J. Math. Anal. Appl. 368 (2010) 247–253

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

On convexity of solutions of ordinary differential equations

M. Keller-Ressel [a], E. Mayerhofer [b,c,1], A.G. Smirnov [d,*,2]

[a] ETH Zürich, Departement Mathematik, HG G 50.1, Rämistrasse 101, 8092 Zürich, Switzerland
[b] Vienna Institute of Finance, University of Vienna, Austria
[c] Vienna University of Economics and Business Administration, Heiligenstädterstrasse 46-48, 1190 Vienna, Austria
[d] I.E. Tamm Theory Department, P.N. Lebedev Physical Institute, Leninsky Prospect 53, Moscow 119991, Russia

ARTICLE INFO

Article history:
Received 10 June 2009
Available online 10 March 2010
Submitted by R. Kiesel

Keywords:
Initial value problem
Convexity
Ordered linear spaces
Quasi-monotonicity

ABSTRACT

We prove a result on the convex dependence of solutions of ordinary differential equations on an ordered finite-dimensional real vector space with respect to the initial data.

1. Introduction

Let E be a finite-dimensional real vector space ordered by a closed proper cone[3] C.

Let $0 < T \leqslant \infty$, $U \subset E$ be a nonempty open set, and $f : [0, T) \times U \to E$ be a locally Lipschitz continuous map. For any $x \in U$, the differential equation

$$\dot{\psi}(t) = f\big(t, \psi(t)\big) \tag{1}$$

has a unique maximally extended solution $\psi_f(\cdot, x)$ satisfying $\psi_f(0, x) = x$. This solution is defined on a semi-interval $[0, \theta_f(x))$, where $0 < \theta_f(x) \leqslant T$. For any $t \geqslant 0$, we set $\mathcal{D}_f(t) = \{x \in U\colon\ t < \theta_f(x)\}$.

Let $D \subset E$. A map $g : D \to E$ is called quasi-monotone increasing [11] if the implication

$$x \leqslant y, \quad l(x) = l(y) \quad \Longrightarrow \quad l\big(g(x)\big) \leqslant l\big(g(y)\big)$$

holds for all $x, y \in D$ and $l \in C^*$, where $C^* = \{l \in E^*\colon\ l(x) \geqslant 0 \text{ for any } x \in C\}$ is the dual cone of C (E^* is the dual space of E). A map $g : D \to E$ is called convex if D is convex and

$$g\big(\lambda x + (1 - \lambda)y\big) \leqslant \lambda g(x) + (1 - \lambda)g(y) \tag{2}$$

for all $x, y \in D$ and $\lambda \in [0, 1]$. A set $D \subset E$ is said to be order regular if the relations $x \in D$ and $y \leqslant x$ imply that $y \in D$.

* Corresponding author.
E-mail addresses: kemartin@math.ethz.ch (M. Keller-Ressel), eberhard.mayerhofer@vif.ac.at (E. Mayerhofer), smirnov@lpi.ru (A.G. Smirnov).

1 E.M. gratefully acknowledges financial support from WWTF (Vienna Science and Technology Fund).

2 The research of A.G.S. was supported by the Russian Foundation for Basic Research (Grant No. 09-01-00835) and the Program for Supporting Leading Scientific Schools (Grant No. LSS-1615.2008).

3 A set C in a real vector space E is called a cone if $\lambda C \subset C$ for any $\lambda > 0$. A cone C is said to be proper if $C + C \subset C$ and $C \cap (-C) = \{0\}$. A cone C induces a partial order on E if and only if it is proper.

0022-247X/$ – see front matter
doi:10.1016/j.jmaa.2010.03.008

Our aim is to prove the next theorem.

Theorem 1. *Let $U \subset E$ be a nonempty order-regular convex open set. Let $0 < T \leqslant \infty$ and $f : [0, T) \times U \to E$ be a continuous map. If $f(t, \cdot)$ is quasi-monotone increasing and convex for all $t \in [0, T)$, then $\mathcal{D}_f(t)$ is convex for any $t \in [0, T)$, and $\psi_f(t, \cdot)$ is convex thereon.*

In the formulation of Theorem 1, we do not require the local Lipschitz continuity of f because the latter is ensured by continuity and convexity (see Lemma 2 below). Note that the quasi-monotonicity of f is a sufficient but not necessary condition for Theorem 1 to hold. For example, if $f(t,x) = f(x)$ is a linear map, then $\psi_f(t,x)$ is linear and hence convex in x, but f may be not quasi-monotone increasing in this case. On the other hand, at least in the autonomous case $f(t,x) = f(x)$, the convexity of f is necessary to maintain the validity of Theorem 1. Indeed, let f be locally Lipschitz, $x, y \in U$ and $z = \lambda x + (1-\lambda)y$ with $0 \leqslant \lambda \leqslant 1$. Suppose $\mathcal{D}_f(t)$ is convex for any $t \in [0, T)$, and $\psi_f(t, \cdot)$ is convex thereon. Then we have

$$\frac{\psi_f(t,z) - z}{t} \leqslant \lambda \frac{\psi_f(t,x) - x}{t} + (1-\lambda)\frac{\psi_f(t,y) - y}{t}$$

for t small enough. Passing to the limit $t \to 0$ in this inequality, we get $f(z) \leqslant \lambda f(x) + (1-\lambda) f(y)$, i.e., f is convex.

