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A Global Theory for Linear-Quadratic Differential Games*

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INTRODUCTION AND SUMMARY

It is the element of conflict and cooperation which lies at the root of game theory and distinguishes differential games from control theory. Rufus Isaacs focused attention upon these problems with the publication of his book Differential Games [5] in 1965 and recently mathematicians have been studying a wide variety of problems in this area [1-4, 8, 9].

Most investigations have concentrated their efforts upon two person games of the pursuit-evasion type and upon zero-sum games in which one player tries to minimize a prescribed cost functional while the other plays so as to maximize the functional. Although considerable progress has been made on two person games the theory is nowhere near completion. In this paper we study an n-person nonzero-sum game which belongs to a class for which few results are known.

In the games considered here, each player selects a cost functional which measures his lack of success in a play of the game and tries to minimize that cost. This leads to the definition of an equilibrium strategy whose existence, uniqueness, stability and synthesis are studied in this paper.

Since differential games cover control theory as the special case of a one player game it is not surprising that along with an enrichment of phenomena

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it inherits all the difficulties of control—and more. Evidence to this effect is given by the example of a nonplayable game presented in Section 2. The problem shows up as the singularity of the decision operator defined in Section 1. Nevertheless, many of the differential games studied here, with linear dynamics and quadratic costs, do possess an equilibrium and this extended concept of a minimum can be analyzed in some detail.

In Section 1 the existence and uniqueness of open-loop equilibrium strategies is analyzed in a Hilbert space setting where the concept of a playable game is most easily abstracted. Various sufficiency conditions for playability are derived in Section 2 and in Section 3 the important idea of a stable equilibrium is developed.

Beginning with Section 4 we study differential games. These concrete problems give rise to spaces H, H_1, H_2,..., H_n of infinite dimension, thus accounting for the generality of our treatment. All the differential games considered are shown to be stable, and hence playable, over sufficiently short time intervals. A Hamilton-Jacobi approach is developed to prove the important fact that the open-loop equilibrium strategies of a playable game can be synthesized by feedback control functions and consequently an efficient means is found for computing the strategies in terms of the state of the game.

Notation. We employ notation and its abuses common in functional analysis. For the inner product of two vectors x and y we use $x \cdot y$ in finite dimensional real number spaces R^k and (x, y) in real Hilbert spaces H, H_1, H_2,..., H_n, etc. The induced norm is written as $|x|$ and the norm of a real matrix or bounded operator B is defined by $\|B\| = \sup_{|x|=1} |Bx|$. B^* denotes the transposed matrix or adjoint operator. The notation $W \geqslant 0$ $(W > 0)$ means W is a symmetric positive semi-definite (definite) matrix, i.e. $x \cdot Wx \geqslant 0$ $(x \cdot Wx > 0)$ for all vectors $x \neq 0$. We also use the convenient notation $|x|_W^2 = x \cdot Wx$ with similar notations when W is a bounded self-adjoint operator. To the extent that it is convenient we use Latin script $\mathscr{B}_i$, $\mathscr{D}$, $\mathscr{S}$, $\mathscr{U}_i$, $\mathscr{W}_i$,... for operators and Latin letters B_i, A, U_i, W_i,... for matrices. I_i denotes the identity operators on the ith space.

1. n-Person Games in a Hilbert Space

Let H, H_1, H_2,..., H_n be real Hilbert spaces. We consider a process

$$x = x_0 + \mathscr{B}_1 u_1 + \mathscr{B}_2 u_2 + \cdots + \mathscr{B}_n u_n \tag{1.1}$$

where the state of the game x and the initial state x_0 lie in H, $u_i \in H_i$, and $\mathscr{B}_i$ denotes a bounded linear transformation

$$\mathscr{B}_i : H_i \to H \qquad (i = 1, 2,..., n). \tag{1.2}$$

It is assumed that the control vector (strategy) u_i can be chosen at will by a player whom we shall call P_i. Each player P_i has, as his objective, the minimization of a certain quadratic cost functional of x and u_i,

$$C_i = | x - \bar{x}_i |^2_{\mathscr{W}_i} + | u_i |^2_{\mathscr{U}_i} . \tag{1.3}$$

Here the target vectors $\bar{x}_i \in H$, the bounded linear operators $\mathscr{W}_i \geqslant 0$ on H and $\mathscr{U}_i > 0$ on H_i are given $(i = 1, 2,..., n)$.

Since x is given by (1.1), (1.3) becomes

$$C_i(u_1, u_2,..., u_n) = | \mathscr{B}_1 u_1 + \mathscr{B}_2 u_2 + \cdots + \mathscr{B}_n u_n - (\bar{x}_i - x_0)|^2_{\mathscr{W}_i} + | u_i |^2_{\mathscr{U}_i} . \tag{1.4}$$

If the strategies $u_1, u_2,..., u_{i-1}, u_{i+1},..., u_n$ are fixed, then a simple completion of squares shows that the minimizing $\hat{u}_i$ is the unique value of u_i at which the Fréchet derivative with respect to u_i of the expression in (1.4) is zero. Thus $\hat{u}_i$ satisfies

$$\begin{aligned} \mathscr{B}_i{}^*\mathscr{W}_i\mathscr{B}_1 u_1 + \mathscr{B}_i{}^*\mathscr{W}_i\mathscr{B}_2 u_2 + \cdots + (\mathscr{U}_i + \mathscr{B}_i{}^*\mathscr{W}_i\mathscr{B}_i)\, \hat{u}_i + \cdots + \mathscr{B}_i{}^*\mathscr{W}_i\mathscr{B}_n u_n \\ = \mathscr{B}_i{}^*\mathscr{W}_i(\bar{x}_i - x_0) \end{aligned} \tag{1.5}$$

where $\mathscr{B}_i{}^* : H \to H_i$ is defined by

$$(\mathscr{B}_i{}^* x, u_i) = (x, \mathscr{B}_i u_i), \qquad x \in H, \quad u_i \in H_i . \tag{1.6}$$

Since $\mathscr{U}_i + \mathscr{B}_i{}^*\mathscr{W}_i\mathscr{B}_i > 0$, the optimal strategy $\hat{u}_i$ is uniquely determined if the other strategies $u_1, u_2,..., u_{i-1}, u_{i+1},..., u_n$ are known.

The most important question in the theory of games is: Is it possible for all players to play optimal strategies simultaneously? Such a n-tuple of strategies $\hat{u}_1, \hat{u}_2,..., \hat{u}_n$ is called an *equilibrium strategy* and is defined by the inequalities

$$\mathscr{C}_i(\hat{u}_1, \hat{u}_2,..., \hat{u}_n) \leqslant \mathscr{C}_i(\hat{u}_1, \hat{u}_2,..., \hat{u}_{i-1}, u_i, \hat{u}_{i+1},..., \hat{u}_n) \tag{1.7}$$

for all $u_k \in H_i$, $(i = 1, 2,..., n)$. It is evidently true that there is an equilibrium strategy if the equations

$$\begin{aligned} \mathscr{B}_i{}^*\mathscr{W}_i\mathscr{B}_1 \hat{u}_1 + \mathscr{B}_i{}^*\mathscr{W}_i\mathscr{B}_2 \hat{u}_2 + \cdots + (\mathscr{U}_i + \mathscr{B}_i{}^*\mathscr{W}_i\mathscr{B}_i)\, \hat{u}_i + \cdots + \mathscr{B}_i{}^*\mathscr{W}_i\mathscr{B}_n \hat{u}_n \\ = \mathscr{B}_i{}^*\mathscr{W}_i(\bar{x}_i - x_0), \qquad (i = 1, 2,..., n) \end{aligned} \tag{1.8}$$

are uniquely solvable. We are thus led to investigate the properties of a linear operator

$$\mathscr{D} : (H_1 \oplus H_2 \oplus \cdots \oplus H_n) \to (H_1 \oplus H_2 \oplus \cdots \oplus H_n) \tag{1.9}$$

which is given as a matrix of operators:

$$\mathscr{D} = \mathscr{U} + \mathscr{B}^*\mathscr{W}\mathscr{B} \tag{1.10}$$

where

$$\mathscr{U} = \text{diag.}[\mathscr{U}_1, \mathscr{U}_2, ..., \mathscr{U}_n] \qquad \mathscr{B} = \text{diag.}[\mathscr{B}_1, \mathscr{B}_2, ..., \mathscr{B}_n]$$

$$\mathscr{W} = \begin{pmatrix} \mathscr{W}_1 & \mathscr{W}_1 & \cdots & \mathscr{W}_1 \\ \mathscr{W}_2 & \mathscr{W}_2 & \cdots & \mathscr{W}_2 \\ \vdots & \vdots & & \vdots \\ \mathscr{W}_n & \mathscr{W}_n & \cdots & \mathscr{W}_n \end{pmatrix}.$$

We will refer to $\mathscr{D}$ as the *decision operator* for the game in question. When $\mathscr{D}$ is nonsingular the system (1.8) has a unique n-tuple $\hat{u}_1, \hat{u}_2, ..., \hat{u}_n$ of solution vectors and we will say that the game is *playable*. If $\mathscr{D}$ is singular we will say that the game is *not playable*.

Our first task then, will be to investigate the playability of the game by determining whether the decision operator $\mathscr{D}$ is nonsingular. This problem is treated in Section 2.

