subfinite* if it is a subset of a finite set. Our definition of finite sets is equivalent to that of *Kuratowski finite set:* $\mathcal{P}_{\mathrm{fin}} A$ is the join-semilattice generated by A.

The following is a useful observation: a finite set is either empty or inhabited. For suppose $e : \{1, \ldots, n\} \to A$ is a listing of A. If $n = 0$ then A is empty, and if $n \neq 0$ then A is inhabited by $e(1)$.

Proposition 3.1 *A finite set may be listed without repetitions if, and only if, it is decidable.*

Proposition 3.2 *If A is decidable then so are $\mathsf{Seq}\, A$ and $\mathcal{P}_{\mathrm{fin}} A$.*

3.2 Enumerable Sets

The computably enumerable sets, formerly known as the recursively enumerable sets, play a central role in classical computability theory. They are a basic concept in the synthetic theory as well, except that they are now just “ordinary” enumerable sets.

Definition 3.3 A set A is *enumerable*, or *countable*, if there exists a surjection $e : \mathbb{N} \to \mathbf{1} + A$, called an *enumeration* of A. A set is *subenumerable*, or *subcountable*, if it is a subset of an enumerable set.

The reason for mapping $\mathbb{N}$ onto $\mathbf{1} + A$ rather than onto A is that we also want to include the empty set among the enumerable ones. The role of the special element $\star \in \mathbf{1}$ is to enumerate nothing, so that the empty set may be enumerated by $\star, \star, \star, \ldots$ When A is inhabited, we may enumerate it by an onto map $e : \mathbb{N} \twoheadrightarrow A$.

For reference we list basic facts about enumerable sets, and leave it to the reader to prove them (constructively):

(i) Finite sets are enumerable.

(ii) Enumerable sets are closed under quotients, disjoint sums, finite cartesian products, and decidable subsets.

(iii) The union of an enumerable family of enumerable sets is enumerable.

(iv) The intersection of two enumerable subsets of a decidable set is enumerable.

(v) The finite sequences of an enumerable set form an enumerable set.

(vi) The finite subsets of an enumerable set form an enumerable family.

We say that $e : \mathbb{N} \to \mathbf{1} + A$ enumerates A *without repetitions* when e is an enumeration and, for all $n, m \in \mathbb{N}$, $e(n) = e(m) \neq \star$ implies $n = m$.

Proposition 3.4 *A set can be enumerated without repetitions if, and only if, it is enumerable and decidable.*

Corollary 3.5 *Every enumerable subset of $\mathbb{N}$ can be enumerated without repetitions.*

Proof. Every subset of $\mathbb{N}$ is decidable. □

We say that a set A *contains an infinite sequence* if there exists an injection $a : \mathbb{N} \to A$. When $A \subseteq \mathbb{N}$, such an injection may always be replaced by a strictly increasing one.

Proposition 3.6 *An inhabited enumerable subset of $\mathbb{N}$ may be enumerated in a strictly increasing order if, and only if, it is a decidable subset of $\mathbb{N}$ and it contains an infinite sequence.*

Proposition 3.7 *A decidable enumerable inhabited set A is a retract of $\mathbb{N}$. Furthermore, if A contains an infinite sequence then it is isomorphic to $\mathbb{N}$.*

Corollary 3.8 *The following sets are isomorphic to $\mathbb{N}$:*

(i) *the set of k-tuples $\mathbb{N}^k$, with $k \geq 1$,*

(ii) *the set of finite sequences $\mathsf{Seq}\,\mathbb{N}$,*

(iii) *the family of finite subsets $\mathcal{P}_{\mathrm{fin}}\mathbb{N}$.*

Proof. These sets are decidable, enumerable and contain infinite sequences. □

The preceding proposition tells us that we may enumerate the elements of a set with k-tuples, finite sequences or finite sets of numbers. We shall do so whenever convenient.

We have exhibited some enumerable sets, but we still do not know whether there are any sets that are *not* enumerable. Let us show that *Cantor space* $2^{\mathbb{N}}$ and *Baire space* $\mathbb{N}^{\mathbb{N}}$ are not enumerable. Recall that a set A has the *fixed point property* if every map $f : A \to A$ has a fixed point, which is an element $x \in A$ such that $f(x) = x$. We use Lawvere's formulation of Cantor's argument.

Proposition 3.9 (Lawvere) *If $e : A \to B^A$ is a surjection then B has the fixed point property.*

Proof. Given $f : B \to B$, there is $x \in A$ such that $e(x) = \lambda y{:}A\,.\,f(e(y)(y))$ because e is surjective. Then $e(x)(x) = f(e(x)(x))$, hence $e(x)(x)$ is a fixed point of f. □

In passing we prove a famous theorem by Cantor.

Corollary 3.10 (Cantor's Theorem) *There is no surjection $A \twoheadrightarrow \mathcal{P}A$.*

Proof. If there were a surjection $A \twoheadrightarrow \mathcal{P}A = \Omega^A$ then Ω would have the fixed point property, but it does not because negation $\neg : \Omega \to \Omega$ does not have a fixed point. $\square$

Corollary 3.11 Cantor space $2^{\mathbb{N}}$ *and* Baire space $\mathbb{N}^{\mathbb{N}}$ *are* not *enumerable.*

Proof. The sets 2 and $\mathbb{N}$ do not have the fixed point property. $\square$

We have just proved our first theorem in computability, namely that total recursive functions and recursive sets cannot be effectively enumerated.

Of particular interest to us is the set of enumerable subsets of $\mathbb{N}$, which we denote by $\mathcal{E}$:

$$\mathcal{E} = \left\{ A \in \mathcal{P}\mathbb{N} \;\middle|\; A \text{ is enumerable} \right\} .$$

It is the least family of subsets of $\mathbb{N}$ that contains $\emptyset$ and $\mathbb{N}$, the singletons and is closed under finite intersections and countable unions. Thus $\mathcal{E}$ is like the topology on $\mathbb{N}$ generated by singletons, i.e., the discrete topology, except that it is closed under enumerable unions rather than arbitrary ones. We shall comment on this topological view of enumerable sets in 3.5.

Recall that the *projection* of a subset $S \subseteq A \times B$ is the set

$$\left\{ x \in A \;\middle|\; \exists\, y \in B \,.\, \langle x, y\rangle \in S \right\} .$$

Theorem 3.12 (Projection Theorem) *A subset of* $\mathbb{N}$ *is enumerable if, and only if, it is the projection of a decidable subset of* $\mathbb{N} \times \mathbb{N}$.