The question of convex dependence of solutions of (1) on initial data was first addressed in [7], and then pursued in [5,4]. In the last two papers, E was assumed to be an ordered Banach space and it was shown (for differentiable f in [5] and for general locally Lipschitz continuous f in [4]) that $\psi_f(t, \cdot)$ is convex on any convex domain contained in $\mathcal{D}_f(t)$ (in Appendix A to this paper, we give a very simple proof of this result). Here, we strengthen this result in the finite-dimensional case by proving the convexity of $\mathcal{D}_f(t)$. Moreover, keeping in mind possible applications (see, e.g., an example in Section 5), we consider arbitrary open convex order-regular domains U rather than the case $U = E$ studied in [5,4].

The paper is organized as follows. In Section 2, we show that the conditions imposed on f in Theorem 1 ensure its local Lipschitz continuity. In Section 3, we prove Theorem 1 in the case, where f is differentiable in the second variable. For this, we combine the technique developed in [5] with the well-known "blow-up property" of ODEs in finite dimensions: as $t \to \theta_f(x)$ for some $x \in U$, the maximal solution $\psi_f(t,x)$ of (1) must approach the boundary of the domain $[0, T) \times U$ on which f is defined. In Section 4, we get rid of the differentiability assumption and prove Theorem 1 in the general case. Finally, in Section 5, we illustrate Theorem 1 by a concrete example of ODEs naturally arising in the theory of stochastic processes.

2. Convexity and local Lipschitz continuity

Let $0 < T \leqslant \infty$ and $\|\cdot\|$ be a norm on E. Let $U \subset E$ be a nonempty open set. Recall that a map $f : [0, T) \times U \to E$ is called locally Lipschitz if

$$L_{t,K}(f) = \sup_{0 \leqslant \tau \leqslant t,\ x_1, x_2 \in K,\ x_1 \neq x_2} \frac{\|f(\tau, x_2) - f(\tau, x_1)\|}{\|x_2 - x_1\|} < \infty \tag{3}$$

for any compact set $K \subset U$ and any $t \in [0, T)$.

Lemma 2. *Let $f : [0, T) \times U \to E$ be a continuous map such that $f(t, \cdot)$ is convex on U for all $t \in [0, T)$. Then f is locally Lipschitz continuous.*

Proof. Since C is closed and $C \cap (-C) = \{0\}$, the set $C \setminus \{0\}$ is contained in an open half-space of E. This implies that the dual cone C^* has a nonempty interior (see, e.g., [10, Section I.4.4, Lemma 1]). Let $l_1, \dots, l_n \in C^*$ be a basis of E^*. Let the real-valued functions $f_1, \dots, f_n$ on $[0, T) \times U$ be defined by the relations $f_j(t,x) = l_j(f(t,x))$. Clearly, f_j are continuous on $[0, T) \times U$ and $f_j(t, \cdot)$ are convex on U for any $t \in [0, T)$. Let $e_1, \dots, e_n \in E$ be the dual basis of $l_1, \dots, l_n$: $l_j(e_k) = \delta_{jk}$. Then we have

$$f(t,x) = \sum_{j=1}^{n} f_j(t,x) e_j.$$

Hence, it remains to prove that f_j are locally Lipschitz continuous, i.e., satisfy (3) with $\|\cdot\|$ in the numerator replaced with $|\cdot|$. Clearly, it suffices to check (3) in the case $K = B_{x,r}$, where $B_{x,r} \subset U$ is a closed ball of radius $r > 0$ centered at $x \in U$. Let $r' > r$ be such that $B_{x,r'} \subset U$. By the continuity of f_j, there is $m > 0$ such that $|f_j(\tau, x)| \leqslant m$ for any $\tau \in [0, t]$ and $x \in B_{x,r'}$. By [12, Corollary 2.2.12], we have

$$\bigl|f_j(\tau, x_2) - f_j(\tau, x_1)\bigr| \leqslant \frac{2m}{r'} \frac{r' + r}{r' - r} \|x_2 - x_1\|$$

for any $x_1, x_2 \in B_{x,r}$ and $\tau \in [0, t]$. The lemma is proved. □

3. The differentiable case

In the rest of the paper, we assume that $T \in (0, \infty]$ is fixed and set $I = [0, T)$.

Our consideration is essentially based on the next comparison result that is a particular case of a more general theorem proved by Volkmann [11] in the setting of normed vector spaces.