Another question which arises is the stability of the game. It is clear from (1.5) that unless the player P_i knows the strategies $u_1, u_2, ..., u_{i-1}, u_{i+1}, ..., u_n$ to be used by the other players, or at least appropriate related information, he can not determine his own optimal strategy $\hat{u}_i$. Typically, then, the strategy selection will proceed through a sequence of adjustments. The players choose initial strategies $\hat{u}_{10}, \hat{u}_{20}, ..., \hat{u}_{n0}$ which become known to all players. Based upon these strategies each player readjusts his strategy according to some systematic procedure, seeking to lower his cost, and thereby new strategies $\hat{u}_{11}, \hat{u}_{21}, ..., \hat{u}_{n1}$ are determined. We assume that P_i is ignorant of or else ignores the procedures used by the other players for readjusting their strategies and acts via (1.5) in a manner which would give the greatest reduction in his cost if the other players repeated their strategies used in the previous stage. That is, in general the strategies $\hat{u}_{1k}, \hat{u}_{2k}, ..., \hat{u}_{nk}$ are determined from $u_{1,k-1}, u_{2,k-1}, ..., u_{n,k-1}$ by

$$\begin{aligned} &\mathscr{B}_i^*\mathscr{W}_i\mathscr{B}_1\hat{u}_{1,k-1} + \cdots + \mathscr{B}_i^*\mathscr{W}_i\mathscr{B}_{i-1}\hat{u}_{i-1,k-1} \\ &\qquad + (\mathscr{U}_i + \mathscr{B}_i^*\mathscr{W}_i\mathscr{B}_i)\,\hat{u}_{i,k} + \mathscr{B}_i^*\mathscr{W}_i\mathscr{B}_{i+1}\hat{u}_{i+1,k-1} \\ &\qquad + \cdots + \mathscr{B}_i^*\mathscr{W}_i\mathscr{B}_n\hat{u}_{n,k-1} = \mathscr{B}_i^*\mathscr{W}_i(\bar{x}_i - x_0), \quad (i = 1, 2, ..., n) \end{aligned} \tag{1.11}$$

which, in matrix-vector form, becomes

$$[\mathscr{U} + \mathscr{B}^*\,\text{d.}(\mathscr{W})\,\mathscr{B}]\,\hat{u}_k + \mathscr{B}^*[\mathscr{W} - \text{d.}(\mathscr{W})]\,\mathscr{B}\hat{u}_{k-1} = \mathscr{B}^*\,\text{d.}(\mathscr{W})\,\bar{x} \tag{1.12}$$

with the notation

$$\hat{u}_k = \begin{pmatrix} \hat{u}_{1,k} \\ \hat{u}_{2,k} \\ \vdots \\ \hat{u}_{n,k} \end{pmatrix}, \quad \bar{x} = \begin{pmatrix} \bar{x}_1 - x_0 \\ \bar{x}_2 - x_0 \\ \vdots \\ \bar{x}_n - x_0 \end{pmatrix} \quad \text{and} \quad \text{d.}(\mathscr{W}) = \text{diag.}[\mathscr{W}_1, \mathscr{W}_2, ..., \mathscr{W}_n].$$

This is a recurrence equation whose solution will converge to the unique solution of (1.8) if the operator

$$\mathscr{S} = [\mathscr{U} + \mathscr{B}^* \,\text{d.}(\mathscr{W})\, \mathscr{B}]^{-1} \mathscr{B}^*[\mathscr{W} - \text{d.}(\mathscr{W})]\, \mathscr{B} \tag{1.13}$$

has the property

$$\| \mathscr{S} \| < 1 \tag{1.14}$$

or, when $H_1, H_2, ..., H_n$ are finite dimensional, if all eigenvalues λ of $\mathscr{S}$ satisfy

$$| \lambda | < 1. \tag{1.15}$$

Notice that each n-tuple $(\mathscr{T}_1, \mathscr{T}_2, ..., \mathscr{T}_n)$ of bounded, nonsingular, linear transformations

$$\mathscr{T}_i : H_i \rightarrow H_i \tag{1.16}$$

induces, through (1.1) and (1.3), a similarity transformation on $\mathscr{S}$. It follows that if (1.14) or (1.15) hold after some appropriate change of control variables (1.16) then every solution $\hat{u}_k$ of (1.12) will converge to the unique equilibrium strategy as $k \rightarrow \infty$ and it is for this reason we call such games *stable.*

This concept of stability is particularly useful since many important games (hopefully the arms race, e.g.) are never actually played out. Rather, the players continually determine new strategies based upon information concerning the current strategies of the other players.

We close this section by pointing out that playability as well as stability is invariant under a change of control variables of type (1.16) since it induces a congruence transformation on $\mathscr{D}$. Both $\mathscr{D}$ and $\mathscr{S}$ are independent of translations and nonsingular linear transformations of state. Consequently in the next section we need not worry about stating sufficiency conditions for playability or stability in coordinate systems which are special.

2. Playability

In this section we study the possibility of inverting the decision operator

$$\mathscr{D} = \mathscr{U} + \mathscr{B}^* \mathscr{W} \mathscr{B} \tag{2.1}$$

on $H_1 \oplus H_2 \oplus \cdots \oplus H_n$ defined by (1.10) where in

$$\mathscr{W} = \begin{pmatrix} \mathscr{W}_1 & \mathscr{W}_1 & \cdots & \mathscr{W}_1 \\ \mathscr{W}_2 & \mathscr{W}_2 & \cdots & \mathscr{W}_2 \\ \vdots & \vdots & & \vdots \\ \mathscr{W}_n & \mathscr{W}_n & \cdots & \mathscr{W}_n \end{pmatrix}$$

$\mathscr{W}_i \geqslant 0$ on H and where in $\mathscr{U} = \text{diag.}[\mathscr{U}_1, \mathscr{U}_2, ..., \mathscr{U}_n]$ we continue to assume $\mathscr{U}_i > 0$ on H_i $(i = 1, 2, ..., n)$.

It is tempting to speculate that under these conditions $\mathscr{D}$ is always nonsingular and hence that the game is playable. The following example shows this is false.

EXAMPLE 1. $\mathscr{D}$ may be singular even if $\mathscr{W}_i > 0$ and $\mathscr{U}_i > 0$ $(i = 1, 2, ..., n)$.

Proof. We take $n = 2$, $H = R^2$, $H_1 = H_2 = R^1$ and

$$\mathscr{W}_1 = \begin{pmatrix} 1 & 1/2 \\ 1/2 & 1 \end{pmatrix}, \qquad \mathscr{W}_2 = \begin{pmatrix} 1 & 2/5 \\ 2/5 & 1/5 \end{pmatrix} \tag{2.2}$$

$$\mathscr{U}_1 = \mathscr{U}_2 = c, \qquad c \text{ a positive scalar,} \tag{2.3}$$

$$\mathscr{B}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \qquad \mathscr{B}_2 = \begin{pmatrix} 0 \\ 8 \end{pmatrix}. \tag{2.4}$$

Then

$$\mathscr{D} = \begin{pmatrix} c & 0 \\ 0 & c \end{pmatrix} + \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 8 \end{pmatrix} \begin{bmatrix} \begin{pmatrix} 1 & 1/2 \\ 1/2 & 1 \end{pmatrix} & \begin{pmatrix} 1 & 1/2 \\ 1/2 & 1 \end{pmatrix} \\ \begin{pmatrix} 1 & 2/5 \\ 2/5 & 1/5 \end{pmatrix} & \begin{pmatrix} 1 & 2/5 \\ 2/5 & 1/5 \end{pmatrix} \end{bmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 8 \end{pmatrix}$$

$$= \begin{pmatrix} c + 3 & 12 \\ \frac{24}{5} & c + \frac{64}{5} \end{pmatrix}. \tag{2.5}$$

The determinant of this matrix is

$$c^2 + \frac{79}{5} c - \frac{96}{5} \tag{2.6}$$

which vanishes for $c = c_0$, where

$$c_0 = \frac{-\frac{79}{5} + \sqrt{\left(\frac{79}{5}\right)^2 + 4 \cdot \frac{96}{5}}}{2} > 0, \tag{2.7}$$

and thus, by example, we have shown that $\mathscr{D}$ may be singular.

We now present sufficient conditions for the nonsingularity of $\mathscr{D}$. You might say we are studying a control problem at another level of abstraction where in (2.1) we steer with $\mathscr{U}$, $\mathscr{B}$ and $\mathscr{W}$ and the target is the set of invertible operators. (Similarly for the problem of stability we control (1.13) and the target is the set of contraction operators.) Our first theorem shows $\mathscr{W}$ can exert a strong influence upon the nonsingularity of $\mathscr{D}$.

THEOREM 1. *If there is an operator $\mathscr{W}_0 \geqslant 0$ and positive constants μ_1, μ_2,..., μ_n such that*

$$\mathscr{W}_i = \mu_i \mathscr{W}_0 \qquad (i = 1, 2,..., n) \tag{2.8}$$

then $\mathscr{D}$ is nonsingular.

Proof. Since $\mathscr{W}_0 \geqslant 0$ it has a square root $\mathscr{W}_0^{1/2} > 0$ and using (2.8) we compute

$$\mathscr{D} = \text{diag.}[\mu_1 I_1, \mu_2 I_2, ..., \mu_n I_n] \left\{ \text{diag.} \left[\frac{\mathscr{U}_1}{\mu_1}, \frac{\mathscr{U}_2}{\mu_2}, ..., \frac{\mathscr{U}_n}{\mu_n} \right] + \mathscr{B}_0{}^* \mathscr{B}_0 \right\} \tag{2.9}$$

where

$$\mathscr{B}_0 = [\mathscr{W}_0^{1/2} \mathscr{B}_1, \mathscr{W}_0^{1/2} \mathscr{B}_2, ..., \mathscr{W}_0^{1/2} \mathscr{B}_n].$$

The first operator factor in (2.9) is nonsingular since $\mu_i > 0$ and the second is positive definite since $\mathscr{U}_i/\mu_i > 0$ and $\mathscr{B}_0{}^*\mathscr{B}_0 \geqslant 0$; hence $\mathscr{D}$ is nonsingular.

While Theorem 1 points out that the nonsingularity of $\mathscr{D}$ may be maintained in a nontrivial way through $\mathscr{W}$ alone the next result shows that together $\mathscr{B}$ and $\mathscr{U}$ can play a similar role in the special case where all players choose their control variables from the same space.

THEOREM 2. *In the special case where $H_1 = H_2 = \cdots = H_n$ let $\mathscr{B}_0 : H_1 \to H$ be a bounded linear transformation and μ_1, μ_2,..., μ_n be positive constants. If*

$$\mathscr{B}_i = \mu_i \mathscr{B}_0 \mathscr{U}_i^{1/2} \qquad (i = 1, 2,..., n) \tag{2.10}$$

then $\mathscr{D}$ is nonsingular.