Proof. Assume first that A is enumerated by $e : \mathbb{N} \to 1 + A$. For the set S we simply take the graph of e, $S = \left\{ \langle m, n\rangle \in \mathbb{N} \times \mathbb{N} \;\middle|\; e(m) = n \right\}$, which is a decidable set because $\mathbb{N}$ has decidable equality. Conversely, suppose $S \subseteq \mathbb{N} \times \mathbb{N}$ is decidable and let $A = \left\{ m \in \mathbb{N} \;\middle|\; \exists\, n \in \mathbb{N} \,.\, \langle m, n\rangle \in S \right\}$. Define a map $e : \mathbb{N} \times \mathbb{N} \to 1 + \mathbb{N}$ by $e\langle i, j\rangle = \texttt{if}\ \langle i, j\rangle \in S\ \texttt{then}\ i\ \texttt{else}\ \star$. Then e is well defined because S is decidable, and it obviously enumerates A. $\square$

3.3 The Semidecidable Truth Values

The computably enumerable sets are also known as the “semidecidable” sets. In this section we show that also in synthetic computability the enumerable sets are semidecidable in a precise sense: we find a set $\Sigma \subseteq \Omega$ of “semidecidable” truth values such that $\mathcal{E} = \Sigma^{\mathbb{N}}$.

Recall that by the Projection Theorem every $A \in \mathcal{E}$ is the projection of a decidable subset $D \subseteq \mathbb{N} \times \mathbb{N}$. Let $d : \mathbb{N} \times \mathbb{N} \to 2$ be the characteristic function of D. Then the characteristic function $\chi_A : \mathbb{N} \to \Omega$ of A is $\chi_A(m) = (\exists\, n \in \mathbb{N} \,.\, d\langle m, n\rangle)$. $\exists\, n \in \mathbb{N} \,.\, f(n)$ where $f : \mathbb{N} \to 2$ is $f(n) = d\langle m, n\rangle$. So we may define the set of *semidecidable* truth values

$$\Sigma = \left\{ p \in \Omega \;\middle|\; \exists\, f \in 2^{\mathbb{N}} \,.\, (p \iff (\exists\, n \in \mathbb{N} \,.\, f(n))) \right\} \ .$$

As a first observation, note that the decidable truth values are semidecidable, $2 \subseteq \Sigma$. In general, we define a *semidecidable* subset $S \subseteq A$ to be one whose characteristic map $\chi_S : A \to \Omega$ maps into Σ, that is $(x \in S) \in \Sigma$ for all $x \in A$.

Proposition 3.13 *A subset of $\mathbb{N}$ is enumerable if, and only if, it is semidecidable.*

Proof. By definition of Σ. □

The definition of Σ is well known in *synthetic domain theory* and appears in [18]. It is a *dominance*, which means that it satisfies $\top \in \Sigma$ and Rosolini's *dominance axiom*

$$\forall\, p \in \Sigma \,.\, \forall\, q \in \Omega \,.\, ((p \implies (q \in \Sigma)) \implies (p \wedge q) \in \Sigma) \ .$$

The set Σ inherits a partial order from the complete Heyting algebra Ω, but it itself is not a complete Heyting algebra, only a lattice with countable suprema. (Recall that a lattice is a poset L with least and greatest elements, and binary infima and suprema.)

Definition 3.14 A *σ-frame* is a non-trivial lattice L in which suprema of enumerable sets exist and binary infima distribute over enumerable suprema. A *morphism* between σ-frames is a map which preserves the lattice structure and enumerable suprema.

The definition of σ-frames was given by Rosolini [18] who called them "σ-algebras". We prefer not to call them σ-algebras in order to avoid confusion with measure-theoretic σ-algebras.

Proposition 3.15 *Σ is the initial σ-frame: for every σ-frame L there exists a unique morphism $\Sigma \to L$.*

We mention one more characterization of Σ, which we could use as the definition of Σ in a predicative system (one without powersets and Ω).

Proposition 3.16 *Σ is a quotient of $\mathbb{N}^+$ via the map $q : \mathbb{N}^+ \to \Sigma$ defined by $q(x) = (x < \infty)$.*

Proof. Recall that $x \in \mathbb{N}^+$ is a binary sequence which is smaller than ∞ when it contains a 1. So $x < \infty$ is equivalent to $\exists\, n \in \mathbb{N} \,.\, x(n) = 1$, which is semidecidable. The map q is surjective because any $p \in \Sigma$ with $p \iff \exists\, n \in \mathbb{N} \,.\, f(n) = 1$ is equivalent to $x < \infty$ where $x(n) = (\exists\, k \leq n \,.\, f(k) = 1)$. □

3.4 Markov Principle

If $a_0, a_1, a_2, \ldots$ is a sequence of zeros and ones, not all of which are zeros, must there be a one in the sequence? An affirmative answer is known as *Markov Principle* and has several equivalent forms.

Proposition 3.17 *The following are equivalent:*

(i) *Markov Principle: for every* $a : \mathbb{N} \to 2$,

$$\neg(\forall\, n \in \mathbb{N} \,.\, a_n = 0) \implies \exists\, n \in \mathbb{N} \,.\, a_n = 1 \,.$$

(ii) *For all* $x \in \mathbb{N}^+$, *if* $x \neq \infty$ *then* $x \in \mathbb{N}$.

(iii) *Semidecidable truth values are classical,* $\Sigma \subseteq \Omega_{\neg\neg}$.

(iv) *Semidecidable subsets are classical.*

(v) *Semidecidable subsets of* $\mathbb{N}$ *are classical.*

While it may seem intuitively clear that Markov Principle holds, it cannot be proved constructively. A number of results in computability rely on its validity. Therefore, we accept it as an axiom.

Axiom 3.18 (Markov Principle) *A binary sequence which is not constantly* 0 *contains a* 1*.*

Proposition 3.19 *Equality and the partial order on* Σ *are classical.*

Proof. Because $\Sigma \subseteq \Omega_{\neg\neg}$ and $\Omega_{\neg\neg}$ has classical equality and partial order, Σ does as well. $\square$

The following is a useful consequence of Markov Principle.

Lemma 3.20 *For any classical predicate* $\psi : \Sigma \to \Omega_{\neg\neg}$, $(\forall\, p \in \Sigma \,.\, \psi(p)) \iff \psi(\bot) \wedge \psi(\top)$.

Proof. One direction is obvious. For the other, suppose $\psi(\bot)$ and $\psi(\top)$. Then, for any $p \in \Sigma$, $(p = \top \vee p = \bot) \implies \psi(p)$, hence $\neg\neg(p = \top \vee p = \bot) \implies \neg\neg\psi(p)$. But $\neg\neg(p = \top \vee p = \bot) = \neg\neg(p \vee \neg p) = \neg(\neg p \wedge \neg\neg p) = \neg\bot = \top$ and $\neg\neg\psi(p) = \psi(p)$, therefore $\psi(p)$ for every $p \in \Sigma$. $\square$

Every book on computability theory contains (the special case $A = \mathbb{N}$ of) the following theorem.