Lemma 3. *Let $U \subset E$ be an open set. Let $f : I \times U \to E$ be a continuous locally Lipschitz map such that $f(t, \cdot)$ is quasi-monotone increasing on U for all $t \in I$. Let $0 < t_0 \leqslant T$ and $x, y : [0, t_0) \to U$ be differentiable maps such that $x(0) \leqslant y(0)$ and*

$$\dot{x}(t) - f\big(t, x(t)\big) \leqslant \dot{y}(t) - f\big(t, y(t)\big), \quad 0 \leqslant t < t_0.$$

Then we have $x(t) \leqslant y(t)$ for all $t \in [0, t_0)$.

In fact, this comparison statement is essentially equivalent to quasi-monotonicity [9], but the above formulation is enough for our purposes. The next lemma is a simple generalization of a well-known result for scalar-valued convex functions.

Lemma 4. *Let $U \subset E$ be an open convex set. A differentiable function $g : U \to E$ is convex on U if and only if*

$$g(y) - g(x) \geqslant g'(x)(y - x), \quad x, y \in U. \tag{4}$$

Proof. Let $h = y - x$ and $\lambda \in (0, 1)$. If g is convex on U, then

$$g(x + \lambda h) = g\big((1 - \lambda)x + \lambda y\big) \leqslant (1 - \lambda)g(x) + \lambda g(y).$$

This implies that

$$\frac{g(x + \lambda h) - g(x)}{\lambda} \leqslant g(y) - g(x).$$

In view of the closedness of C, passing to the limit $\lambda \to 0$ yields (4). Conversely, let (4) hold and $z = \lambda x + (1 - \lambda)y$. Then we have

$$g(x) - g(z) \geqslant -(1 - \lambda)g'(z)h, \qquad g(y) - g(z) \geqslant \lambda g'(z)h.$$

Multiplying the left and right estimates by λ and $1 - \lambda$ respectively and summing the resulting inequalities, we obtain (2). The lemma is proved. □

For differentiable functions, we have the following characterization of quasi-monotonicity [3, Theorem 5].

Lemma 5. *Let $U \subset E$ be open and convex. A differentiable function $g : U \to E$ is quasi-monotone increasing on U if and only if the linear map $g'(x) : E \to E$ is quasi-monotone increasing for any $x \in U$.*

Suppose $f : I \times U \to E$ is a continuous map such that $f(t, \cdot)$ is differentiable on U for all $t \in I$ and the derivative $f'(t, \cdot)$ is continuous on $I \times U$ (here and below, $f'(t, \cdot)$ denotes the derivative of the map $x \to f(x, t)$ with respect to x for fixed t). Then f is locally Lipschitz, and we have

$$L_{t,K}(f) = \sup_{0 \leqslant \tau \leqslant t,\ x \in K} \big\| f'(\tau, x) \big\| \tag{5}$$

for any $t \in I$, and for any compact convex set $K \subset U$ with a nonempty interior. Given $x \in U$ and $0 \leqslant t < \theta_f(x)$, we define the linear map $B^x(t) : E \to E$ by setting

$$B^x(t) = f'\big(t, \psi_f(t, x)\big). \tag{6}$$

For $x \in U$ and $y \in E$, we denote by $w_y^x(t)$ the solution of the initial value problem

$$\dot{w}_y^x(t) = B^x(t) w_y^x(t), \quad 0 \leqslant t < \theta_f(x), \qquad w_y^x(0) = y. \tag{7}$$

Clearly, w_y^x is linear in y. For the norm of w_y^x, we have the standard bound (see, e.g., [2, Chapter IV, Lemma 4.1])

$$\big\| w_y^x(t) \big\| \leqslant \|y\| \exp\left(\int_0^t \big\| B^x(\tau) \big\| \, d\tau \right), \quad 0 \leqslant t < \theta_f(x). \tag{8}$$

Lemma 6. *Let $U \subset E$ be a convex open set and $f : I \times U \to E$ be a continuous map such that $f(t,\cdot)$ is differentiable on U for all $t \in I$ and the derivative $f'(t,\cdot)$ is continuous on $I \times U$. Suppose $f(t,\cdot)$ is convex and quasi-monotone increasing on U for all $t \in I$. For any $x, y \in U$, we have*

$$w^x_{y-x}(t) \leqslant \psi_f(t,y) - \psi_f(t,x) \leqslant w^y_{y-x}(t), \quad 0 \leqslant t < t_0, \tag{9}$$

where $t_0 = \min(\theta_f(x), \theta_f(y))$.

Proof. It suffices to prove the left inequality in (9) because it implies the right one after interchanging x and y. Let $s(t) = \psi_f(t,y) - \psi_f(t,x)$. By Lemma 4, we have

$$\dot{s}(t) = f\big(t, \psi_f(t,y)\big) - f\big(t, \psi_f(t,x)\big) \geqslant B^x(t)s(t), \quad 0 \leqslant t < t_0.$$

By Lemma 5, the map $B^x(t)$ is quasi-monotone increasing for any $t \in [0,t_0)$ and, therefore, the desired inequality follows from (7) and Lemma 3. The lemma is proved. □

Since E is finite-dimensional, the closed ordering cone C is normal. In terms of the partial order induced by C, this means that there exists $\mu_C > 0$ such that the implication

$$0 \leqslant x \leqslant y \quad \Longrightarrow \quad \|x\| \leqslant \mu_C \|y\| \tag{10}$$

holds for all $x, y \in E$.