Proof. Using (2.10) in (2.1) we compute

$$\mathscr{D} = \text{diag.} \left[\frac{\mathscr{U}_1^{1/2}}{\mu_1}, \frac{\mathscr{U}_2^{1/2}}{\mu_2}, ..., \frac{\mathscr{U}_n^{1/2}}{\mu_n} \right] \tilde{\mathscr{W}} \, \text{diag.}[\mu_1 \mathscr{U}_1^{1/2}, \mu_2 \mathscr{U}_2^{1/2}, ..., \mu_n \mathscr{U}_n^{1/2}] \tag{2.11}$$

where

$$\tilde{\mathscr{W}} = \begin{pmatrix} I + \tilde{\mathscr{W}}_1 & \tilde{\mathscr{W}}_1 & \cdots & \tilde{\mathscr{W}}_1 \\ \tilde{\mathscr{W}}_2 & I + \tilde{\mathscr{W}}_2 & \cdots & \tilde{\mathscr{W}}_2 \\ \vdots & & & \\ \tilde{\mathscr{W}}_n & \tilde{\mathscr{W}}_n & \cdots & I + \tilde{\mathscr{W}}_n \end{pmatrix},$$

$$\tilde{\mathscr{W}}_i = \mu_i^2 \mathscr{B}_0^* \mathscr{W}_i \mathscr{B}_0 \qquad (i = 1, 2,..., n).$$

The diagonal operators in (2.11) are nonsingular since $\mathscr{U}_i^{1/2} > 0$ and $\mu_i > 0$. Therefore the proof will be complete if $\tilde{\mathscr{W}}$ is nonsingular. The operator matrix $\tilde{\mathscr{W}}$ may be rewritten

$$\begin{pmatrix} I & 0 & \cdots & \tilde{\mathscr{W}}_1 \\ 0 & I & \cdots & \tilde{\mathscr{W}}_2 \\ \vdots & \vdots & & \vdots \\ -I & -I & \cdots & I + \tilde{\mathscr{W}}_n \end{pmatrix} \begin{pmatrix} I & 0 & \cdots & 0 \\ 0 & I & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ I & I & \cdots & I \end{pmatrix} \tag{2.12}$$

and we note it is sufficient to show the nonsingularity of the operator on the left in (2.12). Now to solve

$$\begin{pmatrix} I & 0 & \cdots & \tilde{\mathscr{W}}_1 \\ 0 & I & \cdots & \tilde{\mathscr{W}}_2 \\ \vdots & \vdots & & \vdots \\ -I & -I & \cdots & I + \tilde{\mathscr{W}}_n \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{pmatrix} = \begin{pmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{pmatrix} \tag{2.13}$$

we compute

$$u_i = -\tilde{\mathscr{W}}_i u_n + w_i \qquad (i = 1, 2,..., n-1) \tag{2.14}$$

$$(I + \tilde{\mathscr{W}}_n)\, u_n = u_1 + u_2 + \cdots + u_{n-1} + w_n\,. \tag{2.15}$$

Substituting (2.14) into (2.15) we have

$$(I + \tilde{\mathscr{W}}_1 + \tilde{\mathscr{W}}_2 + \cdots + \tilde{\mathscr{W}}_n)\, u_n = w_1 + w_2 + \cdots + w_n\,. \tag{2.16}$$

Since $\tilde{\mathscr{W}}_i \geqslant 0$ $(i = 1, 2,..., n)$ the operator $(I + \tilde{\mathscr{W}}_1 + \tilde{\mathscr{W}}_2 + \cdots + \tilde{\mathscr{W}}_n)^{-1}$ exists and is bounded on H_1 and we can solve (2.16) for u_n. Then (2.14) gives us u_1, u_2,..., u_{n-1} and thus the operator in (2.13), and hence $\mathscr{D}$, is nonsingular. This completes the proof of Theorem 2.

We remark that in case H_1, H_2,..., H_n all have the same dimension then since they will be isometric they can be taken to be the same. In fact all that is required is that the $\mathscr{B}_i$ have ranges of the same dimension in H. We simply replace H_i by the orthogonal complement of the null space of $\mathscr{B}_i$ and replace $\mathscr{B}_i$ by its restriction to that subspace.

THEOREM 3. *In the special case where* $H_1 = H_2 = \cdots = H_n$ *let*

$\mathscr{B}_0 : H_1 \to H$ be a bounded linear transformation and $\mu_1, \mu_2, ..., \mu_n$ be positive constants. If

$$\mathscr{W}_i \mathscr{B}_i = \mu_i \mathscr{B}_0 \mathscr{U}_i^{1/2} \qquad (i = 1, 2, ..., n) \tag{2.17}$$

then $\mathscr{D}$ is nonsingular.

Proof. The preliminary change of variables $u_i = \mathscr{U}_i^{-1/2} \tilde{u}_i$ in (1.1) and (1.3) produces new operators $\tilde{\mathscr{B}}_i = \mathscr{B}_i \mathscr{U}_i^{1/2}$, $\tilde{\mathscr{U}}_i = I$ and (2.17) becomes

$$\mathscr{W}_i \tilde{\mathscr{B}}_i = \mu_i \mathscr{B}_0 \qquad (i = 1, 2, ..., n). \tag{2.18}$$

Using (2.18) we compute

$$\begin{aligned} \mathscr{D} &= \tilde{\mathscr{U}} + \tilde{\mathscr{B}}^* \mathscr{W} \tilde{\mathscr{B}} \\ &= \text{diag.}[\mu_1 I, \mu_2 I, ..., \mu_n I]\, \tilde{\mathscr{W}} \, \text{diag.} \left[\frac{I}{\mu_1}, \frac{I}{\mu_2}, ..., \frac{I}{\mu_n}\right] \end{aligned} \tag{2.19}$$

where

$$\tilde{\mathscr{W}} = \begin{pmatrix} I + \tilde{\mathscr{W}}_1 & \tilde{\mathscr{W}}_2 & \cdots & \tilde{\mathscr{W}}_n \\ \tilde{\mathscr{W}}_1 & I + \tilde{\mathscr{W}}_2 & \cdots & \tilde{\mathscr{W}}_n \\ \vdots & \vdots & \cdots & \vdots \\ \tilde{\mathscr{W}}_1 & \tilde{\mathscr{W}}_2 & \cdots & I + \tilde{\mathscr{W}}_n \end{pmatrix} \tag{2.20}$$

and

$$\tilde{\mathscr{W}}_i = \frac{\mathscr{B}_0{}^* \mathscr{B}_i}{\mu_i} \qquad (i = 1, 2, ..., n). \tag{2.21}$$

From (2.18) and (2.19) we see that $\tilde{\mathscr{W}}_i \geqslant 0$ since $\mathscr{W}_i \geqslant 0$ and $\mu_i > 0$. It follows by the same argument used in the proof of Theorem 2 that $\tilde{\mathscr{W}}$ is nonsingular and hence from (2.19) that $\mathscr{D}$ is nonsingular, which completes the proof.

In order to compare the hypotheses of Theorem 2 and Theorem 3 notice that when $\mathscr{W}_i > 0$ (2.17) may be written as $\mathscr{B}_i = \mu_i \mathscr{W}_i^{-1} \mathscr{B}_0 \mathscr{U}_i^{1/2}$. The next theorem shows that the playability of the game can be assured by constraining the norms of $\mathscr{U}^{-1}$, $\mathscr{B}$ and $\mathscr{W}$ in accordance with the number of players.

THEOREM 4. *In the general n-player game the decision operator $\mathscr{D} = \mathscr{U} + \mathscr{B}^* \mathscr{W} \mathscr{B}$ is nonsingular if*

$$\| \mathscr{B} \|^2 \| \mathscr{W} \| \| \mathscr{U}^{-1} \| < 1, \tag{2.22}$$

or, in particular, if

$$\max_{1 \leqslant i,j,k \leqslant n} \| \mathscr{B}_i \|^2 \| \mathscr{W}_j \| \| \mathscr{U}_k^{-1} \| < \frac{1}{n^{3/2}}. \tag{2.23}$$

Proof. Since $\mathscr{U}_i > 0$ $(i = 1, 2, ..., n)$ we may rewrite $\mathscr{D}$ as

$$\mathscr{D} = \mathscr{U}^{1/2}[I + (\mathscr{B}\mathscr{U}^{-1/2})^* \mathscr{W} (\mathscr{B}\mathscr{U}^{-1/2})]\, \mathscr{U}^{1/2} \tag{2.24}$$

where here I denotes the identity operator on

$$H_1 \oplus H_2 \oplus \cdots \oplus H_n, \qquad \mathscr{U}^{1/2} = \text{diag.}[\mathscr{U}_1^{1/2}, \mathscr{U}_2^{1/2}, ..., \mathscr{U}_n^{1/2}]$$

and $\mathscr{U}_i^{1/2}$ is the positive definite square root of $\mathscr{U}_i$. Since $\mathscr{U}^{1/2}$ is nonsingular it is sufficient to prove that the middle operator factor in (2.24) is nonsingular. Using (2.22) we estimate

$$\begin{aligned}(\mathscr{B}\mathscr{U}^{-1/2})^* \mathscr{W}(\mathscr{B}\mathscr{U}^{-1/2})\| &\leqslant \|(\mathscr{B}\mathscr{U}^{-1/2})^*\| \, \|\mathscr{W}\| \, \|\mathscr{B}\mathscr{U}^{-1/2}\| \\ &= \|\mathscr{B}\mathscr{U}^{-1/2}\|^2 \, \|\mathscr{W}\| \leqslant \|\mathscr{B}\|^2 \, \|\mathscr{W}\| \, \|\mathscr{U}^{-1/2}\|^2 \\ &= \|\mathscr{B}\|^2 \, \|\mathscr{W}\| \, \|\mathscr{U}^{-1}\| < 1. \end{aligned} \tag{2.25}$$

It follows from (2.25) that $[I + (\mathscr{B}\mathscr{U}^{-1/2})^* \mathscr{W}(\mathscr{B}\mathscr{U}^{-1/2})]$ is nonsingular and hence that $\mathscr{D}$ is nonsingular. Now to complete the proof of the theorem we prove that (2.23) implies (2.22).

For

$$u = \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{pmatrix} \qquad \text{in} \qquad H_1 \oplus H_2 \oplus \cdots \oplus H_n,$$

$$\|\mathscr{W}u\|^2 = \sum_{i=1}^n \left| \sum_{j=1}^n \mathscr{W}_i u_j \right|^2 \leqslant \sum_{i=1}^n \left(\sum_{j=1}^n |\mathscr{W}_i u_j| \right)^2 \tag{2.26}$$

which gives

$$\|\mathscr{W}\| \leqslant n^{3/2} \max_{1 \leqslant i \leqslant n} \|\mathscr{W}_i\|. \tag{2.27}$$

Similar elementary estimation shows

$$\|\mathscr{B}\|^2 \leqslant \max_{1 \leqslant i \leqslant n} \|\mathscr{B}_i\|^2 \tag{2.28}$$

and

$$\|\mathscr{U}^{-1}\| \leqslant \max_{1 \leqslant i \leqslant n} \|\mathscr{U}_i^{-1}\|. \tag{2.29}$$

Then inequality (2.22) follows directly from (2.23), (2.27)-(2.29) and the proof of Theorem 4 is complete.

Finding a necessary and sufficient condition for the nonsingularity of $\mathscr{D}$ appears to be quite difficult. However the above theorems indicate that it will be possible for many types of games to prove that $\mathscr{D}$ is nonsingular and hence that these games are playable. We will make some further remarks about this problem in Sections 3 and 4.

In the next section on stability we will refer to games satisfying the hypotheses of Theorems 1, 2 and 3 as being *$\mathscr{W}$-simple*, *$\mathscr{B}$-simple* and *$\mathscr{W}\mathscr{B}$-simple*, respectively.

3. Stability

In this section we will derive sufficient conditions for the stability of our linear-quadratic game by studying the operator $\mathscr{S}$ given by (1.13). Since stability of the game implies its playability these conditions supplement the playability results of Section 2.

