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On minimum metric dimension of honeycomb networks

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Abstract

A minimum metric basis is a minimum set W of vertices of a graph $G(V, E)$ such that for every pair of vertices u and v of G, there exists a vertex $w \in W$ with the condition that the length of a shortest path from u to w is different from the length of a shortest path from v to w. The honeycomb and hexagonal networks are popular mesh-derived parallel architectures. Using the duality of these networks we determine minimum metric bases for hexagonal and honeycomb networks.

Keywords: Metric basis; Metric dimension; Hexagonal network; Honeycomb network; Dual of a graph

1. Introduction and background

Multiprocessor interconnection networks are often required to connect thousands of homogeneously replicated processor-memory pairs, each of which is called a *processing node*. Instead of using a shared memory, all synchronization and communication between processing nodes for program execution is often done via message passing. Design and use of multiprocessor interconnection networks have recently drawn considerable attention due to the availability of inexpensive, powerful microprocessors and memory chips [2].

It is known that there exist three regular plane tessellations, composed of the same kind of regular polygons: triangular, square, and hexagonal. They are the basis for the design of direct interconnection networks with highly competitive overall performance. Grid connected computers and tori (Figs. 1(a) and (b)) are based on regular square tessellations, and are popular and well-known models for parallel processing.

Built recursively using the hexagon tessellation [13], *honeycomb networks* (Fig. 1(c)) are widely used in computer graphics [7], cellular phone base stations [9], image processing, and in chemistry as the representation of benzenoid hydrocarbons. Honeycomb networks are better in terms of degree, diameter, total number of links, cost and the bisection width than mesh connected planar graphs. Stojmenovic [13] has studied the topological properties of honeycomb networks, routing in honeycomb networks and honeycomb torus networks. Parhami [10] gave a unified formulation

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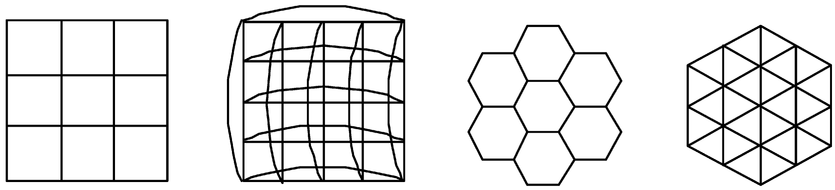


(a) 4 x 4 Grid network (b) Torus network (c) Honeycomb network (d) Hexagonal network

Fig. 1. Graphs obtained by regular plane tessellations.

for the honeycomb and the diamond networks. The embedding of honeycomb networks into trees, hypercubes and other networks was suggested as an open problem in [13].

The triangular tessellation is used to define *Hexagonal network* (Fig. 1(d)) and this is widely studied in [2]. An addressing scheme for hexagonal networks, and its corresponding routing and broadcasting algorithms were proposed by Chen et al. [2].

2. An overview of the paper

A *metric basis* for a graph $G(V,E)$ is a set $W \subseteq V$ such that for each pair of vertices u and v of $V \setminus W$, there is a vertex $w \in W$ such that $d(u,w) \neq d(v,w)$. A *minimum metric basis* is a metric basis of minimum cardinality. The cardinality of a minimum metric basis of G is called *minimum metric dimension* and is denoted by $md(G)$; the members of a minimum metric basis are called *landmarks*. The *minimum metric dimension* (MMD) *problem* is to find a minimum metric basis.

The problem of finding the metric dimension of a graph was first studied by Harary and Melter [5]. Melter and Tomescu [8] studied the metric dimension problem for grid graphs. Khuller et al. [6] describe the application of this problem in the field of computer science and robotics. This problem has been studied for trees, multi-dimensional grids [6], Petersen graphs [1], and Torus Networks [11]. Surprisingly, there is not much relevant work in the literature. The algorithmic complexity status of MMD problem is not known to even simple graphs such as co-graphs, interval graphs, Cayley graphs etc.

It is interesting to learn [6] that a graph has metric dimension 1 if and only if it is a path. The problem of computing the metric dimension of trees is solved in linear time [6]. If G has p vertices then it is clear that $1 \leqslant md(G) \leqslant p-1$. Also $md(K_p) = p-1$, $md(C_p) = 2$, and $md(K_{m,n}) = m+n-2$, where K_p, C_p, and $K_{m,n}$ are the complete graph, the cycle, and the complete bipartite graph respectively [5]. Garey and Johnson [3] proved that this problem is *NP*-complete for general graphs by a reduction from 3-dimensional matching.

