Nilpotent symmetric Jacobian matrices and the Jacobian conjecture

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Abstract

Let $H : \mathbb{C}^n \to \mathbb{C}^n$ be a polynomial map such that the Jacobian $\mathcal{J}H$ of $H$ is nilpotent and symmetric. The symmetric dependence problem, $SDP(n)$, asks whether the rows of the matrix $\mathcal{J}H$ are dependent over $\mathbb{C}$. We show that if $SDP(r)$ has an affirmative answer for all $r \leq n$, then the Jacobian conjecture holds for all $F : \mathbb{C}^n \to \mathbb{C}^n$ of the form $F = x + H$ with $\mathcal{J}H$ nilpotent and symmetric. As a consequence, we deduce the main result of (J. Pure Appl. Algebra, 189/1–3, 123–133), which asserts that the Jacobian conjecture holds for all polynomial maps of the form $F = x + H$, with $\mathcal{J}H$ nilpotent, symmetric and homogeneous, and $n \leq 4$.

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0. Introduction

Write $\mathcal{J}F$ for the Jacobian of a polynomial map $F : \mathbb{C}^n \to \mathbb{C}^n$. The Jacobian conjecture claims that $F$ is an invertible polynomial map in case $\det \mathcal{J}F \in \mathbb{C}^*$. It was shown in [4] that in case $n \leq 4$, the Jacobian conjecture holds for all polynomial maps $F : \mathbb{C}^n \to \mathbb{C}^n$ of the form $F = x + H$, where $H$ is homogeneous and $\mathcal{J}H$ is nilpotent and symmetric. Let $\mathcal{J}f$ be the matrix
defined by
\[
\mathcal{H} f = \begin{pmatrix}
\frac{\partial^2}{\partial x_1 \partial x_1} f & \frac{\partial^2}{\partial x_1 \partial x_2} f & \cdots & \frac{\partial^2}{\partial x_1 \partial x_n} f \\
\frac{\partial^2}{\partial x_2 \partial x_1} f & \frac{\partial^2}{\partial x_2 \partial x_2} f & \cdots & \frac{\partial^2}{\partial x_2 \partial x_n} f \\
\vdots & \vdots & & \vdots \\
\frac{\partial^2}{\partial x_n \partial x_1} f & \frac{\partial^2}{\partial x_n \partial x_2} f & \cdots & \frac{\partial^2}{\partial x_n \partial x_n} f
\end{pmatrix}.
\]

The main ingredient in the proof is a result due to Gordan and Nöther in [5], which asserts the following: if \( n \leq 4 \) and \( h \in \mathbb{C}[x_1, x_2, \ldots, x_n] \) is a homogeneous polynomial such that \( \det \mathcal{H} h = 0 \), then \( h \) is degenerate, i.e. there exists a \( T \in \text{GL}_n(\mathbb{C}) \) such that \( h(Tx) \in C[x_1, x_2, \ldots, x_{n-1}] \).

In this paper we generalize the main result of [4] to the \( n \)-dimensional case. More precisely, we show that if \( F \) is of the form \( F = x + H \) with \( \mathcal{J} H \) nilpotent and symmetric (\( H \) does not need to be homogeneous), then \( F \) is invertible, provided a certain dependence problem (\( \text{SDP}(n) \) in Section 1) has an affirmative answer. Since the Gordan–Nöther theorem implies that the homogeneous dependence problem (\( \text{HSDP}(n) \) in Section 1) has an affirmative answer for \( n \leq 4 \) (Corollary 1.3) and \( \text{SDP}(n) \) has an affirmative answer for \( n \leq 2 \) (Proposition 1.1), our main theorem (Theorem 2.1) implies the main result of [4].

The interest of studying the symmetric case comes from the fact that in [3], the authors have reduced the Jacobian conjecture to this case.

1. The symmetric dependence problem

Throughout this paper \( K \) denotes a field of characteristic zero and \( K[x] = K[x_1, x_2, \ldots, x_n] \) is the polynomial ring in \( n \) indeterminates over \( K \). In search of the Jacobian conjecture the following problem arose naturally (see [7, Conjecture 1, p. 80], [8, Conjecture B, p. 135], [9, Conjecture 11.3], [1] and [2, 7.1.7]).

**Dependence problem** \( \text{DP}(n) \). Let \( H = (H_1, H_2, \ldots, H_n) \in K[x]^n \) such that \( \mathcal{J} H \) is nilpotent. Are the rows of \( \mathcal{J} H \) dependent over \( K \)?

It is not difficult to see that, in case \( H_i(0) = 0 \) for all \( i \), the dependence of the rows of \( \mathcal{J} H \) is equivalent to the linear dependence of the polynomials \( H_1, H_2, \ldots, H_n \) over \( K \).

Due to the embedding lemma (Lefschetz principle) (see [2, Lemma 1.1.13]), we only need to examine the case \( K = \mathbb{C} \) in the above and subsequent dependence problems.

