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Transitive decomposition of symmetry groups for the n-body problem

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Abstract

Periodic and quasi-periodic solutions of the n-body problem are critical points of the action functional constrained to the Sobolev space of symmetric loops. Variational methods yield collisionless orbits provided the group of symmetries fulfills certain conditions (such as the *rotating circle property*). Here we generalize such conditions to more general group types and show how to constructively classify all groups satisfying such hypothesis, by a decomposition into irreducible transitive components. As examples we show approximate trajectories of some of the resulting symmetric minimizers.

Keywords: Symmetric periodic orbits; n-body problem; Minimizers of the Lagrangian action

1. Introduction and main results

Periodic and quasi-periodic orbits for the n-body problem have received much of attention over the last years, also because of the success of variational and topological methods. The starting point can be traced back to the nonlinear analysis works of A. Ambrosetti, A. Bahri, V. Coti Zelati, P. Majer, J. Mawhin, P.H. Rabinowitz, E. Serra and S. Terracini (among others) issued around 1990 [1–4,22,23,25,30]; methods were developed that could deal with singular potentials and particular symmetry groups of the functional. For other approaches one can see also I. Stewart [31] and C. Moore [27]. The next new wave of results has followed the remarkable A. Chenciner and R. Montgomery's proof of the "figure-eight" periodic solution of the three-

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body problem in the case of equal masses [13], where collisions and singularities were excluded by the computation of the action level on test curves and a non-commutative finite group of symmetries was taken as the constraint for a global equivariant variational approach. In order to generalize the equivariant variational method (that is, to restrict the action functional to the space of equivariant loops) to a new range of applicability S. Terracini and the author in [19] could make use of C. Marchal's averaging idea [24] and prove that local minimizers of the action functional are collisionless, provided an algebraic condition on the symmetry group (termed the *rotating circle property*) holds. Meanwhile, symmetry groups and various approaches to level estimates or local variations have been found, together with the corresponding symmetric minimizers, and published by many authors (see for example [6–9,14,15,18,20,28,29,32]). The aim of this article is to provide a unified framework for the construction and classification problem and at the same time to extend the application range of the averaging and blow-up techniques. More precisely, when dealing with the problem of classifying in a constructive way finite symmetry groups for the n-body problem one has to face three issues. First, it is of course preferable to have an equivalence relation defined between groups, which rules out differences thought as non-substantial. Second, one has to find a suitable decomposition of a symmetry group into a sum of (something like) *irreducible* components. The way of decomposing things depends upon the context. In our settings we could choose an orthogonal representations decomposition (as direct sum of G-modules) or a permutation decomposition, or a mixture of the two. Third, it would be interesting to deduce from the irreducible components, from their (algebraic and combinatorial) properties, some consequent properties of action-minimizing periodic orbits (like being collisionless, existence, being homographic or non-homographic, and the like). The purpose of the paper is to give a procedure for constructing all symmetry groups of the n-body problem (in three-dimensional space, but of course the planar case can be done as a particular case) according to these three options, with the main focus on the existence of periodic or quasi-periodic non-colliding solutions. The main result can be used to list groups that might be considered as the elementary building blocks for generic symmetry groups yielding collisionless minimizers.

The first reduction will be obtained by defining the *cover* of a symmetry group (that is, the group acting on the time line instead of the time circle) and considering equivalent groups with the same equivariant periodic trajectories (up to repeating loops and to time rescaling). Also, it is possible to consider equivalent those symmetry groups that differ by a change in the action functional, determined by the angular velocity of the rotating frame. Using this simple *escamotage* it is possible to dramatically reduce the cardinality of the symmetry groups and to deal with a finite number of (numerable) families of groups for every n. This step will be explained at the end of Section 2. The next step is to exploit the fact that any finite permutation representation can be decomposed *à la Burnside* into the disjoint sum of *transitive* (or, equivalently, *homogeneous*) permutation representations. This decomposition requires the definition of a suitably crafted sum of Lagrangian symmetry groups, which will be constructively written in term of Krh and $\widehat{Krh}$ data (to be defined later) yielded by the group. The transitive decomposition allows to state the main result, which can be written as follows. Definition and notation of course refer to the body of the paper and Appendix A.

Theorem A. *Let G be a symmetry group (not bound to collisions) with a colliding G-symmetric Lagrangian local minimizer. If $G_* \subset G$ is the $\mathbb{T}$-isotropy group of the colliding time restricted to the index subset of colliding bodies, then G_* cannot act trivially on the index set; if the permutation isotropy of a transitive component of G_* is trivial, then the image of G_* in $O(3)$ cannot be one of the following: I, C_p (for $p \geqslant 1$), D_p (for $p \geqslant 2$), T, O, Y, P'_{2p}, C_{ph}.*

This result allows to clarify and to extend the above-mentioned *rotating circle property*; in the proof we show how with a simple application of the averaging Marchal technique on space equivariant spheres one can deduce that for the group actions listed in the statement minimizers are collisionless. It is also the case to mention that the transitive decomposition approach has two interesting consequences: from one hand it is possible to determine whether the hypothesis of Theorem A is fulfilled simply by computing the space-representations of the transitive decomposition of the maximal $\mathbb{T}$-isotropy subgroups of the symmetry group (thus making the task of finding rotating circles unnecessary); on the other hand a machinery for finding examples of symmetry groups can be significantly improved by allowing the construction of actions using smaller and combinatorial components. Even if feasible, a complete classification of all symmetry groups satisfying the hypotheses which imply collisionless minimizers and coercivity would just result into an unreasonably long unreadable list. We decided to formulate only the method that can be used for such generation, leaving a few examples in the last section to illustrate it in simple cases. Therefore the paper is basically organized as a multi-step proof of Theorem A, together with the introduction and explanation of the necessary preliminaries, results and definitions. In Section 2 we will review the notation and some basic properties of Lagrangian symmetry groups. Details about Euclidean symmetry groups and notation can be found in Appendix A at the end of the article. In Section 3 the definition of transitive decomposition and disjoint sum of symmetry groups is carried out: this is one of the main steps in the construction process. Furthermore, in Section 4 a simple proof allows to extend the averaging technique to all orientation-preserving finite isotropy groups. Together with the rotating circle property and the classification of finite subgroups of $SO(3)$, this will yield a method for avoiding collisions. The analysis of possible transitive component is then carried out in Section 5, according to the previous definitions and results. At the end, in Section 6 the few examples mentioned above are shown, together with pictures of the corresponding approximate minimizers. The first two examples are chosen to show very simple cases in which the rotating circle property does not hold, while averaging on equivariant spheres implies being collisionless.

2. Preliminaries

We denote by $O(d)$ the orthogonal group in dimension d, that is, the group of $d \times d$ orthogonal matrices over the real field $\mathbb{R}$. The symbol Σ_n denotes the permutation group on n elements $\{1, \dots, n\} = \mathbf{n}$. Space isometries are named *rotation*, *reflection*, *central inversion* and *rotatory reflection* (actually a central inversion is a particular rotatory reflection). We recall, following the terminology and notation of [17, pp. 99, 270–277] and [16, Appendix A, pp. 351–367] (see also [5,26]), that non-trivial finite subgroups of $SO(3)$ are the following: C_p (the cyclic group generated by a rotation of order p, for $p \geqslant 2$, with a single p-gonal axis), D_p (the dihedral group of order $2p$, with p horizontal diagonal axes and a vertical p-gonal axis, with $p \geqslant 2$), $T \cong A_4$ (the tetrahedral group of order 12, with 4 trigonal axes and 3 mutually orthogonal diagonal axes), $O \cong S_4$ (the octahedral group of order 24, with 4 trigonal axes, the same as T, and 3 mutually orthogonal tetragonal axes; it is isomorphic to the orientation-preserving symmetry group of the cube and contains the tetrahedral group as a normal subgroup of index 2) and $Y \cong A_5$ (the icosahedral group of order 60, with 6 pentagonal axes, 10 trigonal axes and 15 diagonal axes). The dihedral group D_2 is a normal subgroup of T of index 3. Further details on generators and the subconjugacy poset of finite space groups can be found in Appendix A.

Let X be *configuration space* of n point particles in $\mathbb{R}^3$: $X = (\mathbb{R}^3)^n$. Let $\mathbb{T}$ be the circle of length $T = |\mathbb{T}|$. A function $\mathbb{T} \to X$ is a T-periodic path in X. By *loops* in X we mean the elements of the Sobolev space $\Lambda = H^1(\mathbb{T}, X)$, i.e. all L^2 functions $\mathbb{T} \to X$ with L^2-derivative. The aim is to find periodic (in an inertial frame or in a uniformly rotating frame) orbits for the n-body problem: they can be obtained as critical points of the Lagrangian action functional

$$\mathcal{A}_{\boldsymbol{\omega}} = \int_{\mathbb{T}} \left(\sum_{i \in \mathbf{n}} \frac{m_i}{2} \left| \dot{x}_i(t) + \Omega x_i(t) \right|^2 + \sum_{\substack{i<j \\ i,j \in \mathbf{n}}} m_i m_j \left| x_i(t) - x_j(t) \right|^{-\alpha} \right) dt, \tag{2.1}$$

where Ω is the anti-symmetric 3×3 matrix defined by the relation $\Omega v = \boldsymbol{\omega} \times v$ for every $v \in \mathbb{R}^3$, with the vector $\boldsymbol{\omega} \in \mathbb{R}^3$ representing the rotation axis of the rotating frame and its norm $|\boldsymbol{\omega}|$ the angular velocity. The domain of the functional $\mathcal{A}_{\boldsymbol{\omega}}$ is $\Lambda = H^1(\mathbb{T}, X)$ (of course, allowing a range with infinite value). Any collisionless critical point is in fact a C^2 solution of the corresponding Euler–Lagrange, or Newton, equations under a homogeneous gravitational potential of degree $-\alpha$, which is periodic in the rotating frame.