The aim of this paper is to find the minimum metric dimension of the honeycomb networks. Hexagonal network $HX(n)$ has a simple distance property like the two-dimensional grid. We first locate a minimum metric basis of $HX(n)$. By making use of the fact that honeycomb networks $HC(n)$ and hexagonal networks $HX(n)$ are dual networks, we derive a minimum metric basis for $HC(n)$. We prove that the minimum metric dimension of the honeycomb networks of size n is 3.

3. Properties of honeycomb networks

Honeycomb networks can be built from hexagons in various ways. The honeycomb network $HC(1)$ is a hexagon. The honeycomb network $HC(2)$ is obtained by adding six hexagons to the boundary edges of $HC(1)$. Inductively, honeycomb network $HC(n)$ is obtained from $HC(n-1)$ by adding a layer of hexagons around the boundary of $HC(n-1)$. For instance, Fig. 1(c) is $HC(2)$. The parameter n of $HC(n)$ is determined as the number of hexagons between the centre and boundary of $HC(n)$. The number of vertices and edges of $HC(n)$ are $6n^2$ and $9n^2 - 3n$ respectively. The diameter is $4n-1$ [13].

In order to view the honeycomb $HC(n)$ as a dual of the hexagonal network $HX(n)$, let us recall the definition of a dual graph. Let G be a planar graph. The dual of G, denoted by $G^\star$, is a graph whose vertex set is the set of faces of G, where two vertices $f^\star$ and $g^\star$ in $G^\star$ are joined by an edge $e^\star$ if the faces f and g are separated by the edge e. Clearly the number of vertices of $G^\star$ is equal to the number of faces of G and the number of edges of $G^\star$ is equal to

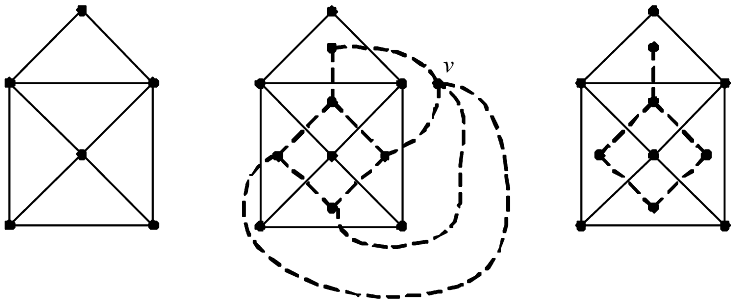
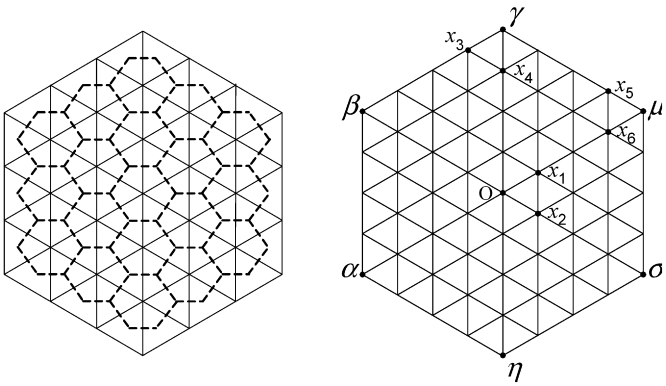


Fig. 2. The bounded dual is obtained by deleting the vertex v from the dual.



(a) (b)

Fig. 3. (a) Bounded dual of $HX(4)$ is $HC(3)$. (b) $HX(5)$ with corner vertices $\{\alpha, \beta, \gamma, \mu, \sigma, \eta\}$, centre vertex O and $md(HX(n)) > 2$.

the number of edges of G. Also any planar graph has exactly one unbounded face. The graph obtained from $G^\star$ by deleting the vertex corresponding to the unbounded face is called a *bounded dual* of G. See Fig. 2.

Lemma 1. [4] *The bounded dual of* $HX(n+1)$ *is* $HC(n)$.

Lemma 1 illustrates the dual relationship between $HX(n)$ and $HC(n)$. See Fig. 3(a).

Though a honeycomb is a bounded degree graph, measurement of distances between vertices is not straightforward. Interestingly, its dual graph, the hexagonal network has a better geometrical structure which enables one to measure easily the distance between any two points. We make use of this geometrical structure of $HX(n)$ to find the MMD of $HX(n)$. From this we easily deduce MMD of $HC(n)$.