In case \( n \leq 2 \), the dependence problem has an affirmative answer, however if \( n \geq 3 \) then there are counterexamples (see [2, Theorem 7.1.7]). On the other hand, if we
additionally assume that each $H_i$ is either zero or homogeneous of a fixed degree $d \geq 1$, then the corresponding problem is still open for all $n \geq 3$:

**Homogeneous dependence problem** $HDP(n)$. Let $H = (H_1, H_2, \ldots, H_n) \in K[x]^n$ be homogeneous of degree $d \geq 1$ such that $\mathcal{J}H$ is nilpotent. Are the rows of $\mathcal{J}H$ dependent over $K$?

In fact a highly non-trivial result obtained by Hubbers in [6] (see also [2, Theorem 7.1.2]) completely classifies all such maps $H$ in case $n=4$ and $d=3$. From this result it follows that the homogeneous dependence problem has an affirmative answer in this case, see [2, Corollary 7.1.4].

In this section we discuss the dependence problem for symmetric nilpotent Jacobian matrices. So let $F=(F_1, F_2, \ldots, F_n) \in K[x]^n$ and assume that $\mathcal{J}F$ is a symmetric matrix. Then $F=(\tilde{f})^\top$ for some $\tilde{f} \in K[x]$ (see for example [2, Lemma 1.3.53]). Consequently, a symmetric Jacobian matrix is of the form

$$
\mathcal{H}f = \begin{pmatrix}
\frac{\partial^2}{\partial x_1 \partial x_1} f & \frac{\partial^2}{\partial x_2 \partial x_1} f & \cdots & \frac{\partial^2}{\partial x_n \partial x_1} f \\
\frac{\partial^2}{\partial x_1 \partial x_2} f & \frac{\partial^2}{\partial x_2 \partial x_2} f & \cdots & \frac{\partial^2}{\partial x_n \partial x_2} f \\
\vdots & \vdots & \ddots & \vdots \\
\frac{\partial^2}{\partial x_1 \partial x_n} f & \frac{\partial^2}{\partial x_2 \partial x_n} f & \cdots & \frac{\partial^2}{\partial x_n \partial x_n} f
\end{pmatrix}.
$$

The above matrix $\mathcal{H}f$ is called the Hessian of $f$. Observe that $\mathcal{H}f = \mathcal{H}\tilde{f}$, where $\tilde{f}$ is obtained from $f$ by subtracting all monomials of degree $\leq 1$. So $\tilde{f}$ has only monomials of degree $\geq 2$. Such a polynomial is called reduced. Furthermore, we call $\tilde{f}$ the reduced part of $f$. Of course, the reduced part of a reduced polynomial is the polynomial itself.

Formulating the (homogeneous) dependence problem for symmetric Jacobian matrices then gives:

**Symmetric dependence problem** $SDP(n)$. Let $h \in K[x_1, x_2, \ldots, x_n]$ such that $\mathcal{H}h$ is nilpotent. Are the rows of $\mathcal{H}h$ dependent over $K$?

**Homogeneous symmetric dependence problem** $HSDP(n)$. Let $h \in K[x_1, x_2, \ldots, x_n]$ be homogeneous of degree $d \geq 2$ such that $\mathcal{H}h$ is nilpotent. Are the rows of $\mathcal{H}h$ dependent over $K$?

The following proposition is an immediate consequence of $DP(2)$, but a direct proof is easier and gives some ideas about nilpotent Hessians.

**Proposition 1.1.** $SDP(2)$ has an affirmative answer.
Proof. Let $M \in \text{Mat}_2(K[x])$ be symmetric and nilpotent ($M$ does not need to be a Hessian matrix). Then $M^2 = 0$. So the first row $v^1$ of $M$ satisfies $v^1 M = 0$. Since $v$ is the first column of $M$, $v^1 \cdot v = 0$. In other words, $v$ is orthogonal to itself with respect to the common bilinear form $(a, b) = \sum_{i=1}^{n} a_i b_i$ (here $n = 2$). Such a vector is called isotropic.

Since $v$ is isotropic, $v_2^2 = -v_1^2$, i.e. $v_2 = \pm iv_1$, where $i = \sqrt{-1}$. Consequently, $(1, \pm i) \cdot M = 0$. In particular, $i \in K$ or $M = 0$. \qed

One can easily see that for $j \geq r/2$ and symmetric $M \in \text{Mat}_n(K[x])$ with $M^r = 0$ and $M^{r-1} \neq 0$, all rows of $M^j$ are orthogonal to each other. In particular, all rows of $M^j$ are isotropic. Furthermore, all nonzero rows of $M^{r-1}$ are eigenvectors of $M$. Isotropic vectors seem to play an important role in the theory of nilpotent Hessians.