Theorem 3.21 (Post) *A subset of a set* A *is decidable if, and only if, it and its complement are semidecidable.*

Proof. The theorem may be rephrased in terms of truth values: a truth value $p \in \Omega$ is decidable if, and only if, p and $\neg p$ are semidecidable. Obviously since $2 \subseteq \Sigma$, a decidable truth value and its complement are semidecidable. Conversely, suppose $p \in \Sigma$ and $\neg p \in \Sigma$. Then by Markov Principle $p \vee \neg p \in \Sigma \subseteq \Omega_{\neg\neg}$, therefore $p \vee \neg p = \neg\neg(p \vee \neg p) = \neg(\neg p \wedge \neg\neg p) = \neg\bot = \top$, hence $p \in 2$. $\square$

3.5 The Topological View

Recall that $\Sigma^{\mathbb{N}}$ is like a topology on $\mathbb{N}$. In fact, for any set A, the family of semidecidable predicates Σ^A is a σ-frame: $\emptyset$ and A are the least and the greatest elements of Σ^A, binary infima are computed as intersections, and countable suprema as unions. It makes sense then to think of Σ^A as a topology on A.

Definition 3.22 The *(intrinsic) topology* of a set A is the set Σ^A of semidecidable subsets, which are also called *open sets*. The *closed sets* are the complements of the open ones.

In the theory of effective topological spaces [20] the intrinsic topology is known as the *Eršov topology*.

Let us compare the situation with classical topology. Recall that the open subsets of a classical topological space X are in bijective correspondence with continuous maps $X \to \mathbb{S}$, where $\mathbb{S}$ is the *Sierpinski space* which consists of two points $\bot$ and $\top$, with $\{\top\}$ open and $\{\bot\}$ closed. In our setting, the correspondence holds by definition, with Σ in place of $\mathbb{S}$ and arbitrary maps in place of continuous ones. However, the maps are not as arbitrary as you might think:

Proposition 3.23 *All maps are continuous.*

Proof. For any $f : A \to B$ and an open set $U : B \to \Sigma$, the inverse image $f^*(U) = \{x \in A \mid f(x) \in U\}$ is open because its characteristic map is $U \circ f : A \to \Sigma$. Therefore f is continuous. $\square$

We are cheating, of course, since we defined topology in such a way that all maps are trivially continuous. For a real challenge we should attempt to show, for example, that all maps $\mathbb{R} \to \mathbb{R}$ are continuous in the usual ϵ–δ sense. Although this turns out to be the case, we are *not* going to prove it here.

While in classical topology a given set may be endowed with many different topologies, in the synthetic world each set has precisely one topology associated with it. At this point you should be worried that certain sets might be equipped with the "wrong topology". For example, there are at least two important topologies on the dual of a Banach space. How can we have both if a set is only allowed to have one? The answer is that sets which are the same classically may be different constructively, and so they may carry different topologies. In the case of the dual of Banach spaces, one might consider the bounded linear functionals versus the normed linear functionals. These might turn out to be different sets, each with its own well known topology.

If we were going to develop the topological point of view further, we would borrow ideas from *Synthetic Domain Theory* [11,18], *Abstract Stone Duality* [22], and *Synthetic Topology* [5]. We leave such an task for the future. However, we shall keep the topological point of view in mind and use its terminology whenever convenient.

4 Basic Computability Theory

In this section we introduce the Enumerability Axiom and derive the basic theorems of computability theory.

4.1 Partial Functions and Partial Values

In classical computability theory the computable partial functions are characterized as precisely those partial functions whose graph is computably enumerable. This characterization helps us find the corresponding notion in synthetic computability.

A *partial function* $f : A \rightharpoonup B$ is a function $f : A' \to B$ defined on a subset $A' \subseteq A$, called the *support* of f. Equivalently, such an f corresponds to a (total) function $g : A \to \widetilde{B}$ where $\widetilde{B}$ is the set of *partial values*

$$\widetilde{B} = \left\{ s \in \mathcal{P}B \;\middle|\; \forall\, x, y \in B\,.\,(x \in s \land y \in s \implies x = y) \right\} ,$$

The connection between f and g is $g(x) = \left\{ f(x) \in B \;\middle|\; x \in A' \right\}$. The empty set represents the value "undefined" and is denoted by $\bot_B = \emptyset$, while a singleton $\{y\}$ represents a "fully defined" value $y \in B$, which we call a *total* value. The singleton map $\{-\} : B \to \widetilde{B}$ is an inclusion that maps the elements of B precisely onto the total values of $\widetilde{B}$. We shall often identify $y \in B$ with its representation as a total value $\{y\} \in \widetilde{B}$. The statement $\exists\, y \in B\,.\,(s = \{y\})$ means "the partial value s is total" and is abbreviated as $s{\downarrow}$.

The *graph* of $f : A \to \widetilde{B}$ is the set

$$\Gamma(f) = \left\{ \langle x, y \rangle \in A \times B \;\middle|\; f(x) = \{y\} \right\} .$$

Among all partial functions $\mathbb{N} \to \widetilde{\mathbb{N}}$ we are only interested in those that have enumerable graphs.

Proposition 4.1 *A partial function $f : \mathbb{N} \to \widetilde{\mathbb{N}}$ has an enumerable graph if, and only if, $f(n){\downarrow}$ is semidecidable for every $n \in \mathbb{N}$.*

Proof. If $e : \mathbb{N} \to 1 + \Gamma(f)$ is an enumeration of $\Gamma(f)$ then $f(n){\downarrow}$ if, and only if,

$$(\exists\, m \in \mathbb{N}\,.\, f(n) = \{m\}) \iff (\exists\, k \in \mathbb{N}\,.\, e(k) \neq \star \land \pi_1(e(k)) = m) \in \Sigma\ .$$

Conversely, suppose $f(n){\downarrow} \in \Sigma$ for every $n \in \mathbb{N}$. Observe that $\langle n, k\rangle \in \Gamma(f)$ if, and only if, $f(n){\downarrow} \land f(n) = \{k\}$. If $f(n){\downarrow}$ then there is a unique $m \in \mathbb{N}$ such that $f(n) = \{m\}$, therefore $f(n) = \{k\}$ is semidecidable, even decidable, because it is equivalent to $m = k$. By the dominance axiom, $f(n){\downarrow} \land f(n) = \{k\}$ is semidecidable so $\Gamma(f)$ is a semidecidable subset of $\mathbb{N} \times \mathbb{N}$, hence enumerable. □

We single out those partial values whose totality is semidecidable.