If f is continuously differentiable in the second variable, Theorem 1 follows from the next lemma.

Lemma 7. *Let U and f be as in Lemma 6 and suppose in addition that U is order-regular. Let $x, y \in U$, $\lambda \in [0,1]$, and $z = \lambda x + (1-\lambda)y$. Let $t_0 = \min(\theta_f(x), \theta_f(y))$. Then we have $\theta_f(z) \geqslant t_0$ and*

$$\psi_f(t,z) \leqslant \lambda\psi_f(t,x) + (1-\lambda)\psi_f(t,y), \quad 0 \leqslant t < t_0. \tag{11}$$

Let $0 \leqslant t < t_0$ and $K \subset U$ be a compact convex set with a nonempty interior such that $\psi_f(\tau,x)$ and $\psi_f(\tau,y)$ lie in K for all $\tau \in [0,t]$. Then

$$\big\|\psi_f(t,z)\big\| \leqslant R_K\big[1 + \mu_C e^{L_{t,K}(f)t}\big], \tag{12}$$

where $R_K = \sup_{\xi \in K} \|\xi\|$.

Proof. Let $\tau_0 = \min(\theta_f(x), \theta_f(y), \theta_f(z))$. Since $z - x = (1-\lambda)(y-x)$ and $z - y = -\lambda(y-x)$, it follows from Lemma 6 that

$$(1-\lambda)w^x_{y-x}(t) \leqslant \psi_f(t,z) - \psi_f(t,x) \leqslant (1-\lambda)w^z_{y-x}(t),$$
$$-\lambda w^y_{y-x}(t) \leqslant \psi_f(t,z) - \psi_f(t,y) \leqslant -\lambda w^z_{y-x}(t),$$

for any $0 \leqslant t < \tau_0$. Multiplying the first and second inequalities by λ and $1-\lambda$ respectively and adding the results, we get

$$-\lambda(1-\lambda)v(t) \leqslant \psi_f(t,z) - u(t) \leqslant 0, \quad 0 \leqslant t < \tau_0, \tag{13}$$

where $u, v : [0,t_0) \to E$ are given by

$$u(t) = \lambda\psi_f(t,x) + (1-\lambda)\psi_f(t,y), \qquad v(t) = w^y_{y-x}(t) - w^x_{y-x}(t). \tag{14}$$

In view of (10), it follows from (13) that

$$\big\|\psi_f(t,z)\big\| \leqslant \big\|u(t)\big\| + \big\|\psi_f(t,z) - u(t)\big\| \leqslant \big\|u(t)\big\| + \mu_C\lambda(1-\lambda)\big\|v(t)\big\|, \quad 0 \leqslant t < \tau_0. \tag{15}$$

Suppose that $\tau_0 < t_0$. Then we obviously have $\tau_0 = \theta_f(z)$. Since both u and v are continuous on $[0,t_0)$, it follows from (15) that $\psi_f(t,z)$ is bounded on $[0,\theta_f(z))$. This implies that we can choose a sequence $t_k \uparrow \tau_0$ such that $\psi_f(t_k,z)$ converge to some $x_0 \in E$ as $k \to \infty$. By (13), we have $\psi_f(t_k,z) \leqslant u(t_k)$ for all k. As C is closed, it follows that $x_0 \leqslant u(\tau_0)$. We hence have $x_0 \in U$ because U is order-regular and $u(\tau_0) \in U$ by the convexity of U. On the other hand, we cannot have $x_0 \in U$ because $\psi_f(t,z)$ is a maximal solution and must approach the boundary of $I \times U$ as $t \to \theta_f(z)$ (see [2, Chapter II, Theorem 3.1]). This contradiction shows that

$$\tau_0 = t_0. \tag{16}$$

Combining this relation with (13) and (14), we obtain (11). Let $t \in [0,t_0)$ and $K \subset U$ be a convex compact set with a nonempty interior such that both $\psi_f(\tau,x)$ and $\psi_f(\tau,y)$ lie in K for any $\tau \in [0,t]$. It follows from (14), (8), (6), and (5) that

$$\big\|v(t)\big\| \leqslant 2\|y-x\|e^{L_{t,K}(f)t} \leqslant 4R_K e^{L_{t,K}(f)t}.$$

In view of (16), inserting this estimate and the obvious inequalities $\|u(t)\| \leqslant R_K$ and $\lambda(1-\lambda) \leqslant 1/4$ in (15) yields (12). The lemma is proved. □

4. Proof of Theorem 1

To pass from continuously differentiable to arbitrary continuous functions, we shall need some results concerning the continuous dependence of solutions of (1) on the map f. Recall that Eq. (1) possesses a maximal solution satisfying a given initial condition if the function $f : I \times U \to E$ is continuous. Note however that such a solution may be not unique if f is not locally Lipschitz continuous.