Now consider a group G acting orthogonally on $\mathbb{T}$, $\mathbb{R}^3$ and permuting the indices in $\mathbf{n}$. In other words, consider three homomorphisms τ, ρ and σ from G to $O(\mathbb{T})$, $O(3)$ and Σ_n respectively. The group G can be seen as subgroup (possibly modulo a normal subgroup) of the direct product $O(\mathbb{T}) \times O(3) \times \Sigma_n$ under the monomorphism $\tau \times \rho \times \sigma$, and the three homomorphisms can be recovered as projections onto the first, second and third factor of the direct product. Given ρ and σ, it is customary to define an action on the configuration space X by the rule $(\forall g \in G)$, $x_{\sigma(g)i} = \rho(g)x_i$. We will often denote by gi the index $\sigma(g)i$, and by gx the value of $g \cdot x$ under this action of G. Furthermore, the action of G on $\mathbb{T}$ and X induces an action on the functions $\mathbb{T} \to X$ by the rule $(\forall g \in G)$, $x(\tau(g)t) = (gx)(t)$, and therefore Λ is a G-equivariant vector space (the action of G is orthogonal under the standard Hilbert metric on Λ).

(2.2) Definition. A subgroup of $O(\mathbb{T}) \times O(3) \times \Sigma_n$ is termed *symmetry group*. It will be termed a *symmetry group* of the Lagrangian action functional $\mathcal{A}_{\boldsymbol{\omega}}$ if it leaves the value of the action $\mathcal{A}_{\boldsymbol{\omega}}$ (2.1) invariant.

Note that if $i, j \in \mathbf{n}$ are indices and $gi = j$ for some element $g \in G$, then it is necessary that $m_i = m_j$. More generally, consider the decomposition of $\mathbf{n}$ into (transitive) G-orbits, also known as *transitive decomposition* of the G-set $\mathbf{n}$. Indices in the same G-orbit must share the value of the mass and, furthermore, the transitive decomposition yields an orthogonal splitting of the configuration space:

$$X = (X_1 + X_{g1} + \cdots) \oplus (X_2 + X_{g2} + \cdots) \oplus \cdots, \tag{2.3}$$

where each X_j is a copy of $\mathbb{R}^3$ and each summand grouped by parentheses is given by a transitive G-orbit in $\mathbf{n}$. This transitive decomposition is nothing but the standard decomposition of a permutation representation in the Burnside ring $A(G)$. We recall that the determinant of a linear representation is the one-dimensional representation obtained by taking the determinant of all the matrices $\rho(g)$, for $g \in G$.

(2.4) Definition. Consider a symmetry group G. A vector $v \in \mathbb{R}^3$ is a *rotation axis* for G if $(\forall g \in G) gv \in \{\pm v\}$ (that is, the line $\langle v \rangle \subset \mathbb{R}^3$ is G-invariant) and the orientation G-representation on the time circle (that is, the determinant representation $\det(\tau)$ of the two-dimensional real representation τ) coincides with the orientation representation on the orthogonal plane of v (which coincides with $\det(\rho)\det(v)$).

We recall from [18, Proposition 2.15] that if $\boldsymbol{\omega}$ is a rotation axis for a symmetry group G (and the values of the masses are compatible with the transitive decomposition (2.3)) then G is a symmetry group of the action functional $\mathcal{A}_{\boldsymbol{\omega}}$. The converse holds, after a straightforward proof, for linear or orthogonal actions.

(2.5) *In case the group has a rotation axis it is termed group* of type R. *If the symmetry group G is not of type R, then all G-equivariant loops have zero angular momentum.*

Proof. The proof of an analogous proposition for 3 bodies can be found in [18, Proposition 4.2]; the details are given for 3 bodies, but it can be trivially generalized to the case of n bodies: if J denotes the angular momentum of the G-equivariant path $x(t)$, for every $g \in G$ the formula

$$J(gt) = \det\bigl(\rho(g)\bigr)\det\bigl(\tau(g)\bigr)\rho(g)J(t)$$

holds, and hence the angular momentum J (which is constant) belongs to the subspace V in $\mathbb{R}^3$ fixed by the G-representation $\det(\tau)\det(\rho)\rho$. But if $V \neq 0$, then there is a non-trivial vector $v \in \mathbb{R}$ with the property that for every $g \in G$, $\det(\tau(g))\det(\rho(g))\rho(g)v = v$. If v denotes the representation on $\langle v\rangle$ and ρ_2 the representation on its orthogonal complement, it follows that

$$\det\bigl(\tau(g)\bigr)\det\bigl(\rho_2(g)\bigr)\det(v)\det(v) = 1,$$

and hence that $\det(\tau) = \det(\rho_2)$: the direction spanned by v is a rotation axis, which contradicts the hypothesis. □

Let $\mathrm{Iso}(\mathbb{R})$ denote the group of (affine) isometries of the time line $\mathbb{R}$, generated by translations and reflections. For every $T > 0$ there is a surjective projection $\mathrm{Iso}(\mathbb{R}) \to O(\mathbb{T})$, where $\mathbb{T} = \mathbb{R}/_{T\mathbb{Z}}$. Let G be a symmetry group and $\widetilde{G}$ its *cover* in $\mathrm{Iso}(\mathbb{R}) \times O(3) \times \Sigma_n$, that is the pre-image of G via the projection

$$\mathrm{Iso}(\mathbb{R}) \times O(3) \times \Sigma_n \to O(\mathbb{T}) \times O(3) \times \Sigma_n.$$

It is easy to see that there is a canonical isomorphism

$$H^1(\mathbb{R}, X)^{\widetilde{G}} \cong H^1(\mathbb{T}, X)^G$$

and hence we can consider solutions of the n-body problem which are $\widetilde{G}$-equivariant loops instead of the periodic solutions of the n-body problem which are G-equivariant. We can, when needed, linearly rescale the time line. Assume now that the symmetry group G has a rotating axis $\boldsymbol{\omega}$, and therefore that $\mathcal{A}_{\boldsymbol{\omega}}$ is G-invariant. In a frame rotating around $\boldsymbol{\omega}$ with fixed angular velocity θ, the equation $x(t) = e^{i\theta t}q(t)$ induces an isomorphism θ_*

$$\theta_* : H^1(\mathbb{R}, X) \to H^1(\mathbb{R}, X)$$

defined by $\theta_* q = x$.

The image $\theta_*(H^1(\mathbb{R}, X)^{\widetilde{G}})$ can be seen as

$$\theta_*\big(H^1(\mathbb{R}, X)^{\widetilde{G}}\big) = H^1(\mathbb{R}, X)^{\widetilde{G}'}$$

for a new symmetry group $\widetilde{G}'$ (still of type R); moreover, by a suitable choice of θ it is possible to obtain a new group G' (whose cover is $\widetilde{G}'$) with the following property: if g is a time translation, then $\rho(g)$ is trivial on the orthogonal complement of the rotation axis $\boldsymbol{\omega}$. Since the following diagram commutes

$$\begin{tikzcd} H^1(\mathbb{R}, X)^{\widetilde{G}} \arrow[rr, "\approx"] \arrow[dr, bend right, "\mathcal{A}_{\omega}"'] & & H^1(\mathbb{R}, X)^{\widetilde{G}'} \arrow[dl, bend left, "\mathcal{A}_{\omega'}"] \\ & \mathbb{R} & \end{tikzcd}$$

(where $\boldsymbol{\omega}'$ is chosen as suggested above) by a suitable change of angular velocity one could reduce the size of the symmetry group G and assume that (if it is of type R, of course) all the time-shifts (i.e. the elements of $\ker(\det\tau)$) act trivially on the orthogonal complement of the rotation axis. See also Section 3 of [7].

(2.6) Example. Let G be the cyclic group of order l with generator g, such that $\rho(g)$ is a rotation of angle $2\pi/k$, and $\sigma(g)$ is the cyclic permutation $(1, 2, \ldots, n)$ on n elements. The action on the time circle (of length 2π) is defined by a time-shift—that is, a rotation of $\mathbb{T}$—of angle $2\pi/\gcd(k, n)$. The group has order $l = \gcd(k, n)$. Let $q(t)$ be an equivariant trajectory: $q_j(t + 2\pi/l) = e^{i2\pi/k} q_{j-1}$, where we denote by $e^{i\alpha}$ a rotation around the rotation axis of angle α and j is meant modulo n. Its parametrization in a rotating frame with angular velocity θ, $x_j(t) = e^{i\theta t} q_j(t)$, fulfills the identity

$$x_j(t + 2\pi/l) = e^{2\pi i(\theta/l + 1/k)} x_{j-1}(t),$$

and θ can be chosen as $-l/k$, for example, in order to obtain the symmetry constraint

$$x_j(t + 2\pi/l) = x_{j-1}(t)$$

in a new rotating frame. Now, the action is redundant (that is, the period is strictly smaller than 2π), since $n \mid l$ and hence the resulting group has elements acting non-trivially on $\mathbb{T}$ but trivially on the index set and on the space; a non-redundant representative for this action can be given by factoring the group and rescaling the time: one obtains the choreography constraint

$$x_j(t + 2\pi/n) = x_{j-1}(t).$$

We finish this section by giving a few definitions of terms that will be used below.