Definition 4.2 The *lifting* $A_\perp$ of A is the set of Σ-*partial values*,

$$A_\perp = \left\{ s \in \widetilde{A} \;\middle|\; s{\downarrow} \in \Sigma \right\}\ .$$

A Σ-*partial function* is a partial function $f : A \to B_\perp$.

The operation $A \mapsto A_\perp$ is a functor. It acts on a map $f : A \to B$ by

$$f_\perp(p) = \left\{ y \in B \;\middle|\; \exists\, x \in A\,.\, x \in p \land f(x) = y \right\}\ .$$

This is well defined because $f_\perp(p){\downarrow} \iff p{\downarrow}$. Thus $f_\perp(\{x\}) = \{f(x)\}$ and $f_\perp(\perp_A) = \perp_B$. For those familiar with category theory we mention that the lifting functor $-_\perp$ is in fact a monad whose multiplication and unit are union and singleton, respectively:

$$\begin{array}{ll} \mu_A : A_{\perp\perp} \to A_\perp & \qquad \eta_A : A \to A_\perp \\ \mu_A : s \mapsto \cup s = \left\{ x \in A \;\middle|\; \exists\, p \in A_\perp\,.\, p \in S \land x \in p \right\} & \qquad \eta_A : x \mapsto \{x\} \end{array}$$

The Σ-partial functions $\mathbb{N} \to \mathbb{N}_\perp$ are the synthetic analogue of partial computable functions. A classical theorem of computability theory claims that the computably enumerable sets are precisely the supports of partial computable functions.

Proposition 4.3

(i) *A partial function is Σ-partial if, and only if, its support is semidecidable.*

(ii) *A subset is semidecidable if, and only if, it is the support of a Σ-partial function.*

Proof.

(i) The support of $f : A \to \widetilde{B}$ is the set $\{x \in A \mid f(x)\downarrow\}$. Clearly then, the support is semidecidable if, and only if, totality is semidecidable.

(ii) We already proved that the support of a Σ-partial function is semidecidable. Conversely, if $S \subseteq A$ is semidecidable then it is the support of its characteristic function $\chi_S : A \to \Sigma = \mathbf{1}_\perp$.

□

Among all semidecidable subsets of $\mathbb{N} \times \mathbb{N}$ we may identify those that are graphs of Σ-partial functions. Recall that a relation $R \subseteq A \times B$ is *single-valued* if $R(x, y)$ and $R(x, z)$ implies $y = z$.

Proposition 4.4 *The graphs of Σ-partial functions $\mathbb{N} \to \mathbb{N}_\perp$ are precisely all the single-valued semidecidable relations on $\mathbb{N} \times \mathbb{N}$.*

Proof. The graph of a Σ-partial function is semidecidable because it is enumerable, and it is clearly single valued. Given a single-valued $R \in \Sigma^{\mathbb{N}\times\mathbb{N}}$, define $f_R : \mathbb{N} \to \mathbb{N}_\perp$ by $f_R(m) = \{n \in \mathbb{N} \mid \langle m, n\rangle \in R\}$. Then $\Gamma(f_R) = R$. □

A *selection* for a binary relation $R \subseteq A \times B$ is a *partial* function $s : A \to \widetilde{B}$ such that, for all $x \in A$,

$$(\exists\, y \in B \,.\, R(x, y)) \implies s(x)\downarrow \wedge R(x, s(x)) \;.$$

The selection function s is like a choice function for R, except that it is defined only at those arguments for which there is something to choose from. A well known theorem in computability theory, the Single Value Theorem, says that every semidecidable relation on $\mathbb{N} \times \mathbb{N}$ has a Σ-partial selection.

Theorem 4.5 (Single Value Theorem) *Every semidecidable binary relation on $\mathbb{N}$ has a Σ-partial selection.*

Proof. Let $e : \mathbb{N} \to \mathbf{1} + R$ be an enumeration of $R \in \Sigma^{\mathbb{N}\times\mathbb{N}}$. Define S by

$$\begin{aligned} S = \{\langle m, n\rangle \in \mathbb{N} \times \mathbb{N} \mid \\ \exists\, k \in \mathbb{N} \,.\, (e(k) = \langle m, n\rangle \wedge \forall\, j < k \,.\, (e(j) \neq \star \implies \pi_1\,(e(j)) \neq m))\} \;. \end{aligned}$$

Thus we put $\langle m, n\rangle$ in S when it is the first pair of the form $\langle m, -\rangle$ enumerated by e. Clearly $S \subseteq R$. To see that it is single-valued, suppose $\langle m, n\rangle \in S$ and $\langle m, n'\rangle \in S$. Then $\langle m, n\rangle = e(k)$ and $\langle m, n'\rangle = e(k')$ for some $k, k' \in \mathbb{N}$. It is impossible that $k' < k$ or $k < k'$, therefore $k = k'$ and so $n = n'$. Lastly, if $\langle m, n\rangle \in R$ then $\langle m, n\rangle = e(k)$ for some $k \in \mathbb{N}$. Now there is a least $j \leq k$ such that $\pi_1\,(e(j)) = m$. Then $e(j) = \langle m, n'\rangle \in S$. A selection for R is the function whose graph is S. □

4.2 The Enumerability Axiom

Everything we have considered so far is consistent with classical logic. Of course, if we interpreted all the definitions and theorems in classical set theory, we would not discover anything interesting, as it would turn out that $2 = \Sigma = \Omega$, all predicates and sets are decidable, all subsets of $\mathbb{N}$ are enumerable, etc. It is time to introduce a genuinely interesting axiom.

Axiom 4.6 (Enumerability) *There are enumerably many enumerable subsets of* $\mathbb{N}$*.*

Let $\mathsf{W}_{-} : \mathbb{N} \twoheadrightarrow \mathcal{E}$ be such an enumeration.

The idea for the Enumerability Axiom comes from the Enumeration Theorem of classical computability theory, which states that there is a computable enumeration of computably enumerable sets. In the classical theory there is also an enumeration theorem for partial computable functions, which we have too.

Proposition 4.7 $\mathbb{N} \to \mathbb{N}_\perp$ *is enumerable.*

Proof. By Enumerability Axiom there is an enumeration $V_{-} : \mathbb{N} \twoheadrightarrow \Sigma^{\mathbb{N}\times\mathbb{N}}$, because $\mathbb{N} \cong \mathbb{N} \times \mathbb{N}$. By Single Value Theorem, for every $n \in \mathbb{N}$ there exists a selection for V_n. By Number Choice there is a choice function $\varphi_{-} : \mathbb{N} \to (\mathbb{N} \to \mathbb{N}_\perp)$ such that φ_n is a selection for V_n. We are done because φ is surjective: for any $f : \mathbb{N} \to \mathbb{N}_\perp$ there is $n \in \mathbb{N}$ such that $V_n = \Gamma(f)$, but then $\varphi_n = f$ because f is the only selection for $\Gamma(f)$. □

Let $\varphi_{-} : \mathbb{N} \twoheadrightarrow (\mathbb{N} \to \mathbb{N}_\perp)$ be an enumeration. Next we derive some basic consequences of the Enumeration Axiom.