4. Minimum metric dimension of hexagonal networks

We begin this section with some topological properties of $HX(n)$. It has $3n^2 - 3n + 1$ vertices and $9n^2 - 15n + 6$ edges, where n is the number of vertices on one side of the hexagon [2]. The diameter is $2n - 2$. There are six vertices of degree three which we call as *corner vertices*. There is exactly one vertex v at distance $n - 1$ from each of the corner vertices. This vertex is called the *centre* of $HX(n)$ and is represented by O. See Fig. 3(b). We solve the MMD problem for hexagonal networks and prove that the minimum metric dimension of hexagonal networks $HX(n)$ is 3. First we estimate the lower bound for $md(HX(n))$. To do this we need the following result of Khuller et al. [6].

Theorem 2. *Let G be a graph with minimum metric dimension* 2 *and let* $\{u, v\} \subset V$ *be a metric basis in* G. *Then the following are true*:

(a) *There is a unique shortest path between* u *and* v.
(b) *The degree of each* u *and* v *is at most* 3.

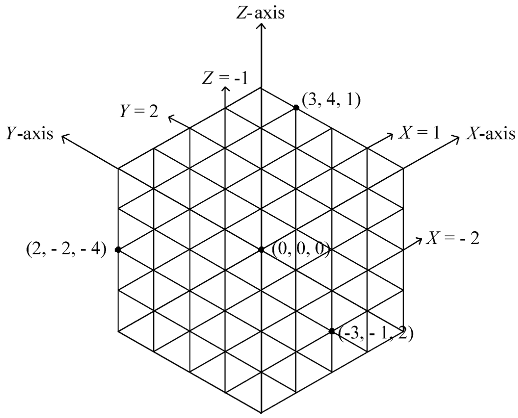


Fig. 4. Coordinates of vertices in $HX(5)$.

Lemma 3. *Let G be a hexagonal network $HX(n)$. Then $md(G) > 2$.*

Proof. In view of Theorem 2, if $\{u, v\} \subset V$ is a metric basis of $HX(n)$, then each u and v is one of the corner vertices since every other vertex of $HX(n)$ is of degree 4 or 6. Since there are only six corner vertices, we can verify each possible pair one by one. Now, $\{\alpha, \beta\}$ is not a metric basis, as x_1 and x_2 are equidistant from α and β; $\{\alpha, \gamma\}$ is not a metric basis, as x_3 and x_4 are equidistant from α and γ. Similarly, considering the vertices x_5 and x_6 we conclude that $\{\alpha, \mu\}$ is not metric basis. See Fig. 3(b). The other possible pairs are ruled out by the symmetrical nature of $HX(n)$. Thus $md(G) > 2$. □

In order to exhibit a metric basis of cardinality three, we require the concept of neighborhood of a vertex. Let V be the vertex set of $HX(n)$. An r-neighborhood of v is defined by $N_r(v) = \{u \in V\colon d(u, v) = r\}$. It is easy to see that the graph induced by vertices in $N_r(v)$ is either a cycle of length $6(r-1)$ or a section of the same cycle.

Stojmenovic [13] proposed a coordinate system for a honeycomb network. This was adapted by Nocetti et al. [9] to assign coordinates to the vertices in the hexagonal network. In this scheme, three axes, X, Y and Z parallel to three edge directions and at mutual angle of 120 degrees between any two of them are introduced, as indicated in Fig. 4. We call lines parallel to the coordinate axes as X-lines, Y-lines and Z-lines. Here $X = h$ and $X = -k$ are two X-lines on either side of the X-axis. Any vertex of $HX(n)$ is assigned coordinates (x, y, z) in the above scheme. See Fig. 4.

We denote by P_X, a segment of an X-line consisting of points (x, y, z), with x coordinate fixed. That is, $P_X = \{(x_0, y, z)/y_1 \leqslant y \leqslant y_2,\ z_1 \leqslant z \leqslant z_2\}$. Similarly $P_Y = \{(x, y_0, z)/x_1 \leqslant x \leqslant x_2, z_1 \leqslant z \leqslant z_2\}$ and $P_Z = \{(x, y, z_0)/x_1 \leqslant x \leqslant x_2,\ y_1 \leqslant y \leqslant y_2\}$.