(2.7) Definition. A symmetry group G is:

bound to collision: if every G-equivariant loop has collisions;

homographic: if every G-equivariant loop is homographic, i.e. constant up to Euclidean similarities;
transitive: if the permutation action of G on the index set is transitive;
fully uncoercive: if for every possible rotation vector $\boldsymbol{\omega}$ the corresponding action functional $\mathcal{A}_{\boldsymbol{\omega}}$ is not coercive if restricted to the space of G-equivariant loops Λ^G.

(2.8) Definition. The kernel $\ker\tau$ is termed the *core* of the symmetry group.

3. Transitive groups and transitive decomposition

Consider a symmetry group $G \subset O(\mathbb{T}) \times O(3) \times \Sigma_n$ and its *cover* $\widetilde{G} \subset \mathrm{Iso}(\mathbb{R}) \times O(3) \times \Sigma_n$, which is a discrete group acting on the time line $\mathbb{R}$ as time-shifts and time-reflections; the kernel of the projection $p : \widetilde{G} \to G$ is a free abelian group of rank 1. Like for τ, ρ, σ, it is possible to define the homomorphisms $\tilde{\tau} : \widetilde{G} \to \mathrm{Iso}(\mathbb{R})$, $\tilde{\rho} : \widetilde{G} \to O(3)$ and $\tilde{\sigma} : \widetilde{G} \to \Sigma_n$. Let us note that the diagram

$$\begin{array}{ccc} \widetilde{G} & \xrightarrow{\tilde{\tau}\times\tilde{\rho}} & \mathrm{Iso}(\mathbb{R}) \times O(3) \\ {\scriptstyle p}\downarrow & & \downarrow \\ G & \xrightarrow[\tau\times\rho]{} & O(\mathbb{T}) \times O(3) \end{array}$$

commutes: the horizontal arrows are monomorphisms and the vertical arrows are epimorphisms. The image $\tilde{\rho}(\widetilde{G}) = \rho(G) \subset O(3)$ is a finite space point group. The group $\widehat{G} = (\tilde{\tau} \times \tilde{\rho})(\widetilde{G}) \subset \mathrm{Iso}(\mathbb{R}) \times O(3)$ is therefore well defined, after a choice of cover $\widetilde{G} \to G$ (depending on the scale of the time line). Hence, by rescaling the time line, it is possible to assume that the time-shifts in $\tilde{\tau}(\widetilde{G})$ are powers of the time shift $t \mapsto t + 1$: to the given symmetry group G we associate the unique *normalized cover* $\widehat{G}$ in $\mathrm{Iso}(\mathbb{R}) \times O(3)$ with such a property. As we noted above, $\tilde{\tau} \times \tilde{\rho} : \widetilde{G} \cong \widehat{G}$, and so there is a unique homomorphism $\hat{\sigma} : \widehat{G} \to \Sigma_n$ such that $\hat{\sigma} \circ (\tilde{\tau} \times \tilde{\rho}) = \tilde{\sigma}$. The group $\widetilde{G}$ is the graph of $\hat{\sigma}$ in $\widehat{G} \times \Sigma_n \subset \mathrm{Iso}(\mathbb{R}) \times O(3) \times \Sigma_n$; $\hat{\sigma}$ is the *permutation homomorphism* from the normalized cover.

(3.1) Definition. Now let G_1 and G_2 be two symmetry groups with the same normalized cover $\widehat{G}_1 = \widehat{G}_2 \subset \mathrm{Iso}(\mathbb{R}) \times O(3)$ and permutation homomorphisms $\hat{\sigma}_1 : \widehat{G}_1 \to \Sigma_{n_1}$, $\hat{\sigma}_2 : \widehat{G}_2 \to \Sigma_{n_2}$. The *disjoint sum* $G_1 + G_2$ is the group having as normalized cover the group $\widehat{G}_1 = \widehat{G}_2$ and as permutation homomorphism the direct product $\hat{\sigma}_1 \times \hat{\sigma}_2 : \widehat{G}_1 = \widehat{G}_2 \to \Sigma_{n_1} \times \Sigma_{n_2} \subset \Sigma_{n_1+n_2}$, where the inclusion of $\Sigma_{n_1} \times \Sigma_{n_2}$ is the standard one.

(3.2) *Let G be a symmetry group. Then there are a finite number of symmetry groups $G_1, \ldots, G_l$ with normalized cover equal to $\widehat{G}$ such that*

$$G = G_1 + G_2 + \cdots + G_l$$

and each G_i acts transitively on its index set.

Proof. The homomorphism $\hat{\sigma} : \widehat{G} \to \Sigma_n$ can be decomposed as $\hat{\sigma} = \hat{\sigma}_1 \times \hat{\sigma}_2 \times \cdots \times \hat{\sigma}_l$, where each $\hat{\sigma}_i : \widehat{G} \to \Sigma_{n_i}$ yields a transitive permutation representation of $\widehat{G}$ on the set of n_i indices. The decomposition is unique up to reordering, and gives rise to a subgroup $\widetilde{G}_i$ of $\mathrm{Iso}(\mathbb{R}) \times O(3) \times \Sigma_{n_i}$ for $i = 1, \dots, l$. The group G_i is obtained as a factor group of $\widetilde{G}_i$. Note that $n = n_1 + \cdots + n_l$. □

We recall an elementary fact about group actions, which is quite relevant to the constructive approach to the transitive decomposition.

(3.3) *Assume that the action of a group G on a set X is transitive. Then for all $i \in X$, the isotropy subgroups $H_i = \{g \in G \mid gi = i\}$ are mutually conjugated in G. Left multiplication by elements in G yields a bijection $G/H_1 \cong X$ between X and the set of left cosets G/H_1, which is G-equivariant* (*that is, a G-bijection*).

As a matter of fact, this property can be used to *define* a group action as a list of subgroups: instead of defining the homomorphism $\sigma : G \to \Sigma_n$, it is possible to consider a transitive decomposition of $\sigma = \sigma_1 \times \cdots \times \sigma_l$ and compute the list of isotropy subgroups $H_1, \dots, H_l$. Conversely, given the group G and l subgroups $H_1, \dots, H_l$ (not necessarily distinct), one can define the permutation action of G on the disjoint sum $\coprod_i G/H_i$, which is nothing but a homomorphism $\sigma : G \to \Sigma_n$, where $n = \sum_i [G : H_i]$. Therefore, a symmetry group G can be constructed by its cover $\widetilde{G}$, by the pair $(\widehat{G}, \hat{\sigma})$ or equivalently by the $(l+1)$-uple $(\widehat{G}; \widehat{H}_1, \dots, \widehat{H}_l)$ consisting of the normalized cover and the l isotropy subgroups of all the transitive components. A transitive symmetry group hence is given by a pair $(\widehat{G}; \widehat{H})$. Now we give a constructive procedure for dealing with these data.

Consider the core of G, $\ker\tau \subset G$ and let K be its image in $O(3)$: $K = \rho(\ker\tau)$. Since the core $\ker\tau$ is normal in G, the space group $\rho(G)$ is a subgroup of the normalizer $N_{O(3)}K$ of K in $O(3)$. The space group $\rho(G)$ is also equal to the projection of $\widehat{G} \subset \mathrm{Iso}(\mathbb{R}) \times O(3)$ onto the second factor. The pre-image of K under this projection is the kernel of $\widehat{G} \xrightarrow{\cong} \widetilde{G} \xrightarrow{\tilde{\tau}} \mathrm{Iso}(\mathbb{R})$. Let us call it $\widehat{K}$. The projection induces an isomorphism $\widehat{K} \cong K$. The homomorphism ρ induces a homomorphism $G/\ker\tau \to W_{O(3)}K$ of $G/\ker\tau$ with image in the Weyl group of K in $O(3)$. In fact, $G/\ker\tau$ is projected onto a subgroup of $O(\mathbb{T}) \times W_{O(3)}K$, while $\widehat{G}/\widehat{K}$ (which is an infinite cyclic or infinite dihedral group) is projected onto a subgroup of $\mathrm{Iso}(\mathbb{R}) \times W_{O(3)}K$.

(3.4) Definition *(Krh data).* Let G be a symmetry group. Then define

(i) $K = \rho(\ker\tau) \subset O(3)$,
(ii) $[r] \in W_{O(3)}K$ as the image in the Weyl group of the generator mod $\ker\tau$ of $\ker\det\tau \subset G/K$ (corresponding to the time-shift with minimal angle, or, equivalently, the generator of the cyclic part of $\widehat{G}/\widehat{K}$). If $\ker\det\tau = K$, then $[r] = 1$.
(iii) $[h] \in W_{O(3)}K$ as the image in the Weyl group of one of the time-reflections (mod $\ker\tau$) in $G/\ker\tau$, in the cases such an element exists. Otherwise it is not defined.