THEOREM 5. *In the general n-player game if*

$$\| \mathscr{B} \|^2 \| \mathscr{W} - \mathrm{d.}(\mathscr{W}) \| \| \mathscr{U}^{-1} \| < 1, \tag{3.1}$$

or, in particular,

$$\max_{1 \leqslant i,j,k \leqslant n} \| \mathscr{B}_i \|^2 \| \mathscr{W}_j \| \| \mathscr{U}_k^{-1} \| \leqslant \frac{1}{(n-1)\, n^{1/2}} \tag{3.2}$$

then $\|\mathscr{S}\| < 1$ *and the game is stable.*

Proof. Starting with (1.13) and rewriting $\mathscr{S}$ we have

$$\begin{aligned} \mathscr{S} &= [\mathscr{U} + \mathscr{B}^* \mathrm{d.}(\mathscr{W}) \mathscr{B}]^{-1} \mathscr{B}^*[\mathscr{W} - \mathrm{d.}(\mathscr{W})] \mathscr{B} \\ &= \mathscr{U}^{-1/2}[I + (\mathscr{B}\mathscr{U}^{-1/2})^* \mathrm{d.}(\mathscr{W}) (\mathscr{B}\mathscr{U}^{-1/2})]^{-1} \mathscr{U}^{-1/2}\mathscr{B}^*[\mathscr{W} - \mathrm{d.}(\mathscr{W})] \mathscr{B} \end{aligned} \tag{3.3}$$

from which we obtain the estimate

$$\| \mathscr{S} \| \leqslant \| \mathscr{B} \|^2 \| \mathscr{U}^{-1} \| \|[I + (\mathscr{B}\mathscr{U}^{-1/2})^* \mathrm{d.}(\mathscr{W}) (\mathscr{B}\mathscr{U}^{-1/2})]^{-1} \| \| \mathscr{W} - \mathrm{d.}(\mathscr{W})\| \tag{3.4}$$

Using the fact that $(\mathscr{B}\mathscr{U}^{-1/2})^* \mathrm{d.}(\mathscr{W}) (\mathscr{B}\mathscr{U}^{-1/2}) \geqslant 0$ it is a simple matter to show that

$$\|[I + (\mathscr{B}\mathscr{U}^{-1/2})^* \mathrm{d.}(\mathscr{W}) (\mathscr{B}\mathscr{U}^{-1/2})]^{-1} \| \leqslant 1 \tag{3.5}$$

which, when applied to (3.4) gives

$$\| \mathscr{S} \| \leqslant \| \mathscr{B} \|^2 \| \mathscr{W} - \mathrm{d.}(\mathscr{W})\| \| \mathscr{U}^{-1} \| . \tag{3.6}$$

The proof of the first part of the theorem then follows directly from (3.6).

For

$$u = \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{pmatrix} \quad \text{in} \quad H_1 \oplus H_2 \oplus \cdots \oplus H_n ,$$

$$| [\mathscr{W} - \mathrm{d.}(\mathscr{W})] \mathscr{U} | = \sum_{i=1}^{n} \Big| \sum_{\substack{j=1 \\ j \neq i}}^{n} \mathscr{W}_i u_j \Big|^2 \leqslant \sum_{i=1}^{n} \Big(\sum_{\substack{j=1 \\ j \neq i}}^{n} | \mathscr{W}_i u_j | \Big)^2 \tag{3.7}$$

from which follows

$$\| \mathscr{W} - \mathrm{d}.(\mathscr{W}) \| \leqslant (n-1)\, n^{1/2} \max_{1 \leqslant j \leqslant n} \| \mathscr{W}_j \| . \tag{3.8}$$

We also have the elementary inequalities

$$\| \mathscr{B} \| \leqslant \max_{1 \leqslant i \leqslant n} \| \mathscr{B}_i \| \tag{3.9}$$

and

$$\| \mathscr{U}^{-1} \| \leqslant \max_{1 \leqslant k \leqslant n} \| \mathscr{U}_k^{-1} \| . \tag{3.10}$$

Substituting (3.8)-(3.10) into (3.6) and then using (3.2) produces the required inequality, $\| \mathscr{S} \| < 1$, and the proof of Theorem 5 is complete.

THEOREM 6. *The $\mathscr{B}$-simple game is stable for $n = 2$.*

Proof. By making a preliminary transformation

$$\tilde{u}_i = \mu_i \mathscr{U}_i^{1/2} u_i \qquad (i = 1, 2) \tag{3.11}$$

we can assume that

$$\mathscr{B}_i = \mathscr{B}_0 \qquad \text{and} \qquad \mathscr{U}_i = \frac{I}{\mu_i^2} \qquad (i = 1, 2). \tag{3.12}$$

From (1.13) we then compute

$$\mathscr{S} = \begin{pmatrix} 0 & \left[\dfrac{I}{\mu_1^2} + \tilde{\mathscr{W}}_1\right]^{-1} \tilde{\mathscr{W}}_1 \\ \left[\dfrac{I}{\mu_2^2} + \tilde{\mathscr{W}}_2\right]^{-1} \tilde{\mathscr{W}}_2 & 0 \end{pmatrix} \tag{3.13}$$

where

$$\tilde{\mathscr{W}}_i = \mathscr{B}_0{}^* \mathscr{W}_i \mathscr{B}_0 \qquad (i = 1, 2).$$

Using (3.13) one may easily verify that

$$\begin{aligned} \| \mathscr{S} \|^2 &= \| \mathscr{S}^* \mathscr{S} \| \\ &= \max \left\{ \left\| \left[\frac{I}{\mu_2^2} + \tilde{\mathscr{W}}_1\right]^{-1} \tilde{\mathscr{W}}_1 \right\|^2, \left\| \left[\frac{I}{\mu_2^2} + \tilde{\mathscr{W}}_2\right]^{-1} \tilde{\mathscr{W}}_2 \right\|^2 \right\} . \end{aligned} \tag{3.14}$$

But $\tilde{\mathscr{W}}_i \geqslant 0$ $(i = 1, 2)$ and from the operational calculus for self-adjoint operators

$$\left[\frac{I}{\mu_i^2} + \tilde{\mathscr{W}}_i\right]^{-1} \tilde{\mathscr{W}}_i = \int_{\sigma(\tilde{\mathscr{W}}_i)} \frac{\lambda}{\dfrac{1}{\mu_i^2} + \lambda}\, dE_i(\lambda) \tag{3.15}$$

and hence

$$\left\| \left[\frac{I}{\mu_i^2} + \tilde{\mathscr{W}}_i\right]^{-1} \tilde{\mathscr{W}}_i \right\| = \max_{\lambda \in \sigma(\tilde{\mathscr{W}}_i)} \frac{\lambda}{\dfrac{1}{\mu_i^2} + \lambda} = \frac{\| \tilde{\mathscr{W}}_i \|}{\dfrac{1}{\mu_i^2} + \| \tilde{\mathscr{W}}_i \|} < 1 \qquad (i = 1, 2) \tag{3.16}$$

since $\quad \| \tilde{\mathscr{W}}_i \| = \text{l.u.b.}\{\lambda \mid \lambda \in \sigma(\tilde{\mathscr{W}}_i)\} \quad$ and $\quad \dfrac{\lambda}{\dfrac{1}{\mu_i^2} + \lambda}$

is monotone increasing for $\lambda \geqslant 0$. The required inequality $\| \mathscr{S} \| < 1$ then follows directly from (3.14) and (3.16) and the proof is complete.

The following example shows that the $\mathscr{B}$-simple, $\mathscr{W}$-simple and the $\mathscr{W}\mathscr{B}$-simple game may be unstable if $n > 2$.

EXAMPLE 2. We take $n = 3$,

$$H = H_1 = H_2 = H_3 = R^1, \qquad \mathscr{B}_1 = \mathscr{B}_2 = \mathscr{B}_3 = \mathscr{W}_1 = \mathscr{W}_2 = \mathscr{W}_3 = 1,$$
$$\mathscr{U}_1 = \mathscr{U}_2 = \mathscr{U}_3 = \epsilon > 0.$$

The decision matrix becomes

$$D = \begin{pmatrix} 1 + \epsilon & 1 & 1 \\ 1 & 1 + \epsilon & 1 \\ 1 & 1 & 1 + \epsilon \end{pmatrix}$$

and [cf. (1.13)] we have

$$S = \begin{bmatrix} 0 & \dfrac{1}{1 + \epsilon} & \dfrac{1}{1 + \epsilon} \\ \dfrac{1}{1 + \epsilon} & 0 & \dfrac{1}{1 + \epsilon} \\ \dfrac{1}{1 + \epsilon} & \dfrac{1}{1 + \epsilon} & 0 \end{bmatrix}. \tag{3.17}$$

Then we compute

$$\det.(S - \lambda I) = -\left(\lambda + \frac{1}{1 + \epsilon}\right)^2 \left(\lambda - \frac{2}{1 + \epsilon}\right) \tag{3.18}$$

and we see that the game is unstable if $0 < \epsilon < 1$.

THEOREM 7. *The $\mathscr{W}$-simple game is stable for $n = 2$.*

Proof. Since $\mathscr{U} + \mathscr{B}^* \text{d.}(\mathscr{W})\, \mathscr{B} > 0$ is a diagonal operator and $\mu_i > 0$ ($i = 1, 2$) we can make the change of variables

$$\tilde{u} = \mathscr{T} u \tag{3.19}$$

where

$$\mathscr{T} = \left[(\mathscr{U} + \mathscr{B}^* \text{d.}(\mathscr{W})\, \mathscr{B}) \text{ diag.} \left(\frac{I}{\mu_1}, \frac{I}{\mu_2}\right)\right]^{1/2} > 0 \tag{3.20}$$

and this produces new operators

$$\tilde{\mathscr{B}} = \mathscr{B}\mathscr{T}^{-1} \qquad \text{and} \qquad \tilde{\mathscr{U}} = \mathscr{T}^{-1}\mathscr{U}\mathscr{T}^{-1}. \tag{3.21}$$

Using (1.13), (2.8) and (3.21) we can easily compute

$$\begin{aligned} \mathscr{S} &= [\tilde{\mathscr{U}} + \tilde{\mathscr{B}}^* \text{d.}(\mathscr{W})\, \tilde{\mathscr{B}}]^{-1}\, \tilde{\mathscr{B}}^*[\mathscr{W} - \text{d.}(\mathscr{W})]\, \tilde{\mathscr{B}} \\ &= \mathscr{T}^{-1}\mathscr{B}^*\tilde{\mathscr{W}}\mathscr{B}\mathscr{T}^{-1} \end{aligned} \tag{3.22}$$

where

$$\tilde{\mathscr{W}} = \text{diag.} \left(\frac{I}{\mu_1}, \frac{I}{\mu_2}\right) [\mathscr{W} - \text{d.}(\mathscr{W})] = \begin{pmatrix} 0 & \mathscr{W}_0 \\ \mathscr{W}_0 & 0 \end{pmatrix}$$

and we note that $\tilde{\mathscr{W}}$ and $\mathscr{S}$ are self-adjoint.