Lemma 4. *In any $HX(n)$, we have $N_r(\alpha) = P_Y \circ P_Z$, $N_r(\beta) = P_X \circ P_Z$, $N_r(\gamma) = P_X \circ P_Y$.*

Proof. It follows from the structure of $HX(n)$ that, $N_r(\alpha)$ is composed of a segment of a Y-line followed by a segment of a Z-line. More precisely, for $1 \leqslant r \leqslant n-1$, we have

$$P_Y = \bigl\{(r-s, -(n-1)+r, -(n-1)+s), 0 \leqslant s \leqslant r\bigr\} \quad \text{and}$$
$$P_Z = \bigl\{(-t, -(n-1)+r-t, -(n-1)+r), 1 \leqslant t \leqslant r\bigr\}.$$

For $n \leqslant r \leqslant 2n-2$,

$$P_Y = \bigl\{((n-1)-s, -(n-1-r), -(2n-2-r)+s), 0 \leqslant s \leqslant n-1\bigr\} \quad \text{and}$$
$$P_Z = \bigl\{(-t, -(n-1-r)-t, -(n-1-r)), 1 \leqslant t \leqslant n-1\bigr\}.$$

Hence, for any r, $N_r(\alpha) = P_Y \circ P_Z$. Similarly $N_r(\beta) = P_X \circ P_Z$, $N_r(\gamma) = P_X \circ P_Y$ can be proved. See Figs. 5 and 6. □

Lemma 5. *For any r_1 and r_2, $N_{r_1}(\alpha) \cap N_{r_2}(\beta)$ is either empty or singleton or a line segment of a Z-line.*

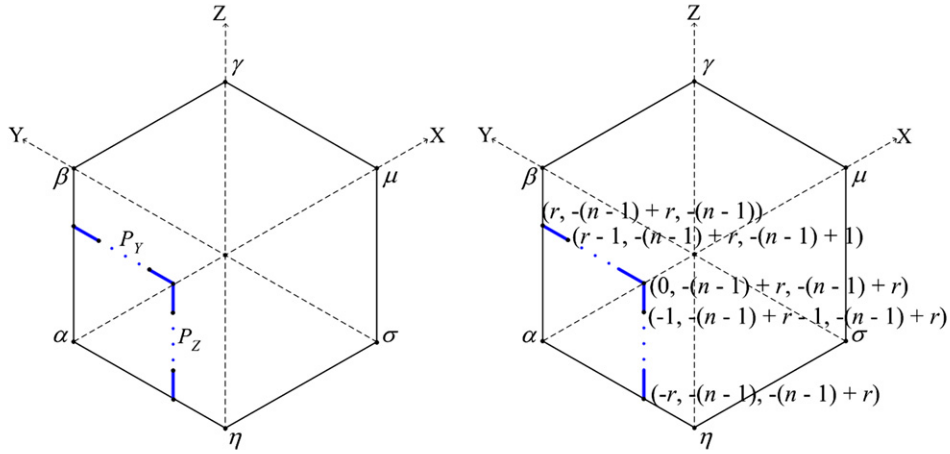


Fig. 5. $N_r(\alpha)$, when $r < n$.

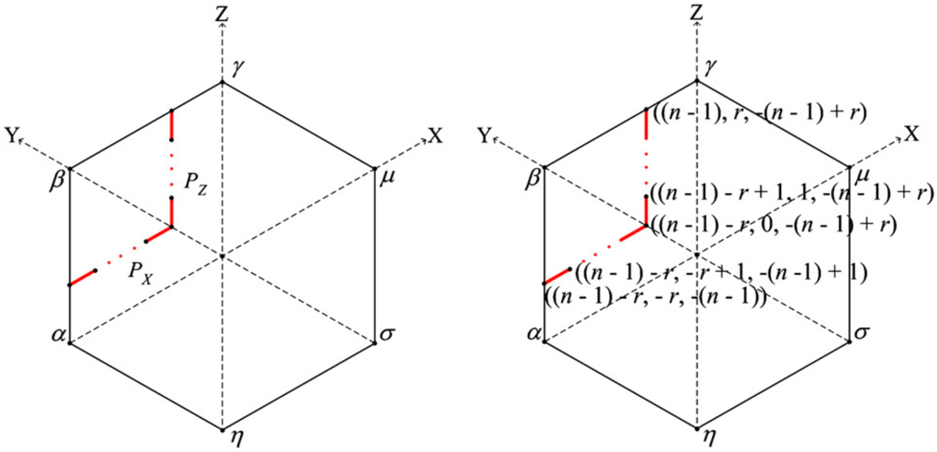


Fig. 6. $N_r(\beta)$, when $r < n$.