Proposition 4.8 Σ *and* $\mathcal{E}$ *have the fixed-point property.*

Proof. By Proposition 3.9 together with the observation that $\mathcal{E}^{\mathbb{N}} = (\Sigma^{\mathbb{N}})^{\mathbb{N}} \cong \Sigma^{\mathbb{N}\times\mathbb{N}} \cong \Sigma^{\mathbb{N}} = \mathcal{E}$. □

Corollary 4.9 *None of the inclusions* $2 \subseteq \Sigma \subseteq \Omega_{\neg\neg} \subseteq \Omega$ *is an equality.*

Proof. Neither 2 nor $\Omega_{\neg\neg}$ has the fixed-point property so they cannot be equal to Σ. If $\Omega_{\neg\neg} = \Omega$ then $2 = \Omega$, but we already have $2 \neq \Sigma \subseteq \Omega$, hence $\Omega_{\neg\neg} \neq \Omega$. □

The Enumerability Axiom invalidates classical logic because it falsifies the Law of Excluded Middle, $2 = \Omega$, which contradicts Corollary 4.9.

Because $2 \neq \Sigma$ the decidable and the semidecidable subsets of $\mathbb{N}$ are not the same. We may explicitly construct a semidecidable subset which is not decidable, namely the well known

$$\mathsf{K} = \left\{ n \in \mathbb{N} \;\middle|\; n \in \mathsf{W}_n \right\} .$$

The set K is not decidable because its complement $\mathbb{N} \setminus \mathsf{K}$ is not semidecidable. If it were there would be some $m \in \mathbb{N}$ such that $\mathsf{W}_m = \mathbb{N} \setminus \mathsf{K}$ and then we would have the usual contradiction

$$m \in \mathsf{K} \iff m \in \mathsf{W}_m \iff m \in \mathbb{N} \setminus \mathsf{K} \iff m \notin \mathsf{K} .$$

Recall that Σ is a σ-frame. Its partial order $p \leq q$ is logical implication $p \implies q$. It is important to know how maps $\Sigma \to \Sigma$ interact with the partial order.

Proposition 4.10 *Every map $f : \Sigma \to \Sigma$ is monotone.*

Proof. The statement of monotonicity of f is classical:

$$\forall\, p, q \in \Sigma \,.\, (p \leq q \implies f(p) \leq f(q)) .$$

By Lemma 3.20 this statement reduces to checking all four combinations of $p, q \in \{\bot, \top\}$. Of these only $p = \bot$, $q = \top$ is nontrivial so that monotonicity of f reduces to $f(\bot) \leq f(\top)$. Suppose the opposite of this, which is $f(\bot) \wedge \neg f(\top)$, were true. Then $f(\bot) = \top$ and $f(\top) = \bot$. Now $p = f(p)$ implies $p \neq \bot$ and $p \neq \top$, which is impossible. Hence f does not have a fixed point, which cannot be because Σ has the fixed-point property. We proved $\neg\neg(f(\bot) \leq f(\top))$, hence $f(\bot) \leq f(\top)$ as desired. □

Proposition 4.11 (Phoa's Principle) *For every $f : \Sigma \to \Sigma$,*

$$f(x) = (f(\bot) \vee x) \wedge f(\top) \qquad \textit{and} \qquad f(x) = f(\bot) \vee (x \wedge f(\top)) .$$

Proof. Phoa's principle is a classical statement so we only need to check $x = \bot$ and $x = \top$. This gives us four equations of which two are trivially true and the other two are $f(\bot) = f(\bot) \wedge f(\top)$ and $f(\top) = f(\bot) \vee f(\top)$. Both of these are equivalent to $f(\bot) \leq f(\top)$, which holds because f is monotone. □

The relevance of Phoa's Principle is revealed by the following corollary.

Corollary 4.12 *Every map $f : \Sigma \to \Sigma$ preserves binary infima and countable suprema.*

Proof. By Phoa's Principle,

$$\begin{aligned} f(p \wedge q) = f(\bot) \vee (p \wedge q \wedge f(\top)) = f(\bot) \vee ((p \wedge f(\top)) \wedge (q \wedge f(\top))) = \\ (f(\bot) \vee (p \wedge f(\top))) \wedge (f(\bot) \vee (p \wedge f(\top))) = f(p) \wedge f(q) , \end{aligned}$$

and

$$f(\textstyle\bigvee_n p_n) = (f(\bot) \vee \bigvee_n p_n) \wedge f(\top) = (\bigvee_n f(\bot) \vee p_n) \wedge f(\top) = \\ \textstyle\bigvee_n (f(\bot) \vee p_n) \wedge f(\top) = \bigvee_n f(p_n)\ .$$

□

4.3 Focal Sets

The lifting operation attaches to a set A an "undefined" value $\bot_A$. Sometimes a set already contains an element which plays the role of "undefined"; for example, in the set of Σ-partial functions $A \to B_\bot$ it is the everywhere undefined function $\lambda x : A\,.\,\bot_B$. Such a special element can be found by attaching $\bot$ to the set and mapping it back in to the original set, without changing the original elements. This idea leads to the following definition.

Definition 4.13 A *focal set* is a set A together with a mapping $\epsilon : A_\bot \to A$, called the *focus map*, such that $\epsilon(\{x\}) = x$ for all $x \in A$. The element $\epsilon(\bot_A)$ is called the *focus point*.

We usually denote the focus point by $\bot$. A lifted set $A_\bot$ is focal. The focus map is the multiplication μ_A for the lifting monad, as defined in the paragraph following Definition 4.2. The focus point is $\bot_A$, as expected.

If B is a focal set with focus map $\delta : B_\bot \to B$ then $A \to B$ is a focal set with focus map $\epsilon : (A \to B)_\bot \to (A \to B)$ defined by

$$\epsilon(p) = \lambda x : A\,.\,\delta(\{y \in B \mid \exists\, f : A \to B_\bot\,.\,(f \in p \wedge f(x) = \{y\})\})\ . \tag{1}$$

The focus point of $A \to B$ is the map that maps every element to the focus point of B. By combining the last two observations, we see that the set of Σ-partial maps $A \to B_\bot$ is focal.