In short, the triple

$$\bigl(K, [r], [h]\bigr)$$

is said the *Krh data* of G. For the sake of simplicity we will often omit such square brackets, if unnecessary. The Krh data are nothing but a constructive representation of $\widehat{G}$: in fact, $\widehat{G}$ can be defined as the subgroup of $\mathrm{Iso}(\mathbb{R}) \times O(3)$ with generators $(0, g)$ as g ranges in K (here 0 denotes the trivial isometry of $\mathbb{R}$) or in a subset of generators of K, together with $(1, r)$ (where 1 represents the time-shift $t \mapsto t+1$) and, if it exists, $(\bar{0}, h)$ (where $\bar{0}$ represents the time-reflection around $0 \in \mathbb{R}$).

Now we can consider the next piece of data in order to obtain the pair $(\widehat{G}; \widehat{H})$: of course, the permutation isotropy subgroup $\widehat{H}$.

(3.5) Definition *($\widehat{Krh}$ data).* Let G be a symmetry group acting transitively on the index set; let $\widehat{G}$ be its normalized cover in $\mathrm{Iso}(\mathbb{R}) \times O(3)$. Let $\widehat{H} \subset \widehat{G}$ and $H \subset G$ denote the isotropy subgroups with respect to the index permutation action, as in (3.3). Consider the following three elements.

(i) Let $\widehat{S}$ denote the subgroup $\widehat{S} = K \cap H \subset K$ (that is, the subgroup of the elements of G fixing the time and indices). Recall that $\widehat{K}$, $\widehat{K} \subset \mathrm{Iso}(\mathbb{R}) \times O(3)$, as defined above, is isomorphic to its projection on the second factor; the isomorphic image of $\widehat{S} \subset K$ in $\widehat{K}$ will be denoted by $\widehat{S}$ with an abuse of notation.

(ii) The image of the isotropy H in $G/\ker\tau$ is isomorphic to the image of $\widehat{H}$ in $\widehat{G}/\widehat{K}$. Its intersection with the cyclic group $\ker\det\tau/\ker\tau \subset G/\ker\tau$ is a cyclic group with a distinguished non-trivial generator (if not trivial), say $r \bmod \ker\tau$. Under the isomorphism in $\widehat{G}/\widehat{H}$ a representative $(k, \hat{r})$ of such a generator can be chosen in $\mathrm{Iso}(\mathbb{R}) \times N_{O(3)}K \subset \mathrm{Iso}(\mathbb{R}) \times O(3)$. As above we denote by k the time-shift $t \mapsto t+k$, and since $\widehat{G}$ is assumed to be normalized, k is an integer.

(iii) If the set $H \setminus \ker\det\tau$ is non-empty (that is, if there are time-reflections in H), then let $\hat{h}$ be the projection in $N_{O(3)}K$ of one of its elements. Otherwise, it is undefined (empty).

The triple

$$\big(\widehat{S}, (k, \hat{r}), \hat{h}\big)$$

is said to be the $\widehat{Krh}$ *data* of G.

As above for the Krh data, the $\widehat{Krh}$ data yields an explicit description of $\widehat{H} \subset \widehat{G}$, as a set of generators of $\widehat{H}$ in $\widehat{G}$. In fact, the choice of $\widehat{S}$ in K yields immediately its isomorphic image in $\widehat{K}$; together with $(k, \hat{r})$ and possibly $(\bar{0}, \hat{h})$ a set of generators for $\widehat{H}$ is obtained. The following proposition follows directly from the definition.

(3.6) *Let G be a transitive symmetry group and* $\left(\begin{smallmatrix} \widehat{S} & (k,\hat{r}) & \hat{h} \\ K & [r] & [h] \end{smallmatrix}\right)$ *the matrix with as first row the $\widehat{Krh}$ data defined above in* (3.5) *and as second row the Krh data defined in* (3.4)*. Then the normalized cover $\widehat{G}$ of G is, up to conjugacy, defined in* $\mathrm{Iso}(\mathbb{R}) \times O(3)$ *by the Krh data. The cover of the isotropy $\widehat{H}$ is defined by the $\widehat{Krh}$ data* (*the first row*)*. Its permutation representation on indices can be deduced by considering the G-set $G/H \cong \widehat{G}/\widehat{H}$.*

We can explicitly describe the disjoint sum of symmetry groups by the following presentation of the generators.

(3.7) *Let G_1 and G_2 be two groups with the same Krh data. Then the disjoint sum $G_1 + G_2$ is defined as follows*: *the normalized covers $\widehat{G}_1$ and $\widehat{G}_2$ are isomorphic and generated in* $\mathrm{Iso}(\mathbb{R}) \times O(3)$ *by the* (*common*) *Krh data. The action of such a resulting $\widehat{G}$ on the index set can be defined by taking the disjoint union of the $\widehat{G}$-sets $\widehat{G}/\widehat{H}_1 + \widehat{G}/\widehat{H}_2$, where $\widehat{H}_1$ and $\widehat{H}_2$ are defined by the $\widehat{Krh}$ data.*

In order to generate all the possible transitive symmetry groups it is now possible to proceed as follows: first define the Krh data (according to the classification of finite subgroups of $O(3)$, as in Appendix A, this can be done in a parametrized finite number of ways). Then choose the subgroup $\widehat{S} \subset K$ and the other $\widehat{Krh}$ data accordingly. By definition, it is only needed to take elements in the Weyl group of the finite subgroups K of $O(3)$ (which in general is finite). Some restrictions apply in order to choose the subgroup $\widehat{S}$, as it is explained by the following proposition.

(3.8) *Let H_1 denote one of the isotropy subgroups defined above in* (3.3) *for the transitive symmetry group G. Assume that* $\ker\tau \neq 1$. *Then one* (*and only one*) *of the following cases can occur*:

$$\begin{cases} \ker\tau \cap H_1 = 1, \\ \ker\tau \cap H_1 = \langle \textit{reflection along a plane} \rangle, \\ \ker\tau \cap H_1 = \ker\tau. \end{cases}$$

Proof. We tacitly assumed that the core $\ker\tau$ is not a reflection along a plane, since otherwise the problem would be a planar n-body problem or bound to collisions. Furthermore, since we assume $\ker\tau \neq 1$, the *only one* part is trivial. Suppose on the other hand that $\ker\tau \cap H_1 \neq 1$. Let $E \subsetneq \mathbb{R}^3$ be the linear fixed subspace

$$E = (\mathbb{R}^3)^{\ker\tau \cap H_1}.$$

The configuration space X can be seen as the space of maps $G/H_1 \to \mathbb{R}^3$, where G/H_1 is seen as a G-set with $[G : H_1]$ elements and $\mathbb{R}^3$ is of course a G-space via ρ. The action on X (as space of maps) is the diagonal action, and configurations in $X^{\ker\tau}$ correspond to $(\ker\tau)$-equivariant maps $G/H_1 \to \mathbb{R}^3$. Now, the number of $(\ker\tau)$-orbits in G/H_1 is also the number of the double cosets $\ker\tau \setminus G/H_1$; since $\ker\tau$ is normal in G, it coincides with the number of H_1-orbits in $G/\ker\tau$, which is $[G : H_1 \ker\tau]$. Any $(\ker\tau)$-map $x : G/H_1 \to \mathbb{R}^3$ (i.e. an element in $X^{\ker\tau}$) can therefore be decomposed into a sum of $[G : H_1 \ker\tau]$ disjoint parts (more precisely, its domain can be) corresponding to the $(\ker\tau)$-orbits in G/H_1. Each map defined on a $\ker\tau$-orbit is conjugated via an element of G to a $\ker\tau$-map of type

$$\ker\tau/(\ker\tau \cap H_1) \to (\mathbb{R}^3)^{\ker\tau \cap H_1} = E$$

(thus yielding $[\ker\tau : \ker\tau \cap H_1]$ particles in E). The space $X^{\ker\tau}$ is isomorphic to a direct sum of $[G : H_1 \ker\tau]$ copies of E, over which the action of G acts via conjugation (actually, it is the induced/inflated module).

Now, consider the hypothesis that $\ker\tau \neq \ker\tau \cap H_1$. The dimension $\dim E$ can be 0, 1 or 2 (it cannot be 3 since by assumption $\ker\tau \cap H_1 \neq 1$ and the action of $\ker\tau$ on $\mathbb{R}^3$ is faithful). If it is 0, then $\ker\tau = \ker\tau \cap H_1$ (since otherwise at each time a collision would occur. If $\dim E = 1$ (and $\ker\tau \neq \ker\tau \cap H_1$, assumption above) then either the group G is fully uncoercive or it is bound to collisions: in fact for one-dimensional E there cannot exist rotation axes, and any symmetry element yielding coercivity would make the group bound to collisions (the complementary of the collision set in E is not connected). It is left the case $\dim E = 2$, i.e. where $\ker\tau \cap H_1$ is the group generated by a single plane reflection. If the plane π fixed by $\ker\tau \cap H_1$ is $\ker\tau$-invariant (that is, $(\ker\tau)\pi = \pi$), then $\ker\tau \cap H_1$ is normal in $\ker\tau$ and the $(\ker\tau)$-representation given by ρ is one of the following:

(1) C_{ph} with $p \geqslant 1$ (the group generated by the reflection around the plane π and p rotations orthogonal to π),
(2) $I \times D_p$ with $p \geqslant 2$ even (the Coxeter group generated by the reflection around π and p "vertical" plane reflections),
(3) D_pC_p with $p \geqslant 2$ (the Coxeter group generated by p plane reflections), and
(4) $D_{2p}D_p$ with $p \geqslant 1$ (generated by D_p and $-\zeta_{2p}$: it is a Coxeter group for p odd).