Proof. By Lemma 4, $N_{r_1}(\alpha) = P_Y \circ P_Z$, and $N_{r_2}(\beta) = P_X \circ P_Z$. Thus $N_{r_1}(\alpha) \cap N_{r_2}(\beta)$ is either empty or a singleton or a line segment of P_Z. □

Corollary 6. *Let $u = (x_1, y_1, z_1)$, $v = (x_2, y_2, z_2)$ be vertices of $HX(n)$ such that $x_1 \neq x_2$, $y_1 \neq y_2$, $z_1 \neq z_2$. Then $N_{r_1}(\alpha) \cap N_{r_2}(\beta)$ contains at most one of u and v.*

Proof. Suppose both $u, v \in N_{r_1}(\alpha) \cap N_{r_2}(\beta)$. By Lemma 5, $N_{r_1}(\alpha) \cap N_{r_2}(\beta)$ is a line segment of P_Z. This means that $z_1 = z_2$, a contradiction. □

Theorem 7. *$\{\alpha, \beta, \gamma\}$ is a metric basis for $HX(n)$.*

Proof. Let $u = (x_1, y_1, z_1)$, $v = (x_2, y_2, z_2)$ be any two vertices of $HX(n)$.

Case 1 ($x_1 = x_2$): Then $u, v \in P_X$. Hence $d(u, \alpha) \neq d(v, \alpha)$.
Case 2 ($y_1 = y_2$): Then $u, v \in P_Y$. Hence $d(u, \beta) \neq d(v, \beta)$.
Case 3 ($z_1 = z_2$): Then $u, v \in P_Z$. Hence $d(u, \gamma) \neq d(v, \gamma)$.
Case 4 ($x_1 \neq x_2$, $y_1 \neq y_2$, $z_1 \neq z_2$): Suppose that $d(u, \alpha) = d(v, \alpha)$. Then $u, v \in N_{r_1}(\alpha)$ for some r_1. We claim that $d(u, \beta) \neq d(v, \beta)$. Assume that $d(u, \beta) = d(v, \beta)$. Then $u, v \in N_{r_2}(\beta)$ for some r_2 and consequently $u, v \in N_{r_1}(\alpha) \cap N_{r_2}(\beta)$. This is a contradiction to Corollary 6. Thus $d(u, \beta) \neq d(v, \beta)$. □

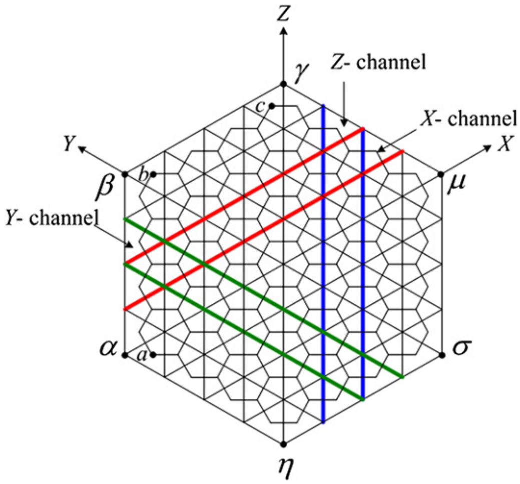


Fig. 7. Channels in $HX(5)$.

5. Minimum metric dimension of honeycomb networks

In this section we prove that the minimum metric dimension of the honeycomb network is 3. We denote the strip between two consecutive X-lines in $HX(n)$ as an X-channel and denote it by C_X. Similarly C_Y and C_Z are defined. See Fig. 7. An X-channel of $HC(n)$ behaves in the same way as an X-line of $HX(n)$. The reader may recall the property that for any two vertices (x_1, y_1, z_1) and (x_2, y_2, z_2) of an X-line, $x_1 = x_2$. In the same way for any two vertices (x_1, y_1, z_1) and (x_2, y_2, z_2) of an X-channel, $x_1 = x_2$. This is applicable for Y-channel and Z-channel too.

In what follows $N_r(v)$ denotes an r-neighborhood of a vertex v of $HC(n)$. The proofs of Lemmas 8 and 9 are similar to those of Lemma 5 and Corollary 6 respectively and hence they are left to the readers.

Lemma 8. *$N_r(a) \subset C_Y \circ C_Z$, $N_r(b) \subset C_X \circ C_Z$, and $N_r(c) \subset C_X \circ C_Y$ where a, b and c are honeycomb vertices as shown in Fig. 7.*

Lemma 9. *For any r_1 and r_2, $N_{r_1}(a) \cap N_{r_2}(b)$ is either empty or singleton or a pair of adjacent vertices or line segment of a Z-channel.*

It follows, as in Theorem 7, that $\{a, b, c\}$ is a metric basis for $HC(n)$. Thus we have

Theorem 10. *Let G be a honeycomb network $HC(n)$. Then $md(G) = 3$.*

6. Two hex derived networks

There are a number of open problems suggested for various interconnection networks. To quote Stojmenovic [12]: '*Designing new architectures remains an area of intensive investigation given that there is no clear winner among existing ones*'.