We now study the resolvent operator for $\mathscr{S}$ which by (3.22) is

$$(\lambda I - \mathscr{S})^{-1} = \mathscr{T}(\lambda\mathscr{T}^2 - \mathscr{B}^*\tilde{\mathscr{W}}\mathscr{B})^{-1}\, \mathscr{T} \tag{3.23}$$

from which we obtain the estimate

$$\|(\lambda I - \mathscr{S})^{-1}\| \leqslant \|\mathscr{T}\|^2\, \|(\lambda\mathscr{T}^2 - \mathscr{B}^*\tilde{\mathscr{W}}\mathscr{B})^{-1}\|. \tag{3.24}$$

Let $u = \binom{u_1}{u_2}$ be in $H_1 \oplus H_2$ and then using (3.20) we get

$$\begin{aligned} (u, [\lambda\mathscr{T}^2 - \mathscr{B}^*\tilde{\mathscr{W}}\mathscr{B}]\, u) &= \frac{\lambda}{\mu_1}(u_1, \mathscr{U}_1 u_1) + \frac{\lambda}{\mu_2}(u_2, \mathscr{U}_2 u_2) \\ &\quad + (\lambda - 1)\, [(u_1, \mathscr{B}_1{}^*\mathscr{W}_0\mathscr{B}_1 u_1) \\ &\quad + (\mathscr{U}_2, \mathscr{B}_2{}^*\mathscr{W}_0\mathscr{B}_2 u_2)] + |\mathscr{B}_1 u_1 - \mathscr{B}_2 u_2|^2_{\mathscr{W}_0} \end{aligned} \tag{3.25}$$

and,

$$\begin{aligned} (u, -[\lambda\mathscr{T}^2 - \mathscr{B}^*\tilde{\mathscr{W}}\mathscr{B}]\, u) &= \frac{(-\lambda)}{\mu_1}(u_1, \mathscr{U}_1 u_1) + \frac{(-\lambda)}{\mu_2}(u_2, \mathscr{U}_2 u_2) \\ &\quad - (\lambda + 1)\, [(u_1, \mathscr{B}_1{}^*\mathscr{W}_0\mathscr{B}_1 u_1) \\ &\quad + (u_2, \mathscr{B}_2{}^*\mathscr{W}_0\mathscr{B}_2 u_2)] + |\mathscr{B}_1 u_1 + \mathscr{B}_2 u_2|^2_{\mathscr{W}_0}. \end{aligned} \tag{3.26}$$

It follows from (3.25) that

$$(u, [\lambda\mathcal{T}^2 - \mathcal{B}^*\tilde{\mathcal{W}}\mathcal{B}]\, u) \geqslant \rho_1 \mid u \mid^2, \qquad \lambda \geqslant 1 \tag{3.27}$$

and from (3.26) that

$$(u, [\lambda\mathcal{T}^2 - \mathcal{B}^*\tilde{\mathcal{W}}\mathcal{B}]\, u) \leqslant - \rho_1 \mid u \mid^2, \qquad \lambda \leqslant - 1 \tag{3.28}$$

where $\rho_1 > 0$ is the smallest of the numbers

$$\inf \left[\frac{(u_i\, , \mathcal{U}_i u_i)}{\mu_i} \mid u_i \in H_i\, , \mid u_i \mid = 1\right] \qquad (i = 1, 2).$$

But u is arbitrary in (3.27), (3.28) and $\lambda\mathcal{T} - \mathcal{B}^*\tilde{\mathcal{W}}\mathcal{B}$ is self-adjoint, hence

$$\| [\lambda\mathcal{T}^2 - \mathcal{B}^*\tilde{\mathcal{W}}\mathcal{B}]^{-1} \| \leqslant \frac{1}{\rho_1} \tag{3.29}$$

which, when applied to (3.24), gives

$$\| (\lambda I - \mathcal{S})^{-1} \| \leqslant \frac{\| \mathcal{T} \|^2}{\rho_1} \equiv \frac{1}{\rho}, \qquad \mid \lambda \mid \geqslant 1. \tag{3.30}$$

Since $\mathcal{S}$ is self-adjoint (3.30) implies that $\sigma(\mathcal{S}) \subseteq [- 1 + \rho, 1 - \rho]$, hence $\| \mathcal{S} \| < 1$ and the stability of the game is established.

THEOREM 8. *The $\mathcal{W}\mathcal{B}$-simple game is stable for $n = 2$.*

Proof. By making the preliminary change of variables $\tilde{u}_i = \mu_i \mathcal{U}_i^{1/2} u_i$ $(i = 1, 2)$ we can take

$$\mathcal{U}_i = \frac{I}{\mu_i^2} \qquad (i = 1, 2) \tag{3.31}$$

and (2.17) becomes

$$\mathcal{W}_i \mathcal{B}_i = \mathcal{B}_0 \qquad (i = 1, 2). \tag{3.32}$$

From (1.13), (3.31) and (3.32) we compute

$$\begin{aligned} \mathcal{S} &= \begin{pmatrix} 0 & \left(\frac{I}{\mu_1^2} + \mathcal{B}_1^* \mathcal{W}_1 \mathcal{B}_1\right)^{-1} \mathcal{B}_1^* \mathcal{W}_1 \mathcal{B}_2 \\ \left(\frac{I}{\mu_2^2} + \mathcal{B}_2^* \mathcal{W}_2 \mathcal{B}_2\right)^{-1} \mathcal{B}_2^* \mathcal{W}_2 \mathcal{B}_1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & \left(\frac{I}{\mu_1^2} + \tilde{\mathcal{W}}_1\right)^{-1} \tilde{\mathcal{W}}_2 \\ \left(\frac{I}{\mu_2^2} + \tilde{\mathcal{W}}_2\right)^{-1} \tilde{\mathcal{W}}_1 & 0 \end{pmatrix} \end{aligned} \tag{3.33}$$

where

$$\tilde{\mathscr{W}}_i = \mathscr{B}_i{}^*\mathscr{W}_i\mathscr{B}_i \geqslant 0 \qquad (i = 1, 2).$$

Using (3.33) we further compute

$$\mathscr{S}^{2k} = \begin{pmatrix} \left[\left(\frac{I}{\mu_1{}^2} + \tilde{\mathscr{W}}_1\right)^{-1} \tilde{\mathscr{W}}_2 \left(\frac{I}{\mu_2{}^2} = \tilde{\mathscr{W}}_2\right)^{-1} \tilde{\mathscr{W}}_1\right]^k & 0 \\ 0 & \left[\left(\frac{I}{\mu_2{}^2} + \tilde{\mathscr{W}}_2\right)^{-1} \tilde{\mathscr{W}}_1 \left(\frac{I}{\mu_2{}^2} + \tilde{\mathscr{W}}_1\right)^{-1} \tilde{\mathscr{W}}_2\right]^k \end{pmatrix} \tag{3.34}$$

from which we obtain the estimate

$$\| \mathscr{S}^{2k} \| \leqslant \left[\left\| \left(\frac{I}{\mu_2{}^2} + \tilde{\mathscr{W}}_1\right)^{-1} \right\| \| \tilde{\mathscr{W}}_1 \| \left\| \left(\frac{I}{\mu_2{}^2} + \tilde{\mathscr{W}}_2\right)^{-1} \right\| \| \tilde{\mathscr{W}}_2 \|\right]^k \qquad (k = 0, 1, 2, \ldots). \tag{3.35}$$

Since $\tilde{\mathscr{W}}_i \geqslant 0$ we can apply the operational calculus to get

$$\begin{aligned} \left\| \left(\frac{I}{\mu_i{}^2} + \tilde{\mathscr{W}}_i\right)^{-1} \right\| \| \tilde{\mathscr{W}}_i \| &= \left[\sup_{\sigma(\tilde{\mathscr{W}}_i)} \left(\frac{1}{\mu_i{}^2 + \lambda}\right)\right] [\sup_{\sigma(\tilde{\mathscr{W}}_i)} \lambda] \\ &= \frac{\| \tilde{\mathscr{W}}_i \|}{\frac{1}{\mu_i{}^2} + \| \tilde{\mathscr{W}}_i \|} \equiv \rho_i\,, \qquad (i = 1, 2) \end{aligned} \tag{3.36}$$

which, when applied to (3.35), gives the result

$$\| \mathscr{S}^{2k} \| \leqslant \rho^{2k} \qquad (k = 0, 1, 2, \ldots) \tag{3.37}$$

where $\rho = \max[\rho_1, \rho_2] < 1$. If $\| \mathscr{S} \| < 1$ the stability of the game is clear and we therefore suppose $\| \mathscr{S} \| \geqslant 1$. It then follows from (3.37) that

$$\| \mathscr{S}^k \| \leqslant \rho^{k-1} \| \mathscr{S} \|, \qquad (k = 1, 2, \ldots). \tag{3.38}$$

We are required to prove the convergence of the sequence $\{\hat{u}_k\}$ defined by (1.12). But $\hat{u}_{k+1} - \hat{u}_k = \mathscr{S}^k(\hat{u}_1 - \hat{u}_0)$ and with the aid of (3.38) we have

$$\begin{aligned} | \hat{u}_{k+p} - \hat{u}_k | &= |(\mathscr{S}^{k+p-1} + \cdots + \mathscr{S}^k)(\hat{u}_1 - \hat{u}_0)| \\ &\leqslant \| \mathscr{S}^k \| [\| \mathscr{S}^{p-1} \| + \cdots + \| \mathscr{S} \| + 1] \, | \hat{u}_1 - \hat{u}_0 | \\ &\leqslant \rho^{k-1} \| \mathscr{S} \| [(\rho^{p-2} + \cdots + 1) \| \mathscr{S} \| + 1] \, | \hat{u}_1 - \hat{u}_0 | \\ &\leqslant \rho^{k-1} \| \mathscr{S} \|^2 \left(\frac{2-\rho}{1-\rho}\right) | \hat{u}_1 - \hat{u}_0 | \end{aligned} \tag{3.39}$$

($k = 1, 2, \ldots; p = 2, 3, \ldots$) which shows $\{\hat{u}_k\}$ is a Cauchy sequence. Therefore the convergence of $\{\hat{u}_k\}$ and hence the stability of the game is established.

4. Linear-Quadratic Differential Games

We are now interested in applying the results of the previous sections to the study of a more concrete problem from the field of differential games. This problem in the one player case reduces to the optimal regulator problem which has been studied extensively in control theory and has had useful applications in control engineering [6, 7].

The data of the problem is given in the form of real matrix-valued functions $A(t)$, $B_i(t)$, $W_i(t) \geqslant 0$, $\hat{W}_i \geqslant 0$, $U_i(t) > 0$, and real, finite dimensional vector valued functions $\bar{y}_i(t)$, $\hat{y}_i$ ($i = 1, 2, \ldots, n$), all defined and continuous on a finite interval $t_0 \leqslant t \leqslant t_1$.