The next lemma easily follows from Theorem 3.2 in Chapter II of [2].

Lemma 8. *Let $U \subset E$ be open. Let $f, f_1, f_2, \dots$ be continuous maps from $I \times U$ to E. Suppose f is locally Lipschitz and f_n converge to f uniformly on all compact subsets of $I \times U$. Let $\psi_n \in C^1([0, \theta_n), U)$ be maximal solutions of*

$$\dot{\psi}_n(t) = f_n\big(t, \psi_n(t)\big) \tag{17}$$

such that $\psi_n(0)$ converge to some $u \in U$ as $n \to \infty$. Then we have

$$\theta_f(u) \leqslant \underline{\lim}\, \theta_n. \tag{18}$$

Let $0 \leqslant a < \theta_f(u)$ and n_0 be such that $\theta_n > a$ for $n > n_0$. Then the sequence $\psi_{n_0+k}(t)$, $k = 1, 2, \dots$, converges to $\psi_f(t, u)$ uniformly on $[0, a]$ as $k \to \infty$.

Lemma 9. *Let $U \subset E$ be open. Let $f, f_1, f_2, \dots$ be continuous maps from $I \times U$ to E. Suppose f is locally Lipschitz and f_n converge to f uniformly on compact subsets of $I \times U$. Let $0 < a < T$ and $\psi_n \in C^1([0, a], U)$ be solutions of (17) such that $\psi_n(0)$ converge to some $u \in U$ as $n \to \infty$. If for some compact set $K \subset U$, $\psi_n(t) \in K$ for all $t \in [0, a]$, then $\theta_f(u) > a$, and we have $\psi_n(t) \to \psi_f(t, u)$ and $\dot{\psi}_n(t) \to \dot{\psi}_f(t, u)$ uniformly on $[0, a]$.*

Proof. Since f_n are uniformly bounded on the compact set $Q = [0, a] \times K$, Eq. (17) implies that $\dot{\psi}_n$ are uniformly bounded. Hence, ψ_n are uniformly equicontinuous. By the Arzelà–Ascoli theorem, it follows that the sequence ψ_n is relatively compact in $C[0, a]$. Let ψ_{n_k} be a subsequence of ψ_n uniformly converging to a function ψ. Obviously, $\psi(0) = u$ and $\psi(t) \in K$ for $t \in [0, a]$. Fix $\varepsilon > 0$. Because f is uniformly continuous on Q, there exists a $\delta > 0$ such that $\|f(t, x_1) - f(t, x_2)\| < \varepsilon/2$ for any $(t, x_i) \in Q$ such that $\|x_2 - x_1\| < \delta$. Let k_0 be such that $\|\psi_{n_k}(t) - \psi(t)\| < \delta$ and $\|f_{n_k}(t, x) - f(t, x)\| < \varepsilon/2$ for all $(t, x) \in Q$ and $k \geqslant k_0$. Then we have

$$\begin{aligned}\big\| f_{n_k}\big(t, \psi_{n_k}(t)\big) - f\big(t, \psi(t)\big)\big\| &\leqslant \big\| f_{n_k}\big(t, \psi_{n_k}(t)\big) - f\big(t, \psi_{n_k}(t)\big)\big\| \\ &\quad + \big\| f\big(t, \psi_{n_k}(t)\big) - f\big(t, \psi(t)\big)\big\| < \varepsilon, \quad t \in [0, a],\end{aligned}$$

for any $k \geqslant k_0$, and in view of (17), the sequence $\dot{\psi}_{n_k}(t)$ converges to $f(t, \psi(t))$ uniformly on $[0, a]$. On the other hand, the uniform convergence of $\dot{\psi}_{n_k}$ implies that ψ is continuously differentiable and $\dot{\psi}$ is the limit of $\dot{\psi}_{n_k}$. This means that ψ satisfies (1). Since f is locally Lipschitz, this implies that ψ is the restriction of $\psi_f(\cdot, u)$ to $[0, a]$ and, therefore, $\theta_f(u) > a$. We thus see that all uniformly converging subsequences of ψ_n have the same limit. As the sequence ψ_n is relatively compact, we conclude that $\psi_n(t) \to \psi_f(t, u)$ uniformly on $[0, a]$. Replacing ψ_{n_k} with ψ_n in the above proof, we obtain the uniform convergence of $\dot{\psi}_n$. The lemma is proved. □

Proof of Theorem 1. For $\kappa > 0$, we set $U(\kappa) = \{\xi \in U\colon\ B_{\xi,\kappa} \subset U\}$, where $B_{\xi,\kappa}$ is the closed ball of radius κ centered at ξ. Clearly, the set $U(\kappa)$ is open, convex, and order-regular for any $\kappa > 0$. Let $t \in I$, $x, y \in \mathcal{D}_f(t)$ and $z = \lambda x + (1 - \lambda) y$ for some $\lambda \in [0, 1]$. We have to show that $\theta_f(z) > t$ and inequality (11) holds. Let $S \subset U$ be a convex compact set whose interior contains $\psi_f(\tau, x)$ and $\psi_f(\tau, y)$ for all $\tau \in [0, t]$. Choose $\kappa > 0$ such that $S \subset U(\kappa)$.