A product of focal sets A and B is focal. The focus map $\epsilon : (A \times B)_\bot \to A \times B$ is $\epsilon(p) = \langle \epsilon_A(\pi_{1\bot}(p)), \epsilon_B(\pi_{2\bot}(p))\rangle$, and the focus point is the pair whose components are the foci of A and B.

A set can have two focal structures, for example, the maps $\delta : \Omega_\bot \to \Omega$ and $\epsilon : \Omega_\bot \to \Omega$ defined by $\delta(s) = \exists\, p \in s\,.\,p$ and $\epsilon(s) = \forall\, p \in s\,.\,p$ both satisfy $\delta(\{q\}) = q$ and $\epsilon(\{q\}) = q$, so they define focal structures on Ω, but $\delta(\emptyset) = \bot$ and $\epsilon(\emptyset) = \top$.

The enumerable focal sets are know in the theory of numbered sets as *Eršov complete sets* [4]. They have good properties.

Lemma 4.14 *(a) If $f : A \to B$ is onto then so is $f_\bot : A_\bot \to B_\bot$. (b) If $f : A \to B$ is onto then so is $f^{\mathbb{N}} : A^{\mathbb{N}} \to B^{\mathbb{N}}$.*

Proposition 4.15 *If A is an enumerable focal set then so is $A^{\mathbb{N}}$.*

Proof. Let $e : \mathbb{N} \twoheadrightarrow A$ be an enumeration and $\epsilon : A_\perp \to A$ the focal map. By the previous lemma, we have a chain of surjections

$$\mathbb{N} \xrightarrow{\varphi} \mathbb{N}_\perp^{\mathbb{N}} \xrightarrow{e_\perp^{\mathbb{N}}} A_\perp^{\mathbb{N}} \xrightarrow{\epsilon^{\mathbb{N}}} A^{\mathbb{N}} \; .$$

This proves that $A^{\mathbb{N}}$ is enumerable. It is focal by (1). □

4.4 Inseparable Sets

A classical theorem of computability theory says that there exists a pair of computably enumerable sets which cannot be separated by a recursive set. We present the same theorem in a domain-theoretic form.

The *domain of partial Booleans* $2_\perp$ is ordered by subset inclusion. The least element is $\perp$, while 0 and 1 are maximal elements. The set $2_\perp$ is enumerable because it is a retract of the enumerable set $\mathbb{N}_\perp$. By Proposition 4.15, it follows that also *Plotkin's universal domain* [16] $2_\perp^{\mathbb{N}}$ is enumerable. One might think, drawing from classical experience, that every element of $2_\perp^{\mathbb{N}}$ is below a maximal one.

Proposition 4.16 *There exists $a \in 2_\perp^{\mathbb{N}}$ such that every maximal $b \in 2_\perp^{\mathbb{N}}$ is inconsistent with a, in the sense that there exists $k \in \mathbb{N}$ such that a_k and b_k are unbounded in $2_\perp$.*

Proof. Let $\psi : \mathbb{N} \twoheadrightarrow 2_\perp^{\mathbb{N}}$ be an enumeration, and $i : 2_\perp \to 2_\perp$ the lifting of negation $\neg : 2 \to 2$. For $n \in \mathbb{N}$ define $a_n = i(\psi_n(n))$ and consider any maximal $b \in 2_\perp^{\mathbb{N}}$. There exists $k \in \mathbb{N}$ such that $b = \psi_k$. Because b_k is total $b_k = \psi_k(k) \neq i(\psi_k(k)) = a_k$, therefore b and a are inconsistent at k. □

The sets $A_0 = a^{-1}(\{0\})$ and $A_1 = a^{-1}(\{1\})$, where a is the element from the proposition, form a pair of disjoint open subsets of $\mathbb{N}$ which cannot be separated by a clopen one.

4.5 Creative Sets

The *exterior* of a subset $S \subseteq X$ is the largest open subset of X disjoint from S. We say that $S \subseteq X$ is *creative* if its complement is inexhaustible by open sets, i.e., for every $U \in \Sigma^X$ such that $S \cap U = \emptyset$ there exists $V \in \Sigma^X$ such that $S \cap V = \emptyset$, $U \subseteq V$, and $V \setminus U$ is inhabited. A creative set does not have an exterior, as any candidate for the exterior may be enlarged. While in classical topology there are no creative sets, the situation is different in synthetic computability.

Proposition 4.17 *There exists an open creative subset of* $\mathbb{N}$*.*

Proof. The set $\mathsf{K} = \{n \in \mathbb{N} \mid n \in \mathsf{W}_n\}$ is open and creative. Indeed, suppose $U \cap \mathsf{K} = \emptyset$ for some $U \in \Sigma^{\mathbb{N}}$. For some $n \in \mathbb{N}$ we have $U = \mathsf{W}_n$. Both $n \in U$ and $n \in \mathsf{K}$ imply $n \in U \cap \mathsf{K}$, so it must be the case that $n \notin \mathsf{K}$ and $n \notin U$. The open set $\{n\} \cup U$ enlarges U and is disjoint from K. □

4.6 Simple and Immune Sets

A subset $S \subseteq X$ is *simple* if S intersects every infinite subset of X and $X \setminus S$ is not finite. A set is *immune* if it is neither finite nor infinite.

Proposition 4.18 *There exists an open simple subset of* $\mathbb{N}$*.*

Proof. Following Post, consider $P = \{\langle m, n\rangle \in \mathbb{N} \times \mathbb{N} \mid n > 2m \land n \in W_m\}$, and let $f : \mathbb{N} \to \mathbb{N}_\perp$ be a selection for P. The image $S = \mathsf{im}(f)$ is open, and because $f(m) > 2m$, $\mathbb{N} \setminus S$ cannot be finite. For any infinite enumerable set $U = \mathsf{W}_m$, we have $f(m)\!\downarrow$ and $f(m) \in \mathsf{W}_m = U$ hence U and $\mathsf{im}(f)$ intersect at $f(m)$. □

Corollary 4.19 *There exists a closed immune subset of* $\mathbb{N}$*.*

Proof. Consider the complement of Post's simple set S. □

4.7 Rice's Theorem

A set A is *connected* if it cannot be decomposed as a disjoint union $A_1 + A_2$ in a non-trivial way. An equivalent way of saying this is that every map $A \to \mathsf{2}$ is constant, which we use as the definition of connectedness.