The dynamics of the differential game are given by the linear system equation in $R^m \times [t_0, t_1]$

$$\frac{dy}{dt} = A(t)\, y + B_1(t)\, u_1 + B_2(t)\, u_2 + \cdots + B_n(t)\, u_n \tag{4.1}$$

where u_i is the m_i-dimensional control variable under the command of the ith player, P_i. P_i is allowed to play any real, Borel measurable control command $u_i = u_i(t)$ satisfying

$$\int_{t_0}^{t_1} |\, u_i(t)|^2 \, dt < \infty. \tag{4.2}$$

With the initial state of the game $y(t_0) = y_0$ fixed, any such n-tuple of control commands $u(t) = (u_1(t), u_2(t), \ldots, u_n(t))$ is called an *open-loop strategy.* An open-loop strategy together with the solution of (4.1) that it generates will be called a *play* of the game.

P_i determines his cost incurred in a play of the game by a functional of the form

$$\mathscr{C}_i = \int_{t_0}^{t_1} [|\, y(t) - \bar{y}_i(t)|_{W_i(t)} + |\, u_i(t)|^2_{U_i(t)}]\, dt + |\, y(t_1) - \hat{y}_i\, |^2_{\hat{W}_i} \tag{4.3}$$

which he wishes to minimize.

In order to place this problem into the framework of Section 1 we consider the spaces

$$L_j^2 = \left\{ f(t) : [t_0, t_1] \to R^j \mid \int_{t_0}^{t_1} |f(t)|^2 \, dt < \infty \right\} \tag{4.4}$$

and rewrite (4.1) using the variation of parameters formula

$$y(t) = \phi(t, t_0)\, y_0 + \int_{t_0}^{t} \phi(t, \tau)\, [B_1(\tau)\, u_1(\tau) + B_2(\tau)\, u_2(\tau) + \cdots + B_n(\tau)\, u_n(\tau)]\, d\tau \tag{4.5}$$

where $\phi(\tau, \tau) = I_m$ and $\phi_t(t, \tau) = A(t)\, \phi(t, \tau)$. We select

$$H = L_m{}^2 \oplus R^m, \qquad H_i = L^2_{m_i}, \qquad (i = 1, 2, \ldots, n) \tag{4.6}$$

and define vectors x, x_0, $\bar{x}_i$, u_i by

$$\begin{aligned} x &= \begin{pmatrix} y(t) \\ y(t_1) \end{pmatrix} \in H \\ x_0 &= \begin{pmatrix} \phi(t, t_0)\, y_0 \\ \phi(t_1, t_0)\, y_0 \end{pmatrix} \in H \\ \bar{x}_i &= \begin{pmatrix} y_i(t) \\ \hat{y}_i \end{pmatrix} \in H \\ u_i &= u_i(t) \in H_i \qquad (i = 1, 2, \ldots, n) \end{aligned} \tag{4.7}$$

and operators $\mathscr{B}_i : H_i \to H$ by

$$\begin{aligned} (\mathscr{B}_i u_i)(t) &= \begin{pmatrix} \int_{t_0}^{t} \phi(t, \tau)\, B_i(\tau)\, u_i(\tau)\, d\tau \\ \\ \int_{t_0}^{t_1} \phi(t_1, \tau)\, B_i(\tau)\, u_i(\tau)\, d\tau \end{pmatrix} \\ (\mathscr{W}_i x)(t) &= \begin{pmatrix} W_i(t)\, y(t) \\ \hat{W}_i\, y(t_1) \end{pmatrix} \\ (\mathscr{U}_i u_i)(t) &= U_i(t)\, u_i(t) \qquad (i = 1, 2, \ldots, n). \end{aligned} \tag{4.8}$$

Then (4.3) may be written as

$$\mathscr{C}_i = |\, x - \bar{y}_i\, |^2_{\mathscr{W}_i} + |\, u_i\, |^2_{\mathscr{U}_i}\,. \tag{4.9}$$

Therefore with these definitions the differential game given by (4.1), (4.3) is equivalent to (1.1), (1.3) and all of the theorems of sections 2 and 3 apply.

Note that the differential game is $\mathscr{B}$-simple if

$$B_i(t) = \mu_i B_0(t)\, U_i^{1/2}(t), \qquad (i = 1, 2, \ldots, n), \tag{4.10}$$

$\mathscr{W}$-simple if

$$W_i(t) = \mu_i W_0(t)$$
$$\hat{W}_i = \mu_i \hat{W}_0, \qquad (i = 1, 2, \ldots, n), \tag{4.11}$$

and $\mathscr{W}\mathscr{B}$-simple if $A(t)$ commutes with $W_i(\tau)$, $\hat{W}_i$ and

$$W_i(t) B_i(\tau) = \mu_i B_0(t, \tau) U_i^{1/2}(\tau)$$
$$\hat{W}_i B_i(\tau) = \mu_i B_0(t_1, \tau) U_i^{1/2}(\tau), \tag{4.12}$$

$t_0 \leqslant t, \tau \leqslant t_1$, $(i = 1, 2, \ldots, n)$. Thus we know that the game (4.1), (4.3) is playable if either (4.10), (4.11) or (4.12) and the commutativity holds. If, in addition, $n = 2$ then the game is stable. These are, of course, sufficient conditions only—not necessary conditions.

We now let $[t_0, t_1]$ denote a compact subinterval of some underlying interval on which $A(t)$, $B_i(t)$, $W_i(t) \geqslant 0$, $U_i(t) > 0$ and $\bar{y}_i(t)$ are defined and continuous. With these functions fixed, our next theorem shows that the differential game (4.1), (4.3) can always be made playable and stable by taking $t_1 - t_0$ sufficiently small.

From (4.8) it is clear that

$$\| \mathscr{W}_i \|^2 \leqslant \sup_{[t_0, t_1]} \| W_i(t) \|^2 + \| \hat{W}_i \|^2 < \infty \tag{4.13}$$

and

$$\| \mathscr{U}_i^{-1} \| \leqslant \sup_{[t_0, t_1]} \| U_i^{-1}(t) \| < \infty. \tag{4.14}$$

As a consequence of (4.13), (4.14) and Theorem 5 we see that to prove "local playability" and "local stability" we need only prove

THEOREM 9. *For t_0, t_1 bounded*

$$\lim_{t_1 - t_0 \to 0} \| \mathscr{B}_i \| = 0, \qquad (i = 1, 2, \ldots, n) \tag{4.15}$$

where $\mathscr{B}_i$ is defined in (4.8).

Proof. Let $[t_\alpha, t_\beta]$ denote the underlying interval containing the intervals $[t_0, t_1]$. From the continuity of the data it follows that there is a number γ which gives a bound on the matrix norm

$$\| \phi(t, \tau) B_i(\tau) \| \leqslant \frac{\gamma}{[(t_\beta - t_\alpha) + 1)^{1/2}} \tag{4.16}$$

for $t_\alpha \leqslant t,\ \tau \leqslant t_\beta$ $(i = 1, 2,..., n)$. Being careful not to confuse the several vector norms involved we compute

$$
\begin{aligned}
| \mathscr{B}_i u_i |^2 &= \int_{t_0}^{t_1} \left| \int_{t_0}^{t} \phi(t, \tau)\, B_i(\tau)\, u_i(\tau)\, d\tau \right|^2 dt + \left| \int_{t_0}^{t_1} \phi(t_1, \tau)\, B_i(\tau)\, u_i(\tau)\, d\tau \right|^2 \\
&\leqslant \int_{t_0}^{t_1} \left[\int_{t_0}^{t} \| \phi(t, \tau)\, B_i(\tau) \|\, | u_i(\tau) |\, d\tau \right]^2 dt \\
&\quad + \left[\int_{t_0}^{t_1} \| \phi(t_1, \tau)\, B_i(\tau) \|\, | u_i(\tau) |\, d\tau \right]^2 \qquad (4.17) \\
&\leqslant \frac{\gamma^2[(t_1 - t_0) + 1]}{[t_\beta - t_\alpha + 1]} \left(\int_{t_0}^{t_1} | u_i(\tau) |\, d\tau \right)^2 \\
&\leqslant \gamma^2 (t_1 - t_0) \int_{t_0}^{t_1} | u_i(\tau) |^2\, d\tau \\
&= \gamma^2 (t_1 - t_0)\, | u_i |^2,
\end{aligned}
$$

whence

$$\| \mathscr{B}_i \| \leqslant \gamma (t_1 - t_0)^{1/2} \tag{4.18}$$

and from this (4.15) follows.

Thus linear-quadratic differential games are always locally playable and locally stable.

We now turn our attention toward the computational aspect of the problem. Our local playability result indicates that we should look for a method of computing the equilibrium strategy for t_0 near t_1. Let t_1 be fixed and let t_0 vary. Define a mapping from $[0, 1]$ into $[t_0, t_1]$ by

$$\tau \to t_0 + \tau (t_1 - t_0). \tag{4.19}$$

Using this mapping the functions f in the space L_j^2, originally defined on $[t_0, t_1]$, can be considered as defined on $[0, 1]$. In this way the spaces $H, H_1, H_2, ..., H_n$ can be taken as fixed spaces, not varying with t_0. It is easy to check, using (4.8), that the decision operator $\mathscr{D} = \mathscr{D}(t_0)$ is then continuous in norm with respect to t_0. $\mathscr{D}^{-1}(t_0)$ will be continuous in norm for t_0 near $\hat{t}_0$, provided $\mathscr{D}^{-1}(\hat{t}_0)$ exists. Applying this fact to (1.4), (1.8) rewritten in the notation of (1.12)

$$\| \mathscr{B}_i \| \leqslant \gamma (t_1 - t_0)^{1/2}$$

$$\hat{u} = \mathscr{D}^{-1}(t_0)\, \mathscr{B}^*(t_0)\, \mathrm{d.}(\mathscr{W}(t_0))\, \bar{x}, \tag{4.20}$$

$$\mathscr{C}_i(\hat{u}) = | \mathscr{B}(t_0)\, \hat{u} - (\bar{x}_i - x_0) |^2_{\mathscr{W}_i(\beta_0)} + | \hat{u}_i |^2_{\mathscr{U}_i(\beta_0)}, \tag{4.21}$$

where

$$\bar{x} = \begin{pmatrix} \bar{x}_1 - x_0 \\ \bar{x}_2 - x_0 \\ \vdots \\ \bar{x}_n - x_0 \end{pmatrix}, \qquad x_0 = \begin{pmatrix} \phi(t, t_0)\, y_0 \\ \phi(t_1, t_0)\, y_0 \end{pmatrix},$$

we have the following theorem.