Cases (2), (3) and (4) do not possibly have rotation axes, and a symmetry group extending $\ker\tau$ and not coercive would be fully uncoercive. Since the bodies are constrained to belong to π (H_1 is the isotropy of the permutation action) and the singular set of π cuts π into different components, a symmetry group extending such $\ker\tau$ cannot be coercive without being bound to collisions. Case (1) is of a different type: the $p = [\ker\tau : \ker\tau \cap H_1]$ bodies are constrained to be vertices of a regular k-agon centered at the origin and contained in π. The direction orthogonal to π is a rotation axis. This is the case in which the reflection along a plane yields possible periodic orbits. □

Note that if $\ker\tau \cap H_1 = 1$, then the isotropy H_1 is isomorphic to its image under ρ (after the composition with the projection onto the $(\ker\tau)$-quotient) in the Weyl group $W_{O(3)}K$.

4. Local variations and averaging techniques over equivariant spheres

The disjoint sum of symmetry groups defined in the previous section allows one to generate all symmetry groups in terms of transitive components. In this section we show how the decomposition is related to colliding trajectories and local variations. The purpose is to extend the range of applicability of the averaging technique of [19] to a wider class of symmetry groups. For details on the blow-up and the averaging technique we refer to Sections 7–9 of [19]. Let $\widehat{G}$ be the normalized cover of a symmetry group and $x = x(t) \in \Lambda = H^1(\mathbb{R}, X)^{\widehat{G}}$ an equivariant local minimizer. Assume that at time $t = 0 \in \mathbb{R}$ the trajectory $x(t)$ collides, and all bodies in a cluster $\mathbf{k} \subset \mathbf{n}$ collide (which means that other bodies might collide, but not with bodies in $\mathbf{k}$). Given the colliding cluster $\mathbf{k} \subset \mathbf{n}$ at time $t = 0$, let $G_* \subset G$ be the following subgroup:

$$G_* = \{g \in G\colon\ g(\mathbf{k}) = \mathbf{k},\ g(0) = 0\},$$

and analogously $\widehat{G}_* = \{g \in \widehat{G}\colon\ g(\mathbf{k}) = \mathbf{k},\ g(0) = 0\}$. Since they do not contain time-shifts, the projection $\widehat{G} \to G$ induces an isomorphism $\widehat{G}_* \cong G_*$. It is the subgroup consisting of those elements fixing the colliding time 0 and sending indices in $\mathbf{k}$ to indices in $\mathbf{k}$. Let $\bar{q}(t)$ be the blow-up of $x(t)$ centered at 0 with respect to $\mathbf{k}$; what is proved in [19] can be rephrased as follows: $\bar{q}(t)$ is a G_*-equivariant local minimizer with respect to compactly supported G_*-equivariant variations of the Lagrangian action $\mathcal{A}$, restricted to the path space $H^1(\mathbb{R}, X_{\mathbf{k}})^{G_*}$. The transitive decomposition is obtained by restricting the permutation action to the colliding particles in $\mathbf{k}$, that is, restricting $\sigma : \widehat{G} \to \Sigma_n$ to $\sigma_* : \widehat{G}_* \to \Sigma_{\mathbf{k}}$ (where we denote by $\Sigma_{\mathbf{k}}$ the permutation group of elements in $\mathbf{k}$). A similar procedure gives the restriction $\hat{\sigma}_*$ of $\hat{\sigma}$. If $\widehat{H}_1, \ldots, \widehat{H}_l$ are the (permutation) isotropy subgroups of $\widehat{G}$ relative to the indices $i_1, \ldots, i_l$, then the permutation isotropy subgroups of $\widehat{G}_*$ are the intersection $\widehat{H}_{i_j} \cap \widehat{G}_*$ for those i_j corresponding to indices in $\mathbf{k}$. It is important to understand that the configuration space $X_{\mathbf{k}}$ of the particles in $\mathbf{k}$ can be decomposed as orthogonal sum of components corresponding to the $\hat{\sigma}_*$-orbits of $\widehat{G}_*$ in $\mathbf{k}$, and hence the components are $\widehat{G}_*$-invariant. As a consequence, if with an abuse of notation we denote again with $\widehat{H}_1, \ldots, \widehat{H}_l$

the isotropy subgroups in $\widehat{G}_*$ relative to the indices in $\mathbf{k}$ instead of those relative to $\mathbf{n}$, the index set can be decomposed as $\mathbf{k} = \coprod_{i=1,\dots,l} \mathbf{k}_i$, where for each i one has $\mathbf{k}_i \cong \widehat{G}_*/\widehat{H}_i$; therefore $\mathbf{k}$ has $\sum_{i=1}^{l} [\widehat{G}_* : \widehat{H}_i]$ elements. By looking at the orthogonal decomposition analogous to (2.3) (with $\mathbf{k}$ instead of $\mathbf{n}$), one can deduce that the subspace fixed by G_* in the configuration space $X_{\mathbf{k}}$ can be written as $X_{\mathbf{k}}^{G_*} = X_{\mathbf{k}_1}^{G_*} + \cdots + X_{\mathbf{k}_l}^{G_*}$.

Let us define

$$S(s,\delta) = \int_0^\infty \left[\frac{1}{|t^{2/(2+\alpha)}s + \delta|^\alpha} - \frac{1}{|t^{2/(2+\alpha)}s|^\alpha} \right] dt.$$

The following lemma can be found in Section 9 of [19].

(4.1) *Let $\bar{q}(t)$ be a colliding blow-up trajectory and $\bar{s}$ the limiting central configuration in $X_{\mathbf{k}}$. If there exists a symmetric configuration $\delta \in X_{\mathbf{k}}^{\widehat{G}_*}$ (that is, δ is fixed by the isotropy $\widehat{G}_*$) such that for every $i, j \in \mathbf{k}$*

$$S(\bar{s}_i - \bar{s}_j, \delta_i - \delta_j) \leqslant 0$$

and for at least a pair of indices the inequality is strict, then the colliding blow-up trajectory $\bar{q}(t)$ is not a minimizer.

Now we consider three different procedures that can be used to find such a δ. A symmetric variation δ that let the action functional $\mathcal{A}$ decrease on the standard variation is called *V-variation*.

The following proposition is contained in theorem (10.10) of [19] (it is proved in the second part of the proof).

(4.2) *If G_* acts trivially on $\mathbf{k}$ via $\sigma_{\mathbf{k}}$, then a V-variation always exists.*

Note that since G_* acts on the time line fixing the point 0, this lemma is mainly relevant in the case G_* has a time-reflection: the symmetry constraint can be written as "all the point particles at time $t = 0$ belong to a linear subspace of $\mathbb{R}^3$." It might be of interest to see also [10]. As we noted above, it is equivalent to consider G_* or $\widehat{G}_*$.

We recall from [19] that a circle $\mathbb{S} \subset \mathbb{R}^3$ (with center in the origin 0) is called *rotating* under a group G_* for an index i when it is G_*-invariant and $\mathbb{S} \subset (\mathbb{R}^3)^{H_i}$, where $H_i \subset G_*$ is the isotropy of i with respect to the permutation action of G_* on the index set, via σ. Proposition (9.8) of [19] can be rephrased as follows.

(4.3) *If there is an index $i \in \mathbf{k}$ and a circle $\mathbb{S} \subset \mathbb{R}^3$ which is rotating under G_* for the index i, then the average*

$$\int_{\delta \in \iota_i \mathbb{S}} \sum_{j \neq i} S(\bar{s}_i - \bar{s}_j, \delta_i - \delta_j) < 0,$$

is strictly negative, where $\iota_i \mathbb{S} \subset X_{\mathbf{k}}$ is the image of the rotating circle $\mathbb{S}$ under the inclusion ι_i defined as the inclusions in the proof of (3.8). *In other words, if there is a rotating circle under $H_i \subset G_*$ then by averaging it is possible to find a V-variation.*

The next proposition is a new generalization of the rotating circle property. It allows to find V-variations in cases in which there are no rotating circles (for example, if $K \subset O(3)$ is an irreducible representation, or even in the dihedral case $K = D_p$ with $p \geqslant 2$). We will illustrate in Section 6 some easy but non-trivial cases in which proposition (4.3) does not hold (while proposition (4.4) does).