In this section we introduce two new architectures using the hexagonal and honeycomb networks. See Fig. 8. It is known that the bounded dual of $HX(n)$ is $HC(n-1)$. The vertex corresponding to each face (a triangle) of $HX(n)$ is placed in the face itself. Then the vertex is joined to the three vertices of the triangle. The resulting graph is a planar graph and it is called $HDN1$. This graph has $9n^2 - 15n + 7$ vertices and $27n^2 - 51n + 24$ edges. The diameter is $2n - 2$.

The second architecture is obtained from the union of $HX(n)$ and its bounded dual $HC(n-1)$ by joining each honeycomb vertex with the three vertices of the corresponding face of $HX(n)$. The resulting graph is non-planar and it is called $HDN2$. This graph has $9n^2 - 15n + 7$ vertices and $36n^2 - 72n + 36$ edges. The diameter is $2n - 2$.

These two architectures $HDN1$ and $HDN2$ have a few advantages over the hexagonal and honeycomb networks. The vertex-edge ratio of $HDN1$ and $HDN2$ are the same as that of hexagonal and honeycomb networks. However,

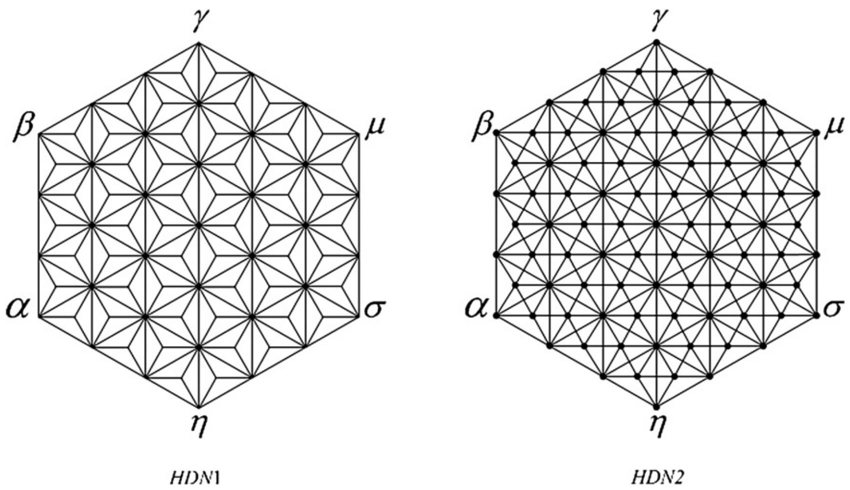


Fig. 8. The two hex derived networks.

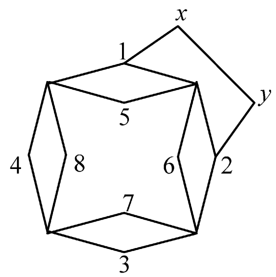


Fig. 9. A graph containing a butterfly as an induced subgraph.

both these hexagonal and honeycomb networks are simulated by these hex derived networks with no extra cost. This is possible since the hex derived networks contain hexagonal and honeycomb. *HDN*1 is planar and it accommodates in a given space more processors and wires than hexagonal and honeycomb. We conjecture that the minimum metric dimension of these hex derived networks *HDN*1 and *HDN*2 lies between 3 and 5. Thus we pose the following

Open Problem. Let G be *HDN*1 or *HDN*2. Then $3 \leqslant md(G) \leqslant 5$.

7. Conclusion

It is interesting to note that if H be an induced subgraph of a graph G then $md(H)$ need not be less than or equal to $md(G)$. For example the graph in Fig. 9 has metric dimension 3, but it has a two-dimensional butterfly as an induced subgraph which has metric dimension 4.

However a symmetrical network such as hexagonal and honeycomb networks satisfy a nice hereditary property in the sense that any sub hexagonal or a sub honeycomb network has the same minimum metric dimension as the parent network. This result helps us to provide a lower bound for $md(HX(n))$. Higher dimensional hexagonal and honeycomb networks, as well as the corresponding torii are interconnection networks for which the MMD problem is open.

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