THEOREM 10. *If $\mathscr{D}(t_0)$ is nonsingular for each $t_0 \in [t_\alpha, t_1]$, t_α fixed, $t_\alpha < t_1$, then there is a constant K_α^1 such that*

$$\| \mathscr{D}^{-1}(t_0) \| \leqslant K_\alpha^1, \qquad t_0 \in [t_\alpha, t_1] \tag{4.22}$$

and hence further constants K_α^2, K_α^3 such that

$$| \hat{u} | \leqslant K_\alpha^2 | \bar{x} | \tag{4.23}$$

and

$$\mathscr{C}_i(\hat{u}) \leqslant K_\alpha^3 | \bar{x} |^2 \quad (i = 1, 2,..., n) \qquad \textit{for all} \qquad y_0 \in R^m, \quad t_0 \in [t_\alpha, t_1]. \tag{4.24}$$

Proof. The proof is a direct consequence of the remarks preceding Theorem 10 and continuity considerations.

Differential games satisfying the hypothesis of Theorem 10 in which it is required that $\mathscr{D}(t_0)$ be nonsingular for all t_0 in an interval $(t_\alpha, t_1]$ and not just at one point will be called *playable on the interval* $(t_\alpha, t_1]$. We note that if the data matrices of the game and $\bar{y}_i(t)$ are defined on $(-\infty, t_1]$ and the differential game is $\mathscr{B}$-simple, $\mathscr{W}$-simple or $\mathscr{W}\mathscr{B}$-simple there, then it is playable on each interval $(t_\alpha, t_1]$ and hence (4.24) will hold for all $t_\alpha < t_1$ although of course we may have some or all of the K_α^i tending to ∞ as t_α tends to $-\infty$.

Thus far our only means for computing the equilibrium strategy of a playable game, (4.20), is a rather unsatisfactory one since it involves the abstract operator $\mathscr{D}^{-1}(t_0)$. Fortunately, the feedback synthesis procedure which we are about to develop provides a simple method for computing the strategy from the instantaneous state of the game. We shall see shortly that the main computation involves the integration of a system of ordinary differential equations, [(4.44), (4.46)], depending upon only the data of the problem.

As a notational convenience we define, in terms of the given matrices $B_i(t)$, $U_i(t) > 0$, additional matrix functions

$$\tilde{U}_i(t) = B_i(t)\, U_i^{-1}(t)\, B_i^*(t) \geqslant 0 \qquad (i = 1, 2,..., n). \tag{4.25}$$

The feedback synthesis which we now develop is suggested by the maximum principle and Hamilton-Jacobi formalisms which were applied in [6] to study the regulator control problem.

Our basic assumption is that the differential game (4.1), (4.3) is playable on a fixed interval $(t_\alpha, t_1]$. We proved in the above sections that for each $t_0 \in (t_\alpha, t_1]$, $y_0 \in R^m$ there exists a unique equilibrium strategy $\hat{u} = \hat{u}(t, y_0, t_0)$ for the differential game on $[t_0, t_1]$ with initial state $\hat{y}(t_0, y_0, t_0) = y_0$. From the uniqueness it is clear that $\hat{u}$, $\hat{y}$ have the important semi-group property

$$\begin{aligned} \hat{u}(t, \hat{y}(t_2, y_0, t_0), t_2) &= \hat{u}(t, y_0, t_0) \\ \hat{y}(t, \hat{y}(t_2, y_0, t_0), t_2) &= \hat{y}(t, y_0, t_0) \end{aligned} \tag{4.26}$$

holding for all $y_0 \in R^m$, $t_\alpha < t_0 \leqslant t_2 \leqslant t \leqslant t_1$. For the purpose of ultimately constructing the equilibrium strategy from the solutions of differential equations we define Hamiltonian functions

$$\begin{aligned} H^i = J_t^i + \left[A(t)\, y + B_i(t)\, u_i + \sum_{j=1}^{n} B_j(t)\, \hat{u}_j(t, y_0, t_0)\right] \cdot J_y^i \\ + |\, y - \bar{y}_i\,|^2_{W_i(t)} + |\, u_i\,|^2_{U_i(t)} \qquad (i = 1, 2,..., n) \end{aligned} \tag{4.27}$$

and by formally setting $H^i_{u_i} = 0$, $H^i = 0$ are led to considering the system of Hamilton-Jacobi partial differential equations in the dependent variables $J^i = J^i(y, t, y_0, t_0)$

$$\begin{aligned} J_t^i + \left[A(t)\, y - \tfrac{1}{2}\, \tilde{U}_i(t)\, J_y^i + \sum_{j \neq i}^{n} B_j(t)\, \hat{u}_j(t, y_0, t_0)\right] \cdot J_y^i \\ + |\, y - \bar{y}_i(t)|^2_{W_i(t)} + \tfrac{1}{4}\, |\, J_y^i\,|^2_{\tilde{U}_i(t)} = 0, \qquad y \in R^m, \qquad \text{a.e.} \qquad t \in (t_0, t_1] \\ J^i(y, t_1, y_0, t_0) = |\, y - \hat{y}_i\,|^2_{\hat{W}_i}, \qquad y \in R^m \qquad (i = 1, 2,..., n). \end{aligned} \tag{4.28}$$

As a preliminary step in proving (4.28) has a solution we make a formal separation of variables in (4.28)

$$J^i = y \cdot Q_i y + 2y \cdot q_i + c_i \tag{4.29}$$

where $Q_i = Q_i(t)$ is a symmetric $m \times m$ matrix function, $q_i = q_i(t, y_0, t_0)$ is an m-dimensional vector function and $c_i = c_i(t, y_0, t_0)$ is a scalar function

($i = 1, 2,\ldots, n$), all to be determined so that (4.29) satisfies (4.28). This gives rise to the system of ordinary differential equations

$$-\dot{Q}_i = W_i(t) + A^*(t)Q_i + Q_iA(t) - Q_i\tilde{U}_i(t)Q_i, \qquad Q_i(t_1) = \hat{W}_i, \qquad (i = 1, 2,\ldots, n); \tag{4.30}$$

$$-\dot{q}_i = [A^*(t) - Q_i\tilde{U}_i(t)]\, q_i + Q_i \sum_{\substack{j=1\\ \neq i}}^{n} B_j(t)\, \hat{u}_j(t, y_0, t_0) - W_i(t)\, \bar{y}_i(t), \qquad q_i(t_1, y_0, t_0) = -\hat{W}_i\hat{y}_i, \qquad (i = 1, 2,\ldots, n); \tag{4.31}$$

$$-\dot{c}_i = |\,\bar{y}_i(t)|^2_{W_i(t)} - |\,q_i\,|^2_{\tilde{U}_i(t)} + 2q_i \cdot \sum_{\substack{j=1\\ \neq i}}^{n} B_j(t)\, \hat{u}_j(t, y_0, t_0), \qquad c_i(t_1, y_0, t_0) = |\,\hat{y}_i\,|^2_{\hat{W}_i}, \qquad (i = 1, 2,\ldots, n). \tag{4.32}$$

Equation (4.30) is the well-known Kalman-Riccati matrix differential equation. This nonlinear equation has a solution $Q(t) \geqslant 0$ on $[t_\alpha, t_1]$. This may be verified by applying the inequality

$$-\dot{Q}_i \leqslant W_i(t) + A^*(t)Q_i + Q_iA(t), \qquad (i = 1, 2,\ldots, n) \tag{4.33}$$

to extend the local solution of (4.30) about $t = t_1$ to $[t_\alpha, t_1]$ or by referring to [6]. Of course (4.31), (4.32) then also have solutions on $[t_0, t_1]$ and we substitute these solutions into (4.29) to obtain a solution to (4.28).

For the time being we fix $i \in \{1, 2,\ldots, n\}$. Select arbitrary $t_2 \in [t_0, t_1]$, $u_i = u_i(t) \in L^2_{m_i}[t_2, t_1]$ and let $y(t)$ denote the solution of

$$\dot{y} = A(t)\, y + B_i(t)\, u_i(t) + \sum_{\substack{j=1\\ \neq i}}^{n} B_j(t)\, \hat{u}_j(t, y_0, t_0) \tag{4.34}$$

on $[t_2, t_1]$ for which $y(t_2) = y_2 = \hat{y}(t_2, y_0, t_0)$. By differentiation, using (4.28), (4.34), one can easily verify the equation

$$J^i(y_2, t_2, y_0, t_0) - |\,y(t_1) - \hat{y}_i\,|^2_{\hat{W}_i} - \int_{t_2}^{t_1} [\,|\,y(\tau) - \bar{y}_i(\tau)|^2_{W_i(\tau)} + |\,u_i(\tau)|^2_{U_i(\tau)}]\,d\tau$$

$$= -\int_{t_2}^{t_1} |\,u_i(\tau) + \tfrac{1}{2}\, U_i^{-1}(\tau)\, B_i^*(\tau)\, J_y{}^i(y(\tau), \tau, y_0, t_0)|^2_{U_i(\tau)}\, d\tau. \tag{4.35}$$

From (4.35) we conclude that

$$J^i(y_2, t_2, y_0, t_0) \leqslant \int_{t_2}^{t_1} [|y(\tau) - \bar{y}_i(\tau)|^2_{W_i(\tau)} + |u_i(\tau)|^2_{U_i(\tau)}]\, d\tau + |y(t_1) - \hat{y}_i|^2_{\hat{W}_i} \tag{4.36}$$

for all $u_i \in L^2_{m_i}[t_2, t_1]$. Moreover, if we let $\tilde{y}(t)$ denote the solution of

$$\dot{\tilde{y}} = A(t)\,\tilde{y} - \tfrac{1}{2}\,\tilde{U}_i(t)\, J^i_{\tilde{y}}(\tilde{y}, t, y_0, t_0) + \sum_{\substack{j=1\\ \neq i}}^{n} B_j(t)\,\hat{u}_j(t, y_0, t_0) \tag{4.37}$$

for which $\tilde{y}(t_2) = y_2$ it follows immediately from (4.34)-(4.36) that the control function

$$\tilde{u}_i(t) = -\tfrac{1}{2}\, U_i^{-1}(t)\, B_i^*(t)\, J^i_{\tilde{y}}(\tilde{y}(t), t, y_0, t_0) \tag{4.38}$$

is the unique control in $L^2_{m_i}[t_2, t_1]$ minimizing the cost functional defined by (4.34) and the right side of (4.36). By (4.26) and the definition of an equilibrium strategy the restriction of $\hat{u}_i(t, y_0, t_0)$ to $[t_2, t_1]$ minimizes the same cost functional. Hence $\tilde{u}_i(t) = \hat{u}_i(t, y_0, t_0)$ on $[t_2, t_1]$ and $\tilde{y}(t) = \hat{y}(t, y_0, t_0)$ there.