(4.4) *Let G_* be as above the symmetry group of a blow-up solution $\bar{q}$. If $\det \rho(G_*) = 1$ (i.e. G_* acts orientation-preserving on the space $\mathbb{R}^3$) and for one of the indices $i \in \mathbf{k}$ the permutation isotropy H_i (restricted to G_*) is trivial, then there exists a V-variation, obtained by averaging over a 2-sphere.*

Proof. Let $S^2 \subset \mathbb{R}^3$ be a 2-sphere centered in 0. If $H_i = H_i \cap G_* = 1$, then the space $E = (\mathbb{R}^3)^{G_* \cap H_i}$ is equal to $\mathbb{R}^3$ and it contains the sphere S^2. As explained also in the proof of (3.8), the fixed configuration space X^{G_*} can be decomposed into a sum of copies of E (exactly $|G_*|$, since the isotropy is trivial) and an orthogonal complement (which depends on the indices which are not in the same homogeneous part of the index i): hence there is an embedding $\iota_i : S^2 \to X_{\mathbf{k}}$ defined by the group action. Now, all elements of G_* by hypotheses act by rotations on $\mathbb{R}^3$. Next, consider the average

$$A = \int_{\delta \in \iota_i S^2} \sum_{i<j} S(\bar{s}_i - \bar{s}_j, \delta_i - \delta_j).$$

The sum is equal to the sum of terms like

$$A_g = \int_{\delta_i \in S^2} S\big(\bar{s}_i - \bar{s}_j, (1-g)\delta_i\big)$$

where g ranges in G_*. But since g acts as rotation in $\mathbb{R}^3$, $(1-g)$ is the projection onto the plane orthogonal to the line l fixed by g, composed with a rotation around l and a dilation. Therefore for each $g \in G_*$ there is a positive constant $c_g > 0$ such that

$$A_g = c_g \int_{\delta_i \in \mathbb{S}} S(\bar{s}_i - \bar{s}_j, \delta_i) \tag{4.5}$$

obtained as in the case of the integration on a disc (see also [11]). The proof of this fact is simple: write $\mathbb{R}^3$ as the orthogonal sum $\mathbb{C} \oplus l$, where l is the line fixed by g; δ_i can be written as $(\sin\varphi\, e^{i\theta}, \cos\varphi)$, where $\theta \in [0, 2\pi]$ and $\varphi \in [0, \pi]$. Let z_g denote the complex number acting by left multiplication on $\mathbb{C}$ exactly as the rotation g: we can write

$$\int_{\delta_i \in S^2} S\big(\bar{s}_i - \bar{s}_j, (1-g)\delta_i\big) = \int_{\delta \in \mathbb{S}} \int_{\varphi \in [0,\pi]} \sin\varphi\, S\big(\bar{s}_i - \bar{s}_j, \sin\varphi\,(1-z_g)\delta\big),$$

where $\mathbb{S}$ is the unit circle in $\mathbb{C}$; by homogeneity $S(\xi, \mu\delta) = \mu^{1-\alpha/2} S(\xi, \delta)$, and hence the last term can be written as

$$\int_{\delta \in \mathbb{S}} S(\bar{s}_i - \bar{s}_j, \delta) \cdot \int_0^{\pi} \left(\sin\varphi |1 - z_g|\right)^{1-\alpha/2} \sin\varphi \, d\varphi$$

from which Eq. (4.5) follows. Since such terms are always strictly negative, the conclusion follows. □

Note that (4.3) and (4.4) hold true if and only if an hypothesis is fulfilled on one of the transitive components. In other words, a V-variation obtained by averaging over an equivariant circle or an equivariant sphere exists if and only if it is possible to obtain a V-variation by averaging over a circle only in one of the transitive components in which $\widehat{G}_*$ can be subdivided. Moreover, if the hypothesis holds for $\widehat{G}_*$ then it will hold for all the subgroups of $\widehat{G}_*$ and hence also for all $\mathbf{k} \subset \mathbf{n}$. So if *all* transitive components of the possible $\widehat{G}_* \subset \widehat{G}$ fulfill one of the hypotheses then G-equivariant local minimizers are surely collisionless.

5. Transitive components of groups with collisionless minimizers

In this section we try to analyse which transitive groups can be taken as building blocks for the generation of non-colliding minimizers. We begin by listing all the properties that we would like a symmetry group to have. The list will promptly imply Theorem A.

(5.1) Definition. We say that a group G has property (5.1) if it is:

(i) not bound to collision,
(ii) not fully uncoercive,
(iii) not homographic, and at last that
(iv) for all maximal time-isotropy subgroups $G_* \subset G$ at least one of the propositions (4.2), (4.3) or (4.4) can be applied (that is, either G_* acts trivially on indexes, or there is a transitive component with a rotating circle or G_* acts by rotations on the Euclidean space $\mathbb{R}^3$).

According to its definition, if the group is not fully uncoercive, then possibly considering a non-zero angular velocity vector $\boldsymbol{\omega}$, local minima always exist in the rotating frame. Of course, we exclude the groups bound to collisions; we exclude homographic groups for the obvious reason that we are looking for collisionless solutions which are not homographic. Now, if furthermore property (iv) (which can be easily tested only on the transitive components, as noted above) holds, the existence of a V-variation implies that all local minimizers are collisionless, which is our goal.

We start by considering the possible cores for G (not considering at the moment the permutations on the indices), as the first entry in the Krh data (K, r, h). All finite subgroups of $SO(3)$ listed in Table 1 at page 782 (and the trivial group, not listed) can be cores by (4.4), as far as the isotropy ($\widehat{S}$ in the $\widehat{Krh}$ data of the corresponding component) of one of the indices is trivial. Then, of the groups of Table 2, the central inversion group I and the central prism/antiprism $I \times C_p$ group have a rotating circle and can be considered. The groups $I \times D_p$ with $p \geqslant 2$ are generated by plane reflections for p even and do not contain rotating circles: the action restricted to invariant planes is never consisting of rotations. The only possible hypothesis for the existence of a V-variation is the triviality of the permutation action: but the subspace of $\mathbb{R}^3$ fixed is 0, and hence with more than one particle the group would be bound to collisions. The remaining groups $I \times T$, $I \times O$ and $I \times Y$ of the table act on $\mathbb{R}^3$ without invariant planes (the representation is irreducible) and hence they must be excluded. The same is true for the full tetrahedron group OT of Table 3. Of the three remaining groups in the same table, the prism/antiprism group $C_{2p}C_p$ clearly has a rotating circle and must be added to the list. The groups D_pC_p (the p-gonal planes reflection group) and $D_{2p}D_p$ (for $p \geqslant 2$) do not have rotating circles and have reflections: not only none of (4.2), (4.3) and (4.4) can be applied, but all the symmetry groups with this core are bound to collisions or fully uncoercive.

(5.2) *The groups satisfying* (5.1) *are the following*: (1) C_p (*for* $p \geqslant 1$), (2) $I \times C_p$ (*for* $p \geqslant 1$), (3) $C_{2p}C_p$ (*for* $p \geqslant 2$), (4) D_p (*for* $p \geqslant 2$), (5) T, (6) O, (7) Y.

For the same reason this is also the list of projections on $O(3)$ of the (possible) maximal time-isotropy groups and of the cores. This concludes the proof of Theorem A.

It is interesting, however, to consider extensions (of index 2) of such cores as possible time-isotropy groups for times fixed by reflections. The method used for obtaining the existence of V-variations sets constraints on the type of admissible extensions: a group with V-variations obtained only by averaging on spheres and without rotating circles cannot be extended other than in $SO(3)$, as in the case of the last four items in the list.

(5.3) *The index* 2 *extensions satisfying* (5.1) *of cores satisfying* (5.1) *are the following*: (1) C_1: *I, C_2C_1, C_2*; (2) C_p (*for* $p \geqslant 2$): C_{2p}, D_p, $I \times C_p$, $C_{2p}C_p$; (3) P'_{2p} (*for* $p \geqslant 1$): $I \times C_{2p}$; (4) D_p (*for* $p \geqslant 2$): D_{2p}; (5) T: O; (6) O: *nothing*; (7) Y: *nothing*.

Proof. The index 2 extensions of the trivial group are of course the groups of order 2 in the list (5.2): I, C_2C_1 and C_2. The extensions of C_p in $p \geqslant 2$ in $SO(3)$ are C_{2p} and D_p. The remaining groups in (5.2) of order $2p$ containing C_p are $I \times C_p$ and $C_{2p}C_p$. Now consider the two prism/antiprism family of groups $I \times C_p$ and $C_{2p}C_p$: they have a rotating plane and are not orientation-preserving: hence they can be extended without restrictions on the orientation once the rotating plane is preserved. On the other hand, if $\mathbb{R}^3$ is disconnected by the collision subspaces, then it is not possible to assume coercivity and being collisionless. Hence groups with fixed planes must be eliminated: of the two families $I \times C_p$ and $C_{2p}C_p$ only the antiprism family of groups P'_{2p} survives, with normalizer $I \times C_{2p} (= C_{2ph})$. The group D_p (without rotating circles) can be only in the orientation-preserving group D_{2p}. The group T can be extended only in O, while O and Y do not have index 2 extensions in $SO(3)$. □

We end the section by exhibiting two examples of such groups (in terms of their Krh data). Recall that the matrices of Krh and $\widehat{Krh}$ data are $\left(\begin{smallmatrix} \widehat{S} & (k,\hat{r}) \\ K & [r] \end{smallmatrix}\right)$ (for the cyclic type) or $\left(\begin{smallmatrix} \widehat{S} & (k,\hat{r}) & \hat{h} \\ K & [r] & [h] \end{smallmatrix}\right)$ (for brake or dihedral type), as defined in (3.6).