Let ρ be a nonnegative smooth function on E such that $\rho(\xi) = 0$ for $\|\xi\| > 1$ and $\int_E \rho(\xi)\, d\xi = 1$. For any positive $\varepsilon \leqslant \kappa$, we define the map $f_\varepsilon : I \times U(\kappa) \to E$ by setting

$$f_\varepsilon(\tau, \xi) = \int_E f(\tau, \xi - \varepsilon\eta) \rho(\eta)\, d\eta.$$

Let ϕ denote the restriction of f to $I \times U(\kappa)$. Clearly, f_ε are smooth in the second variable and converge to ϕ uniformly on compact subsets of $I \times U(\kappa)$ as $\varepsilon \to 0$. It is straightforward to check that f_ε are convex quasi-monotone increasing maps on $U(\kappa)$ such that

$$L_{t,S}(f_\varepsilon) \leqslant L_{t,S_\kappa}(f), \tag{19}$$

where S_κ is the closed κ-neighborhood of S. Our choice of κ ensures that $t < \min(\theta_\phi(x), \theta_\phi(y))$. Let $t_\varepsilon = \min(\theta_{f_\varepsilon}(x), \theta_{f_\varepsilon}(y))$. By Lemma 8, there exists $0 < \varepsilon_0 \leqslant \kappa$ such that $t_\varepsilon > t$ for any $0 < \varepsilon \leqslant \varepsilon_0$ and $\psi_{f_\varepsilon}(\cdot, x) \to \psi_f(\cdot, x)$ and $\psi_{f_\varepsilon}(\cdot, y) \to \psi_f(\cdot, y)$ uniformly on $[0, t]$ as $\varepsilon_0 \geqslant \varepsilon \to 0$. Decreasing ε_0 if necessary, we can ensure that $\psi_{f_\varepsilon}(\tau, x)$ and $\psi_{f_\varepsilon}(\tau, y)$ lie in S for all $\tau \in [0, t]$ and $\varepsilon \in (0, \varepsilon_0]$. It follows from Lemma 7 that $\theta_{f_\varepsilon}(z) \geqslant t_\varepsilon > t$ and

$$\psi_{f_\varepsilon}(\tau, z) \leqslant \lambda\psi_{f_\varepsilon}(\tau, x) + (1-\lambda)\psi_{f_\varepsilon}(\tau, y), \tag{20}$$

$$\left\|\psi_{f_\varepsilon}(\tau, z)\right\| \leqslant R_S\left[1 + \mu_C e^{tL_{t,S_\kappa}(f)}\right] \tag{21}$$

for any $0 \leqslant \tau \leqslant t$ and $0 < \varepsilon \leqslant \varepsilon_0$. Let $r > 0$ and $K = (S - C) \cap \{\xi \in E\colon \|\xi\| \leqslant r\}$. Since S is compact and C is closed, $S - C$ is closed and, therefore, K is compact. The order-regularity of $U(\kappa)$ implies that $K \subset U(\kappa)$. By (20) and (21), we have $\psi_{f_\varepsilon}(\tau, z) \in K$ for all $0 \leqslant \tau \leqslant t$ if r is large enough. It follows from Lemma 9 that $\theta_\phi(z) > t$ and $\psi_{f_\varepsilon}(\cdot, z) \to \psi_\phi(\cdot, z)$ uniformly on $[0, t]$. Obviously, $\mathcal{D}_\phi \subset \mathcal{D}_f$ and ψ_ϕ is the restriction of ψ_f to $\mathcal{D}_\phi$. Hence $\theta_f(z) \geqslant \theta_\phi(z) > t$ and passing to the limit $\varepsilon \to 0$ in inequality (20) for $\tau = t$ yields (11). The theorem is proved. □

5. Example

As an illustration, we give an example of a system of ODEs that arises naturally in the theory of stochastic processes and satisfies all conditions of Theorem 1. We consider a so-called *affine process* evolving on the state space $C := \mathbb{R}^d_{\geqslant 0}$ (see [1]). Such a process $X = (X_t)_{t\geqslant 0}$, can be regarded as a multi-type extension of the singe-type continuously branching process of [6], which arises as a continuous-time limit of a classical Galton–Watson branching process. X is defined as a stochastically continuous, time-homogeneous Markov process starting at $X_0 \in C$, with the property that the moment generating function is of the form

$$\mathbb{E}\left[e^{x\cdot X_t}\right] = e^{\psi(t,x)\cdot X_0} \tag{22}$$

for all $(t, x) \in \mathbb{R}_{\geqslant 0} \times \mathbb{R}^d$, and where $\psi : \mathbb{R}_{\geqslant 0} \times \mathbb{R}^d \to \mathbb{R}^d \cup \{\infty\}$.[4] We assume that the time-derivative of $\psi(t, x)$ at $t = 0$,

$$f(x) := \frac{\partial}{\partial t}\psi(t, x)\bigg|_{t=0}$$

exists and is a continuous function on the set $U = \{x \in \mathbb{R}^d\colon\ f(x) < \infty\}$. In this case the map $\psi(t, x)$ satisfies the following differential equation:

$$\frac{\partial}{\partial t}\psi(t, x) = f\big(\psi(t, x)\big), \qquad \psi(0, x) = x. \tag{23}$$