Proposition 4.20 Σ *is connected.*

Proof. Consider a map $h : \Sigma \to \mathsf{2}$. Let $r : \mathsf{2} \to \Sigma$ be the map

$$r(p) = \texttt{if } p = h(\perp) \texttt{ then } \top \texttt{ else } \perp .$$

By Proposition 4.10 the map $r \circ h$ is monotone so that for every $x \in \Sigma$ we have $\top = r(h(\perp)) \leq r(h(x)) \leq \top$. Thus $r(h(x)) = \top$, hence $h(x) = h(\perp)$ by definition of r. □

Lemma 4.21 *Let A be a focal set with focus $\perp_A$. For every $x \in A$ there exists $f : \Sigma \to A$ such that $f(\perp) = \perp_A$ and $f(\top) = x$.*

Proof. Let $g : \mathsf{1} \to A$ be the map $g(\star) = x$. Define $f(s) = \epsilon(g_\perp(s))$ where $\epsilon : A_\perp \to A$ is the focal map. Then $f(\perp) = \epsilon(g_\perp(\perp)) = \epsilon(\perp) = \perp_A$ and $f(\top) = \epsilon(g_\perp(\top)) = \epsilon(x) = x$, as required. □

Theorem 4.22 (Rice's Theorem) *A focal set is connected.*

Proof. Let $h : A \to 2$ be an arbitrary map. We show that $h(x) = h(\bot_A)$ for every $x \in A$. As in Lemma 4.21, let $f : \Sigma \to A$ be such that $f(\bot) = \bot_A$ and $f(\top) = x$. Because Σ is connected $h \circ f$ is constant, but this means $h(x) = h(f(\top)) = h(f(\bot)) = h(\bot_A)$. $\square$

Classical Rice's Theorem states that there are no non-trivial decidable subsets of $\mathcal{E}$. This follows immediately from our theorem as $\mathcal{E}$ is focal.

4.8 Recursion Theorem

A *multivalued function* $f : A \rightrightarrows B$ is a function $f : A \to \mathcal{P}B$ such that $f(x)$ is inhabited for every $x \in A$. The graph of a multivalued function $\Gamma(f) \subseteq A \times B$, defined by

$$\Gamma(f) = \left\{ \langle x, y \rangle \in A \times B \mid y \in f(x) \right\} ,$$

is a total relation. Every total relation $R \subseteq A \times B$ determines a multivalued function $f_R : A \rightrightarrows B$ by $f_R(x) = \left\{ y \in B \mid R(x, y) \right\}$, hence multivalued functions and total relations are two equivalent notions.

A *fixed point* of a multivalued function $f : A \rightrightarrows A$ is $x \in A$ such that $x \in f(x)$.

Theorem 4.23 (Recursion Theorem) *Every multivalued function on an enumerable focal set has a fixed point.*

Proof. Let $f : A \rightrightarrows A$ be a multivalued function, let $e : \mathbb{N} \twoheadrightarrow A$ be an enumeration, and $\epsilon : A_\bot \to A$ a focal map. For every $k \in \mathbb{N}$ there exists $m \in \mathbb{N}$ such that $e(m) \in f(e(k))$. By Number Choice there is a map $c : \mathbb{N} \to \mathbb{N}$ such that $e(c(k)) \in f(e(k))$ for every $k \in \mathbb{N}$. It suffices to find k such that $e(c(k)) = e(k)$ since then we can take $x = e(k)$.

For every $m \in \mathbb{N}$ there is $n \in \mathbb{N}$ such that $\epsilon(e_\bot(c_\bot(\varphi_m(m)))) = e(n)$. By Number Choice there is $g : \mathbb{N} \to \mathbb{N}$ such that $\epsilon(e_\bot(c_\bot(\varphi_m(m)))) = e(g(m))$ for every $m \in \mathbb{N}$. There is $j \in \mathbb{N}$ such that $g = \varphi_j$. Let $k = g(j)$. Then

$$e(k) = e(g(j)) = \epsilon(e_\bot(c_\bot(\varphi_j(j)))) = \epsilon(e_\bot(c_\bot(g(j)))) = e(c(g(j))) = e(c(m)) .$$

$\square$

The classical Recursion Theorem is indeed a consequence of what we just proved.

Corollary 4.24 *For every $f : \mathbb{N} \to \mathbb{N}$ there exists $n \in \mathbb{N}$ such that $\varphi_{f(n)} = \varphi_n$.*

Proof. Apply Recursion Theorem to the enumerable focal set $\mathbb{N} \to \mathbb{N}_\perp$ and the multivalued function h, defined by

$$h(u) = \left\{ v : \mathbb{N} \to \mathbb{N}_\perp \;\middle|\; \exists\, n \in \mathbb{N} \,.\, u = \varphi_n \land v = \varphi_{f(n)} \right\} ,$$

to obtain a fixed point $u \in h(u)$. By definition of h there exists $n \in \mathbb{N}$ such that $u = \varphi_n$ and $u = \varphi_{f(n)}$, hence $\varphi_n = \varphi_{f(n)}$. □

The following consequence of Recursion Theorem is a generalization of Berger's Branching Lemma [1].

Lemma 4.25 (Berger) *Let A be an enumerable focal set, $U : A \rightrightarrows \Sigma$ a multivalued open set, and $x : \mathbb{N}^+ \to A$ a sequence with a limit. If $U(x_\infty) = \{\top\}$ then $\top \in U(x_n)$ for some $n \in \mathbb{N}$.*

Proof. Recall that by Proposition 3.16 for every $s \in \Sigma$ there exists $p \in \mathbb{N}^+$ such that $s = \top \iff p < \infty$. Therefore, for every $y \in A$ there is $p \in \mathbb{N}^+$ such that $(p < \infty) \in U(y)$. Consequently, for every $y \in A$ there is $z \in A$ such that

$$\exists\, p \in \mathbb{N}^+ \,.\, ((p < \infty) \in U(y) \land z = x_p) \ . \tag{2}$$

By Recursion Theorem there is $y = z$ satisfying (2). For such y, p is not equal to ∞ because $p = \infty$ implies $y = x_\infty$ and $\perp = (p < \infty) \in U(y) = U(x_\infty) = \{\top\}$, contradiction. By Proposition 3.17, $p \in \mathbb{N}$ so we have $x_p = y$ and $\top = (p < \infty) \in U(x_p)$, as required. □

The single-valued version of Berger's Lemma, in which the multi-valued function is replaced by an open set, follows more easily from the following observation.

Proposition 4.26 *If $U \subseteq \mathbb{N}^+$ is open and $\infty \in U$, then $n \in U$ for some $n \in \mathbb{N}$.*

Proof. By Markov Principle it suffices to show that not for all $n \in \mathbb{N}$, $U(n) = \perp$. If we had $U(n) = \perp$ for all $n \in \mathbb{N}$, we could define a map $f : \Sigma \to \Sigma$ by $f([p]) = U(p)$, where $[-] : \mathbb{N}^+ \to \Sigma$ is the quotient map $p \mapsto (p < \infty)$. Since Σ has the fixed-point property, there exists $s \in \Sigma$ such that $f(s) = s$. However, since $f(\top) = \perp$ and $f(\perp) = \top$, s is different from both $\perp$ and $\top$, which is impossible. □

It is natural to ask whether the conclusion of Proposition 4.26 may be strengthened to the existence of m such that $n \in U$ for every $n \geq m$. An adaptation of Friedberg's counterexample [7], see also [9, 15.3.31], shows this is impossible, as Alex Simpson has observed.