(5.4) Example *(Cyclic type and trivial core).* Let us now consider the simpler case of trivial core with cyclic action type. By definition $K = 1$ and hence $\widehat{S} = 1$, which implies $\mathbb{Z} \cong \widehat{G} = \langle (1, r) \rangle \subset \operatorname{Iso}(\mathbb{R}) \times O(3)$ in the cyclic case. About the pair $(k, \hat{r})$ generating the cover $\widehat{H}$ of the permutation isotropy, it must be a power of the generator $(1, r)$ and hence of the form (k, r^k).

If the action is of cyclic type, then the Krh can be written as $\left(\begin{smallmatrix}1 & (k,r^k)\\ 1 & r\end{smallmatrix}\right)$, where up to rotating frames r can be chosen with order at most 2 (it is not difficult to see that every cyclic symmetry group is of type R). Since if $r=1$, then it must be $\hat{r}=1$, we have for every $k\geqslant 1$ the *choreographic* symmetry

$$\begin{pmatrix}1 & (k,1)\\ 1 & 1\end{pmatrix},$$

which acts transitively on the set of k bodies. Of course, the constraints can be written also as the better known form $x_1(t+i)=x_i(t)$ for $i=1,\dots,k$ for k-periodic loops.

If r is the reflection $-\zeta_2$, then the Krh is

$$\begin{pmatrix}1 & (k,(-\zeta_2)^k)\\ 1 & -\zeta_2\end{pmatrix},$$

which acts again on set of k indices, but with a resulting cyclic group G with $2k$ elements. Any other choice of r would give rise to one of these groups, up to a change of rotating frame.

(5.5) Example *(Dihedral type and trivial core).* Following the same argument as in Section 6 of [18], one can see that the Krh for a dihedral group of type R can be chosen of the following forms (for h_1 and h_2 integers):

$$\begin{pmatrix}1 & (k,1) & *\\ 1 & 1 & (-1)_1^h\zeta_2^{h_2}\end{pmatrix} \quad\text{or}\quad \begin{pmatrix}1 & (k,(-\zeta_2)^k) & *\\ 1 & -\zeta_2 & (-1)_1^h\zeta_2{}^{h_2}\end{pmatrix}.$$

Groups not of type R can be found in a similar fashion.

6. A few simple examples

In this last section we give some simple examples of minimizers symmetric with respect to groups generated by Krh data as explained above. The first two examples have been selected because they illustrate exactly under which extent Theorem A is a real generalization of the results in [19]. In fact, examples (6.1) and (6.2) are the simplest cases in which the rotating circle property does not hold (as we have mentioned above, if K is irreducible in $O(3)$ there cannot be rotating circles) but, since K is a rotation group and at the same time the maximal $\mathbb{T}$-isotropy subgroup of the symmetry group (the action is of cyclic type), proposition (4.4) can be applied. Example (6.2) for $k=2$ yields a symmetry group for 4 bodies in the space, which is the simplest example on which the results of [19] could not be applied. Example (6.1) (also called the *buckyball*) has been selected mainly because it generates quite interesting collisionless periodic orbits of 60 bodies with equal masses and icosahedral symmetry.

(6.1) Example. Consider the icosahedral group Y of order 60. The group G with Krh data

$$\begin{pmatrix}1 & (1,-1)\\ Y & -1\end{pmatrix}$$

is isomorphic to the direct product $I\times Y$ of order 120, and acts on the Euclidean space $\mathbb{R}^3$ as the full icosahedron group. The action on $\mathbb{T}$ is cyclic and given by the fact that $\ker\tau = 1\times Y$.

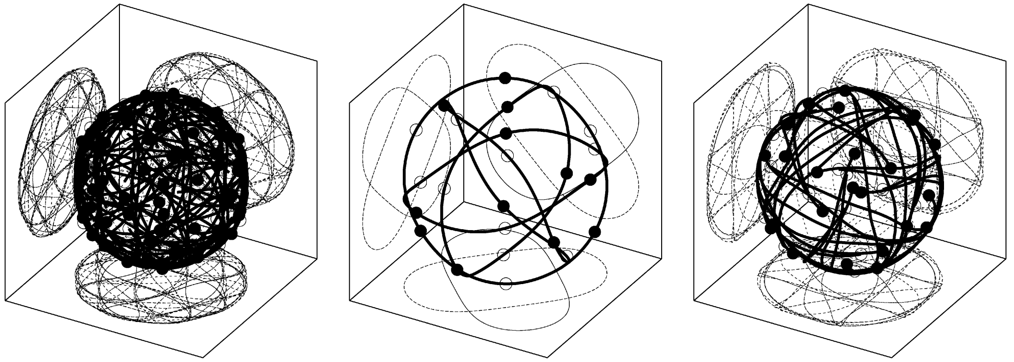

Fig. 1. 60-icosahedral Y, 12-tetrahedral T and 24-octahedral O periodic minimizers (chiral).

Since G (more properly, its isomorphic image in $O(3)$ under ρ) is generated by π_3, π_3' and -1 (see Appendix A about notation on generators and finite subgroups of $O(3)$), the action in terms of generators can be described as follows: for each $t \in \mathbb{T}$, $\tau(\pi_3)(t) = \tau(\pi_3')(t) = t$, $\tau(-1)(t) = t + T/2$ (assuming period T). The isotropy is generated by the central inversion -1, and hence the set of bodies is $G/I \cong Y$. Thus at any time t the 60 point particles are constrained to be a Y-orbit in $\mathbb{R}^3$ (which does not mean they are vertices of a icosahedron, simply that the configuration is Y-equivariant). The permutation action is given by left multiplication: $\sigma(\pi_3)(y) = \pi_3 y$, $\sigma(\pi_3')(y) = \pi_3' y$ and $\sigma(-1)(y) = y$ for each $y \in Y$. After half period every body is in the antipodal position: $x_i(t + T/2) = -x_i$ (in other words, the group contains the *anti-symmetry*, also known as *Italian symmetry*—see [2,3,11,12]). Of course, the group Y is just an example: one can choose also the tetrahedral group T or the octahedral O and obtain anti-symmetric orbits for 12 (tetrahedral) or 24 (octahedral) bodies, as depicted in Fig. 1. The action is by its definition transitive and coercive; local minimizers are collisionless since the maximal $\mathbb{T}$-isotropy group acts as a subgroup of $SO(3)$ (i.e. orientation-preserving).

(6.2) Example. Let G be the group with Krh data

$$\begin{pmatrix} 1 & (1,-1) \\ D_k & -1 \end{pmatrix},$$

where D_k is the rotation dihedral group of order $2k$. As in the previous example, the action is such that the action functional is coercive and its local minima are collisionless. At every time instant the bodies are D_k-equivariant in $\mathbb{R}^k$ and the anti-symmetry holds. Approximations of minima can be seen in Fig. 2. The group action in terms of generators is as follows: the generators of D_k are ζ_k and κ (see Table 1), which generate G together with -1 (G is therefore isomorphic to $I \times D_k$). One has, for every $t \in \mathbb{T}$, $\tau(\zeta_k)(t) = t$, $\tau(\kappa)(t) = t$ and $\tau(-1)(t) = t + T/2$ (where T is the period); the permutation action is given by left multiplication of the $2k$ elements of D_k, as above.

The next example is meant to illustrate a simple case of sum of groups, and the connection with the shape of the corresponding equivariant minimizers. It is interesting to see that the disjoint sum of groups has minimizers made of perturbations of the minimizers of the single transitive components in the decomposition, as it is illustrated in Fig. 3.

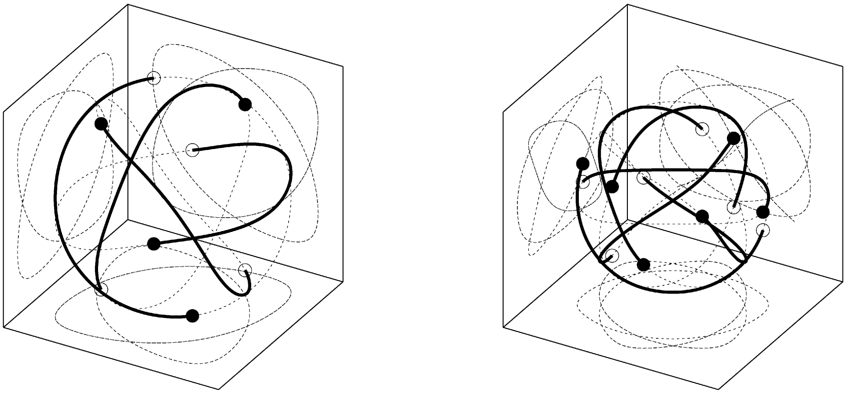

Fig. 2. 4-dihedral D_2 and 6-dihedral D_6 symmetric periodic minimizers.

(6.3) Example. To illustrate the case of non-transitive symmetry group, consider the following (cyclic) Krh data

$$\begin{pmatrix} 1 & (3,-1) \\ 1 & -1 \end{pmatrix},$$

which yield a group of order 6 acting cyclically on 3 bodies, and with the antipodal map on $\mathbb{R}^3$. Since $\ker\tau$ is trivial and the group is of cyclic type, local minima are collisionless. Now, by adding k copies of such group one obtains a symmetry group having k copies of it as its transitive components, where still local minimizers are collisionless and the restricted functional is coercive. In terms of generators, consider a cyclic group of order 6 generated by an element g; the action on the time line is given by $\tau(g)(t) = t + T/6$ for each t, where T is the period. The action on the set of $6k$ indices is given by cyclic permutations on all the k blocks, and the action on the space (via ρ) is given by $\rho(g)(x) = -x$ for all x. Let us recall that the minimizers of the components are rotating Lagrange configurations. Some possible minima (numerically approximated) can be found in Fig. 3, for $k = 3, 4$.