Moreover, the components of the map $f(x)$ are of so-called Levy–Khintchine type (cf. [8, Theorem 8.1]):

$$f_i(x) = \frac{\alpha_i}{2}x_i^2 + x\cdot\beta^i - c_i + \int\limits_{C\setminus\{0\}} \left(e^{x\cdot\xi} - 1 - x\cdot\xi\mathbf{I}_{|\xi|\leqslant 1}\right)\mu_i(d\xi),$$

with $\mathbf{I}$, the indicator function, where, for all $i \in \{1, \dots, d\}$,

- $\alpha_i \in \mathbb{R}_{\geqslant 0}$;
- $\beta^i \in \mathbb{R}^d$ with $\beta^i_k - \int_{|\xi|\leqslant 1}\xi_k\mu_i(d\xi) \geqslant 0$ for all $k \neq i$;
- $c_i \in \mathbb{R}_{\geqslant 0}$;
- $\mu_i(d\xi)$ are Borel measures on $C \setminus \{0\}$ assigning finite mass to the set $\{\xi \in C\colon\ |\xi| > 1\}$ and satisfying the integrability condition

$$\int\limits_{\xi\in C,\, 0<|\xi|\leqslant 1} \left(\sum_{k\neq i}|\xi_k| + |\xi_i|^2\right)\mu_i(d\xi) < \infty$$

  on its complement.

The above conditions are both necessary and sufficient for the existence of X and referred to as *admissibility* conditions (see [1]).

In the following we consider the ordering on $\mathbb{R}^d$ induced by the cone $\mathbb{R}^d_{\geqslant 0}$.

Proposition 10. *The domain U is convex and order-regular and the map $f(x)$ is convex and quasi-monotone increasing thereon.*

Proof. We make use of the following representations of $f_i(x)$:

$$f_i(x) = \log\int\limits_{\mathbb{R}^d} e^{x\cdot\xi}\,p_i(d\xi) = f_i^\dagger(x) + \int\limits_{C\setminus\{0\},\,|\xi|>1} \left(e^{x\cdot\xi} - 1\right)\mu_i(d\xi), \tag{24}$$

[4] We set $\psi(t, x) = \infty$, whenever the left side of (22) is infinite. Note that for $(t, x) \in \mathbb{R}_{\geqslant 0} \times (-\infty, 0]^d$ it is always guaranteed that $\psi(t, x)$ is finite.

where $p_i(d\xi)$ is an infinitely divisible, substochastic measure on $\mathbb{R}^d$, and $f_i^\dagger(x)$ is a function on $\mathbb{R}^d$, that can be extended to an entire function on $\mathbb{C}^d$. The representation as $\log \int_{\mathbb{R}^d} e^{x\cdot\xi}\, p_i(d\xi)$ is an immediate consequence of the Levy–Khintchine formula, and its analytic extension to exponential moments [8, Theorem 8.1, Theorem 25.17]. The second representation of $f_i(x)$ follows directly from [8, Lemma 25.6]. To show that $f_i(x)$ is convex, apply Hölder's inequality:

$$f_i\big(\lambda x + (1-\lambda)y\big) = \log \int_{\mathbb{R}^d} e^{\lambda x\cdot\xi} e^{(1-\lambda)y\cdot\xi}\, p_i(d\xi) \leqslant \lambda \log \int_{\mathbb{R}^d} e^{x\cdot\xi}\, p_i(d\xi) + (1-\lambda)\log \int_{\mathbb{R}^d} e^{y\cdot\xi}\, p_i(d\xi)$$
$$= \lambda f_i(x) + (1-\lambda) f_i(y)$$

for all $x, y \in \mathbb{R}^d$ and $\lambda \in (0,1)$. We show next that the domain U is order-regular. Assume that $x \in U$, i.e. $f_i(x) < \infty$ for all i, and let $y \leqslant x$. Using the second representation in (24) it is clear that $f_i^\dagger(y) < \infty$. But also the integral with respect to $\mu_i(d\xi)$ is finite, since the integrand is dominated by $(e^{x\cdot\xi} - 1)\mathbf{1}_{|\xi|\geqslant 1}$, whose integral is finite by assumption. We conclude that $f_i(y) < \infty$, and thus that $y \in U$, i.e., U is order-regular. Finally we show that $f(x)$ is also quasi-monotone increasing. Assume that $y \leqslant x$ with $y_i = x_i$ for some $i \in \{1,\dots,d\}$. It follows that