Initially $i \in \{1, 2,..., n\}$ was arbitrary and therefore our above conclusions hold for $i = 1, 2,..., n$. Furthermore, from (4.35), (4.37), (4.38) and the fact that $\tilde{u}_i(t) = \hat{u}_i(t, y_0, t_0)$ on $[t_2, t_1]$ we see that $J^i(y_2, t_2, y_0, t_0)$ $(i = 1, 2,..., n)$ are the costs of $\hat{u}(t, y_0, t_0)$ on $[t_2, t_1]$ with $\hat{y}(t_2, y_0, t_0) = y_2$. Finally we note that the solution $J^i(y, t, y_0, t_0)$ $(i = 1, 2,..., n)$ of (4.28) defined by (4.29)-(4.32) is unique for otherwise we could repeat the above argument to contradict the fact that the costs of $\hat{u}$ are given by the J^i. We summarize these results in the next theorem.

THEOREM 11. *For a differential game* (4.1), (4.3) *which is playable on* $(t_\alpha, t_1]$ *there exists a unique solution to* (4.28), $J^i = J^i(y, t, y_0, t_0)$ $(i = 1, 2,..., n)$, *defined for all* $t_0 \in (t_\alpha, t_1]$, $y_0 \in R^m$, $t \in [t_0, t_1]$, $y \in R^m$. *This solution is related to the equilibrium strategy and trajectory functions* $\hat{u} = \hat{u}(t, y_0, t_0)$, $\hat{y} = \hat{y}(t, y_0, t_0)$, *respectively, by the equations*

$$\dot{\hat{y}} = A(t)\,\hat{y} - \tfrac{1}{2}\sum_{j=1}^{n} \tilde{U}_j(t)\, J^j_{\hat{y}}(\hat{y}, t, y_0, t_0), \qquad \hat{y}(t_0, y_0, t_0) = y_0 \tag{4.39}$$

$$\begin{aligned}\hat{u}_i(t, y_0, t_0) &= -\tfrac{1}{2}\, U_i^{-1}(t)\, B_i^*(t)\, J^i_{\hat{y}}(\hat{y}, t, y_0, t_0)\\ &= -\tfrac{1}{2}\, U_i^{-1}(t)\, B_i^*(t)\, J^i_{y}(y, t, \hat{y}, t)|_{y=\hat{y}}\end{aligned} \tag{4.40}$$

$$J^i(\hat{y}, t, y_0, t_0) = \int_t^{t_1} [|\hat{y}(\tau, y_0, t_0) - \bar{y}_i(\tau)|^2_{W_i(\tau)} + |\hat{u}_i(\tau, y_0, t_0)|^2_{U_i(\tau)}]\, d\tau \\ + |\hat{y}(t_1, y_0, t_0) - \hat{y}_i|^2_{\hat{W}_i}, \qquad (i = 1, 2, ..., n). \tag{4.41}$$

Proof. We consider the second equality in (4.40), which is the only part of Theorem 11 not already proved. If we examine (4.29)-(4.32) to see how the $J^i(y, t, y_0, t_0)$ were constructed and apply the semi-group property of $\hat{u}$ given in (4.26) then we see that

$$J^i(y, t, y_0, t_0) = J^i(y, t, \hat{y}(t, y_0, t_0), t) \tag{4.42}$$

for each $t \in [t_0, t_1]$ and all $y \in R^m$. Differentiating both sides of (4.42) with respect to y and evaluating these derivatives at $y = \hat{y}(t, y_0, t_0)$ we have

$$J_{\hat{y}}^i(\hat{y}, t, y_0, t_0) = J_y^i(y, t, \hat{y}, t)|_{y=\hat{y}} \tag{4.43}$$

$(i = 1, 2,..., n)$ and the proof is complete.

The significance of equations (4.39)-(4.40) is that they show the open-loop equilibrium strategies can be "synthesized by feedback control functions." That is, at each time $t \in (t_\alpha, t_1]$ and state of the game $y \in R^m$ the instantaneous value of $\hat{u}$ is determined by t, y according to the second equality in (4.40) and no memory involving past states of the game nor prediction of future states is required for the computation. Consequently, in Theorem 12 we shall denote the equilibrium strategy as a function of t, y by $\hat{u}(t, y)$.

Of course although Theorem 11 tells us that a feedback synthesis is possible, it does not tell us directly how to compute it since the technique that we used to construct the J^i involved the future values of $\hat{u}$. The next theorem provides a method for constructing the feedback control function $\hat{u}(t, y)$ directly from the data of the problem.

THEOREM 12. *For a differential game* (4.1), (4.3) *which is playable on* $(t_\alpha, t_1]$ *there exists a solution on* $(t_\alpha, t_1]$ *to the* $m \times m$ *matrix differential equations*

$$-\dot{L}_i = W_i(t) + A^*(t)L_i + L_iA(t) - L_i \sum_{j=1}^{n} \tilde{U}_j(t) L_j \\ L_i(t_1) = \hat{W}_i, \qquad (i = 1, 2,..., n). \tag{4.44}$$

At each time $t \in (t_\alpha, t_1]$ *and state of the game* $y \in R^m$ *the equilibrium strategy function* $\hat{u}(t, y)$ *can be computed by*

$$\hat{u}_i(t, y) = -U_i^{-1}(t) B_i^*(t) [L_i(t) y + r_i(t)] \tag{4.45}$$

($i = 1, 2,..., n$) where the $r_i(t)$ are the solutions to the linear vector differential equations

$$\dot{r}_i = - A^*(t)\, r_i + L_i(t) \sum_{j=1}^{n} \tilde{U}_j(t)\, r_j + W_i(t)\, \bar{y}_i(t),$$

$$r_i(t_1) = - \hat{W}_i \hat{y}_i, \qquad (i = 1, 2,..., n); \tag{4.46}$$

hence, (4.45) *provides the feedback synthesis for the game.*

Proof. From (4.29), (4.31), (4.39) and (4.40) we have the solution $J^i(y, t, y_0, t_0)$ ($i = 1, 2,..., n$) to (4.28) given by

$$J^i(y, t, y_0, t_0) = y \cdot Q_i(t)\, y + 2y \cdot q_i(t, y_0, t_0) + c_i(t, y_0, t_0) \tag{4.47}$$

and hence the formula

$$\hat{u}_i(t, y_0, t_0) = - U_i^{-1}(t)\, B_i^*(t)\, [Q_i(t)\, \hat{y}(t, y_0, t_0) + q_i(t, y_0, t_0)] \tag{4.48}$$

($i = 1, 2,..., n$) where the $q_i(t, y_0, t_0)$ are defined by (4.31). Furthermore $\hat{y}(t, y_0, t_0)$ and $q_i(t, y_0, t_0)$ ($i = 1, 2,..., n$) provide a solution to the boundary value problem for the system in $R^{(m+1)n}$

$$\dot{y} = \left[A(t) - \sum_{j=1}^{n} \tilde{U}_j(t)\, Q_j(t)\right] y - \sum_{j=1}^{n} \tilde{U}_j(t)\, q_j \tag{4.49}$$

$$\dot{q}_i = Q_i(t) \sum_{\substack{j=1 \\ \neq i}}^{n} \tilde{U}_j(t)\, Q_j(t)\, y - A^*(t)\, q_i + Q_i(t) \sum_{j=1}^{n} \tilde{U}_j(t)\, q_j + W_i(t)\, \bar{y}_i(t) \tag{4.50}$$

corresponding to the boundary conditions $y = y_0$ at $t = t_0$ and $q_i = - \hat{W}_i \hat{y}_i$ at $t = t_1$, ($i = 1, 2,..., n$).

Let $y = y(t, y_1)$, $q_i = q_i(t)$ ($i = 1, 2,..., n$) be the solution of (4.49), (4.50) for the final value problem $y(t_1, y_1) = y_1$, $q_i(t_1) = - \hat{W}_i \hat{y}_i$, y_1 arbitrary in R^m. From the fact that there is a solution to the previously mentioned boundary value problem it is clear that the map $y_1 \to y(t_0, y_1)$ of R^m into itself is surjective. By writing $y(t, y_1)$, $q_i(t)$ in terms of a system of fundamental solutions for (4.49), (4.50) it follows that there are matrix functions $R_i(t_0)$ and vector functions $r_i(t_0)$ such that

$$q_i(t_0) = R_i(t_0)\, y(t_0, y_1) + r_i(t_0) \qquad \text{for} \qquad t_0 \in (t_\alpha, t_1].$$

In particular,

$$q_i(t, y_0, t_0) = R_i(t)\, \hat{y}(t, y_0, t_0) + r_i(t) \tag{4.51}$$

for all $t_0 \in (t_\alpha, t_i]$, $t \in [t_0, t_1]$, $y_0 \in R^m$ $(i = 1, 2,..., n)$. One may use (4.51) to verify $R_i(t)$, $r_i(t)$ satisfy the equations

$$\begin{aligned} -\dot{R}_i &= Q_i(t)\,\tilde{U}_i(t)\,Q_i(t) + A^*(t)\,R_i + R_i A(t) \\ &\quad - [R_i + Q_i(t)] \sum_{j=1}^{n} \tilde{U}_j(t)\,[R_j + Q_j(t)], \end{aligned} \tag{4.52}$$

$$R_i(t_1) = 0, \qquad (i = 1, 2,..., n);$$

$$\begin{aligned} \dot{r}_i &= -A^*(t)\,r_i + [R_i(t) + Q_i(t)] \sum_{j=1}^{n} \tilde{U}_j(t)\,r_j + W_i(t)\,\bar{y}_i(t), \\ r_i(t_1) &= -\hat{W}_i \hat{y}_i, \qquad (i = 1, 2,..., n). \end{aligned} \tag{4.53}$$

By adding equations (4.52), (4.30) and defining $L_i(t) \equiv R_i(t) + Q_i(t)$ we see that L_i satisfies (4.44) and (4.53) becomes (4.46). The proof is completed by applying (4.48), (4.51) and the definition of $L_i(t)$ to compute

$$\begin{aligned} \hat{u}_i &= -U_i^{-1}(t)\,B_i^*(t)\,[Q_i(t)\,\hat{y} + q_i(t, y_0, t_0)] \\ &= -U_i^{-1}(t)\,B_i^*(t)\,[L_i(t)\,\hat{y} + r_i(t)]. \end{aligned} \tag{4.54}$$

Our theory says that equilibrium will be played automatically whenever all players apply feedback control $\hat{u}(t, y)$. Furthermore, player P_i will suffer an increase in cost if he departs from equilibrium while the remaining $n - 1$ players continue to play their open-loop equilibrium strategies. However if P_i departs from equilibrium with the other players playing feedback then we might suspect that the feedbacks react to increase P_i's cost. In general this is not the case. In Reference [2] linear feedback controls with this "closed-loop equilibrium" property are shown to exist over sufficiently short time intervals. They do not produce control signals which are equilibrium strategies in the sense in which we have defined them.

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