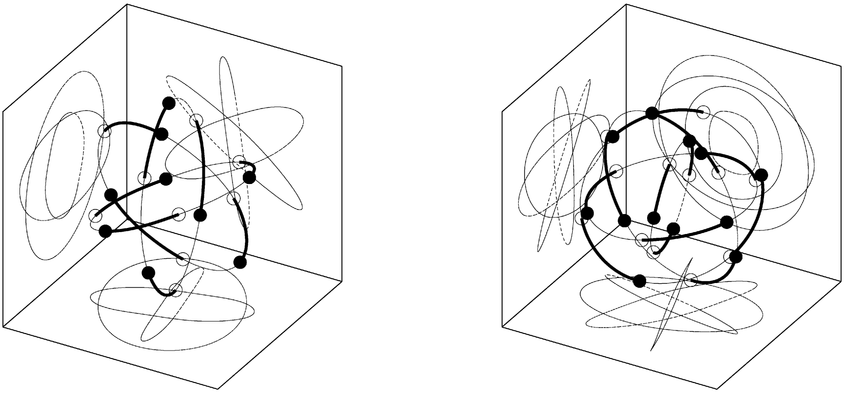

Fig. 3. 9 and 12 bodies in anti-choreographic constraints grouped by 3.

(6.4) Remark. The planar case can be dealt exactly as we did for the spatial case, with a significant simplification: only when the permutation action is trivial or there exists a rotating circle (that is, under these hypotheses the maximal $\mathbb{T}$-isotropy group of all possible colliding times has transitive components which act on the position space as rotations). A transitive decomposition of such planar symmetry group, also, is much simpler since the core has to be a (regular polygon) cyclic group. Nevertheless, also in the planar case many examples can be built using these simple building blocks. It is still an open problem whether there are symmetry groups not bound to collisions with (local or global?) minimizers which are colliding trajectories. It has been proved in [7] that it cannot happen for $n = 3$, but to the author's knowledge there is not yet a general result.

Appendix A. Notation on finite space groups

The finite subgroups of $O(3)$ are index 2 extensions of the groups listed above. Let I denote the group generated by the central inversion $-1 \in O(3)$. Since $O(3) = I \times SO(3)$ and I is the center of $O(3)$, finite groups containing the central inversion are I, $I \times C_p$, $I \times D_p$, $I \times T$, $I \times O$ and $I \times Y$.

The remaining *mixed groups* are those not containing the central inversion: $C_{2p}C_p$ (of order $2p$), D_pC_p (of order $2p$, it is a Coxeter group, i.e. generated by plane reflections; it is the full symmetry group of a p-gonal pyramid), $D_{2p}D_p$ (of order $4p$; it is a Coxeter group if p is odd, full symmetry group of a p-gonal prism or a p-gonal dipyramid) and $S_4A_4 = OT$ (of order 24, it is a Coxeter group: the full symmetry group of a tetrahedron). One word about notation: mixed groups are denoted by a pair GH, where G is a finite rotation group of Table 1, which turns out to be isomorphic to the group under observation but not conjugated to it, and H is a subgroup of index 2 in G. Given such a pair, a group not containing I is obtained as the union (of sets) $H \cup (-1(G \setminus H))$. Let ζ_p and κ be the rotations

$$\zeta_p = \begin{bmatrix} \cos 2\pi/p & -\sin 2\pi/p & 0 \\ \sin 2\pi/p & \cos 2\pi/p & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad \kappa = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix},$$

and, if $\varphi = (\sqrt{5}+1)/2$ denotes the golden ratio, let π_3 and π_3' be the rotations defined by the following matrices:

$$\pi_3 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}, \qquad \pi_3' = \begin{bmatrix} \varphi/2 & (1-\varphi)/2 & 1/2 \\ (\varphi-1)/2 & -1/2 & -\varphi/2 \\ 1/2 & \varphi/2 & (1-\varphi)/2 \end{bmatrix}.$$

Then the generators and normalizers of finite subgroups of $SO(3)$ are listed in Table 1. For more data on the icosahedral group, see also [21].

Note that other symbols might be used: $O^- = \overline{O} = OT$, $D^d_{2p} = D_{2p}D_p$, $D^z_p = D_pC_p$, $Z^-_{2p} = \overline{Z}_{2p} = C_{2p}C_p$, $\mathbb{Z}^c_2 = I$, $Z_p = C_p$, $I = Y$ (here there is a notation clash with $I = \langle -1 \rangle$); the Schönflies notation for crystallographic point groups (or the equivalent Hermann–Mauguin notation) is also another option: for example, $T_d = OT$, $T_h = I \times T$, $O_h = I \times O$, $Y_h = I \times Y$ or $D_pC_p = C_{pv}$. Groups generated by reflections (that is, Coxeter groups) are D_pC_p (with $p \geqslant 1$), $D_{2p}D_p$ (with p odd), $I \times D_p$ (with p even), OT, $I \times O$, $I \times Y$.

Finally, note that the G-orbit of a point in general position in $\mathbb{R}^3$ is a regular p-agon for $G = C_p$ but it is not a regular polygon for $G = D_p$ or if G is a polyhedral group (full or rotation).

Table 1
Finite subgroups of $SO(3)$, their normalizers in $SO(3)$ and generators (the generators of the normalizer are obtaining adding the generator of the fourth column to the generators of the second column)

Name	Symbol	Order	Generators	$N_{SO(3)}G$	Generators
Rotation cyclic	C_p	$p \geqslant 2$	ζ_p	$O(2)$	ζ_*, κ
Rotation three axes	D_2	4	ζ_2, κ	O	ζ_4, π_3
Rotation dihedral	D_p	$2p \geqslant 6$	ζ_p, κ	D_{2p}	ζ_{2p}
Rotation tetrahedral	$T \cong A_4$	12	ζ_2, π_3	O	ζ_4
Rotation octahedral	$O \cong S_4$	24	ζ_4, π_3	O	
Rotation icosahedral	$Y \cong A_5$	60	π_3, π_3'	Y	

Table 2
Finite subgroups of $O(3)$ containing the central inversion

Name	Symbol	Order	Generators	$N_{O(3)}G$	Generators
Central inversion	I	2	-1	$O(3)$	
Prism/antiprism	$I \times C_p$	$2p \geqslant 4$	$-1, \zeta_p$	$I \times O(2)$	ζ_*, κ
Three planes	$I \times D_2$	8	$-1, \zeta_2, \kappa$	$I \times O$	ζ_4, π_3
	$I \times D_p$	$2p \geqslant 6$	$-1, \zeta_p, \kappa$	$I \times D_{2p}$	ζ_{2p}
	$I \times T$	24	$-1, \zeta_2, \pi_3$	$I \times O$	ζ_4
Full octahedron	$I \times O$	48	$-1, \zeta_4, \pi_3$	$I \times O$	
Full icosahedron	$I \times Y$	120	$-1, \pi_3, \pi_3'$	$I \times Y$	

Table 3
Finite subgroups of $O(3)$ of mixed type, their normalizers and generators

Name	Symbol	Order	Generators	$N_{O(3)}G$	Generators
Prism/antiprism	$C_{2p}C_p$	$2p \geqslant 2$	$-\zeta_{2p}$	$I \times O(2)$	$\zeta_*, -1$
Reflections dihedral	D_pC_p	$2p \geqslant 4$	$\zeta_p, -\kappa$	$I \times D_{2p}$	-1
	$D_{2p}D_p$	$4p \geqslant 4$	$\zeta_p, \kappa, -\zeta_{2p}$	$I \times D_{2p}$	-1
Full tetrahedron	OT	24	$\zeta_2, \pi_3, -\zeta_4$	$I \times O$	-1

For the groups $I \times C_p$ and $C_{2p}C_p$, the G-orbit of a point (in general position in $\mathbb{R}^3$) is the set of vertices of a *prism* if p is even and $G = I \times C_p$ or if p is odd and $G = C_{2p}C_p$. It is the set of vertices of an *antiprism* (also known as *twisted prism*) if p is odd and $G = I \times C_p$ or if p is even and $G = C_{2p}C_p$. Therefore such groups might be called prism/antiprism groups correspondingly. In the Schönflies notation the antiprism group of order $2p$ is denoted by S_{2p} and the prism group of order $2p$ by C_{ph}. To avoid possible confusion, we define for $p \geqslant 1$ the antiprism group S_{2p} also as

$$P_{2p}' = \begin{cases} I \times C_p & \text{if } p \text{ is odd,} \\ C_{2p}C_p & \text{if } p \text{ is even.} \end{cases}$$

It is a cyclic group generated by a *rotatory reflection* of order $2p$. The prism group on the other hand is defined for $p \geqslant 1$ as

$$C_{ph} = \begin{cases} I \times C_p & \text{if } p \text{ is even,} \\ C_{2p}C_p & \text{if } p \text{ is odd} \end{cases}$$

and is generated by a rotation of order p together with a reflection (with fixed plane orthogonal to the rotation axis).

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