$$f_i(x) - f_i(y) = \sum_{k\neq i} (x_k - y_k)\cdot\left(\beta_k^i - \int_{\xi\in C,\, 0<|\xi|\leqslant 1} \xi_k\, \mu_i(d\xi)\right) + \int_C \big(e^{x\cdot\xi} - e^{y\cdot\xi}\big)\, \mu_i(d\xi) \geqslant 0,$$

where we have made use of the admissibility conditions given above. □

Appendix A

In this section we give a very simple proof of the convexity result [4] for ODEs in ordered normed spaces. Let E be a real normed space (not necessarily finite-dimensional) ordered by a proper closed cone C. As shown in [11], Lemma 3 holds for E if one of the following conditions is satisfied:

1. C has a nonempty interior,
2. E is complete,
3. C is a distance set (i.e., for every $x \in E$, there is $y \in C$ such that $\|x - y\|$ is equal to the distance from x to C).

As above, let $T \in (0,\infty]$ and $I = [0,T)$. Theorem 1 in [4] follows immediately from the next result.

Theorem 11. *Let E be an ordered normed space such that one of the above conditions is satisfied. Let $U \subset E$ be an open convex set and $f : I \times U \to E$ be a continuous locally Lipschitz map such that $f(t,\cdot)$ is quasi-monotone increasing and convex on U for all $t \in I$. Let $0 < t_0 \leqslant T$ and $x_1, x_2, x_3 : [0,t_0) \to U$ be differentiable maps such that*

$$\dot{x}_i(t) = f\big(t, x_i(t)\big), \quad i = 1,2,3,$$

and $x_3(0) = \lambda x_1(0) + (1-\lambda)x_2(0)$ for some $\lambda \in [0,1]$. Then $x_3(t) \leqslant \lambda x_1(t) + (1-\lambda)x_2(t)$ for all $t < t_0$.

Proof. Set $z(t) = \lambda x_1(t) + (1-\lambda)x_2(t)$ for $t < t_0$. By the convexity of f,

$$\dot{z}(t) - f\big(t, z(t)\big) = \lambda\dot{x}_1(t) + (1-\lambda)\dot{x}_2(t) - f\big(t, \lambda x_1(t) + (1-\lambda)x_2(t)\big)$$
$$\geqslant \lambda\big(\dot{x}_1(t) - f\big(t, x_1(t)\big)\big) + (1-\lambda)\big(\dot{x}_2(t) - f\big(t, x_2(t)\big)\big) = 0 = \dot{x}_3(t) - f\big(t, x_3(t)\big)$$

for all $t < t_0$. Since $z(0) = x_3(0)$, the above-mentioned analogue of Lemma 3 for normed spaces implies that $z(t) \geqslant x_3(t)$. The theorem is proved. □

References

[1] D. Duffie, D. Filipović, W. Schachermayer, Affine processes and applications in finance, Ann. Appl. Probab. 13 (3) (2003) 984–1053.
[2] P. Hartman, Ordinary Differential Equations, John Wiley & Sons, Inc., New York, 1964.
[3] G. Herzog, Quasimonotonicity, Nonlinear Anal. 47 (4) (2001) 2213–2224.
[4] G. Herzog, R. Lemmert, A note on convex dependence of solutions of IVPs relative to initial values, Nonlinear Anal. 68 (12) (2008) 3841–3844.
[5] V. Lakshmikantham, N. Shahzad, W. Walter, Convex dependence of solutions of differential equations in a Banach space relative to initial data, Nonlinear Anal. 27 (12) (1996) 1351–1354.
[6] J. Lamperti, Continuous state branching processes, Bull. Amer. Math. Soc. 73 (1967) 382–386.
[7] A.V. Sarychev, On equation $x^{(n+1)} = f(t, x, \dot{x}, \dots, x^{(n)})$ with convex quasi-monotone right-hand side, Nonlinear Anal. 27 (7) (1996) 785–792.
[8] K. Sato, Lévy Processes and Infinitely Divisible Distributions, Cambridge Stud. Adv. Math., vol. 68, Cambridge University Press, Cambridge, 1999, translated from the 1990 Japanese original, revised by the author.
[9] R. Uhl, Ordinary differential inequalities and quasimonotonicity in ordered topological vector spaces, Proc. Amer. Math. Soc. 126 (7) (1998) 1999–2003.
[10] V.S. Vladimirov, Generalized Functions in Mathematical Physics, Mir, Moscow, 1979, translated from the second Russian edition by G. Yankovskiĭ.
[11] P. Volkmann, Über die Invarianz konvexer Mengen und Differentialungleichungen in einem normierten Raume, Math. Ann. 203 (1973) 201–210.
[12] C. Zălinescu, Convex Analysis in General Vector Spaces, World Scientific Publishing, River Edge, NJ, 2002.