4.9 The Myhill-Shepherdson and Rice-Shapiro Theorems

Recall that a poset $(A, \leq)$ is *ω-chain complete* if every increasing chain $x_0 \leq x_1 \leq x_2 \leq \cdots$ has a supremum $\bigvee_n x_n$. A subset $S \subseteq A$ *generates* A if every element in A is the supremum of a chain in S. A *base* for A is an enumerable subset $S \subseteq A$ that generates A, and $x \leq y$ is semidecidable whenever $x \in S$ and $y \in A$.

Proposition 4.27 *The following sets are ω-chain complete with a base, therefore countably based:*

(i) Σ *with the base* $\{\top\}$,

(ii) $\mathbb{N}_\perp$ *with the base* $\mathbb{N}$,

(iii) $A^\mathbb{N}$, *if A is an ω-chain complete focal set with a base.*

Proof. Only the last part requires some attention. The partial order on $A^\mathbb{N}$ is component-wise: $f \leq g \iff \forall n \in \mathbb{N} \,.\, f(n) \leq g(n)$. This makes $A^\mathbb{N}$ into an ω-chain complete poset with the supremum of a chain computed component-wise. If S is a base for A, we may take as a base for $A^\mathbb{N}$ the set

$$T = \left\{ f \in \mathbb{A}^\mathbb{N} \;\middle|\; \exists n \in \mathbb{N} \,.\, \forall k \geq n \,.\, f(k) = \perp_A \right\} .$$

It is not hard to see that T is enumerable. Given any $f \in A^\mathbb{N}$, there is by Countable Choice a map $g : \mathbb{N} \to S^\mathbb{N}$ such that, for every $m \in \mathbb{N}$, $g(m)$ is a chain in S with supremum $f(m)$. Define $h : \mathbb{N} \to T$ by

$$h(n)(m) = \texttt{if } m < n \texttt{ then } g(m)(n) \texttt{ else } \perp_A .$$

Then h is a chain in T whose supremum is f:

$$\bigvee_n h(n)(m) = \bigvee_{n>m} h(n)(m) = \bigvee_{n>m} g(m)(n) = f(m) .$$

□

By the previous proposition $\Sigma^\mathbb{N}$ and $\mathbb{N} \to \mathbb{N}_\perp$ are ω-chain complete with bases consisting of the finite subsets of $\mathbb{N}$ and the finite Σ-partial maps, respectively. A map $f : \mathbb{N} \to \mathbb{N}_\perp$ is *finite* if it has finite support.

Theorem 4.28 *In an ω-chain complete poset $(A, \leq)$ open subsets are upward closed and inaccessible by chains, i.e., if the supremum of a chain belongs to an open set then already some element of the chain does.*

Furthermore, if $S \subseteq A$ is a base for A, then every open subset of A is an enumerable union of open subsets of the form $\uparrow x = \left\{ y \in A \;\middle|\; x \leq y \right\}$ with $x \in S$.

Proof. For the first claim, suppose $x \in U \in \Sigma^A$ and $x \leq y$. Define a sequence $a : \mathbb{N}^+ \to A$ by $a_p = \bigvee_{n \in \mathbb{N}} (\texttt{if } n < p \texttt{ then } x \texttt{ else } y)$. Then $a_\infty = x \in U$ and by Berger's Lemma there exists $k < \infty$ such that $y = a_k \in U$. For the second claim, suppose $x_0 \leq x_1 \leq \cdots$ is a chain with $\bigvee_n x_n \in U \in \Sigma^A$. Define $b : \mathbb{N}^+ \to A$ by $b_p = \bigvee_{n \in \mathbb{N}} x_{\min(n,p)}$. Then $b_\infty = \bigvee_n x_n \in U$ and once again by Berger's Lemma there exists $k < \infty$ such that $x_k = b_k \in U$.

For the last claim, suppose U is open. Then $T = S \cap U$ is enumerable. By the first claim $\bigcup_{x \in T} \uparrow x \subseteq U$, and the opposite inclusion holds as well: if $y \in U$, then $y = \bigvee_n x_n$ for some chain $x_0 \leq x_1 \leq \cdots$ in S, therefore by the second claim $x_k \in U$ for some $k \in \mathbb{N}$. But then $y \in \uparrow x_k \subseteq \bigcup_{x \in T} \uparrow x$, as required. □

The Myhill-Shepherdson and Rice-Shapiro theorems characterize the topologies of $\mathbb{N} \to \mathbb{N}_\perp$ and $\Sigma^{\mathbb{N}}$, respectively. They follow immediately from the previous proposition and theorem.

Corollary 4.29 (Myhill-Shepherdson) *A subset U of $\mathbb{N} \to \mathbb{N}_\perp$ is open if, and only if, it is an enumerable union $U = \bigcup_{n \in \mathbb{N}} \uparrow f_n$, where each $f_n : \mathbb{N} \to \mathbb{N}_\perp$ has finite support.*

Corollary 4.30 (Rice-Shapiro) *A subset $U \subseteq \Sigma^{\mathbb{N}}$ is open if, and only if, it is an enumerable union $U = \bigcup_{n \in \mathbb{N}} \uparrow S_n$, where each $S_n \subseteq \mathbb{N}$ is finite.*

5 Conclusion

We have only scratched the surface of a large body of work. There are at least two directions to go from here.

First, we could develop recursive topology [20] and recursive analysis in a style that resembles the usual topology and analysis, but with unusual results, such as failure of compactness of the closed interval and the existence of open subsets of Cantor space that are not metrically open. But we could also prove positive results, such as the Kreisel-Lacombe-Shoenfield theorem, which states that all functions between complete separable metric spaces are continuous in the metric sense.

Second, we have not spoken at all about Turing reducibility and Turing degrees. How this can be done with the j-operators in the effective topos was indicated by Hyland [10]. There might be simpler ways to treat Turing degrees. In particular, it is well known that the priority methods are related to Baire category theorem [15, V.3], a connection worth examining in the synthetic setting.

Our axiomatization has its limit: it cannot prove any results in computability theory that fail to relativize to oracle computations. This is so because the theory can be interpreted in a variant of the effective topos built from partial recursive functions with access to an oracle.

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