Now we relate the $\text{HSDP}(n)$ to the Gordan–Nöther theorem mentioned in the introduction. Let $f \in K[x]$ and $T \in \text{GL}_n(K)$. Put $f \circ T = f(Tx)$. Then it is well-known that

$$\mathcal{H}(f \circ T) = T^\dagger (\mathcal{H} f)|_{T^\dagger} T,$$

where $M|_{T^\dagger}$ is the matrix with entries $M_{ij}(Tx)$.

We call $f$ degenerate if there exists a $T \in \text{GL}_n(K)$ such that $f \circ T \in K[x_1, x_2, \ldots, x_{n-1}]$.

Proposition 1.2. Let $f \in K[x_1, x_2, \ldots, x_n]$. Then the following statements are equivalent.

(i) The rows of $\mathcal{H} f$ are dependent over $K$.
(ii) The columns of $\mathcal{H} f$ are dependent over $K$.
(iii) There exists a nonzero $v \in K^n$ such that $\mathcal{H} f \cdot v = 0$.
(iv) There exists a $T \in \text{GL}_n(K)$ such that the last column of $\mathcal{H}(f \circ T)$ equals zero.
(v) The reduced part $\tilde{f}$ of $f$ is degenerate, i.e. there exists a $T \in \text{GL}_n(K)$ such that $\tilde{f} \circ T \in K[x_1, x_2, \ldots, x_{n-1}]$.

Moreover, the $T \in \text{GL}_n(K)$ for which (iv) holds match those for which (v) holds.

Proof. (iii) is a reformulation of (ii). Further, (i) and (ii) are equivalent, since $\mathcal{H} f$ is symmetric. So it suffices to show (iii) $\Rightarrow$ (iv), (iv) $\Rightarrow$ (v), and (v) $\Rightarrow$ (iii).

First assume (iii). Extend $v$ to a $T \in \text{GL}_n(K)$ such that $v$ is the last column of $T$. Then the last column of $\mathcal{H} f \cdot T$ equals $\mathcal{H} f \cdot v = 0$. From (1), (iv) now follows.

Next assume (iv). Since $\mathcal{H}(f \circ T)$ is symmetric, the last row of $\mathcal{H}(f \circ T)$ is zero. So $\partial f / \partial x_i \partial f / \partial x_n(f \circ T) = 0$ for all $i$. Since char$K = 0$, it follows that $\partial f / \partial x_n(f \circ T) \in K$. Notice that the reduced part of $f \circ T$ is just $\tilde{f} \circ T$. Consequently, $\partial f / \partial x_n(\tilde{f} \circ T) = 0$ and (v) follows.

Finally assume (v). Since $\tilde{f} \circ T \in K[x_1, x_2, \ldots, x_{n-1}]$, the last column of $\mathcal{H}(f \circ T) = \mathcal{H}(\tilde{f} \circ T)$ equals zero. So the $n$th standard basis vector $e_n$ satisfies $\mathcal{H}(f \circ T) \cdot e_n = 0$. From (1), it follows that $\mathcal{H} f \cdot T e_n = 0$ and (iii) follows. \qed

Corollary 1.3. $\text{HSDP}(n)$ has an affirmative answer for all $n \leq 4$. 

Proof. Suppose that \( \mathcal{H} \) is nilpotent. Then \( \det \mathcal{H} = 0 \) in particular. By the theorem of Gordan and Nöther mentioned in the introduction, \( h \) is degenerate. If \( \deg h \leq 1 \), then the reduced part of \( h \) equals zero. If \( \deg h \geq 2 \), then \( h \) is reduced, for \( h \) is homogeneous. In either case, the reduced part of \( h \) is degenerate. Now apply Proposition 1.2, (v) \( \Rightarrow \) (i). \( \square \)

In the remainder of this section, we assume that \( K \) is algebraically closed (in fact it is sufficient that \( K \) is closed under taking square roots).

Suppose that \( \mathcal{H} \) is nilpotent and \( T \) is orthogonal, i.e. \( T^t T = 1 \). From (1), it follows that \( \mathcal{H}(h \circ T) \) is nilpotent as well. So an interesting question is whether \( T \) can always be chosen orthogonal in the definition of degenerate, in which case we call \( h \) orthogonally degenerate. The answer is no. Take \( h = (x_1 + ix_2)^2 \) and suppose that \( h \circ T \in K[x_1] \). Then \( \mathcal{H}(h \circ T) \) is of the form

\[
\begin{pmatrix}
a & 0 \\
0 & 0
\end{pmatrix}
\]

So \( \mathcal{H}(h \circ T) \) cannot both be nilpotent and have rank 1.

We call \( f \) isotropically degenerate if there is an orthogonal \( T \in GL_n(K) \) such that \( f \circ T \in K[x_1, x_2, \ldots, x_n] \). Clearly, the above \( h \) with \( n = 2 \) is isotropically degenerate (take \( T = 1 \)).

The following lemma gives a class of \( f \in K[x] \) that are orthogonally degenerate.

**Lemma 1.4.** Let \( f \in K[x_1, x_2, \ldots, x_n] \) be reduced such that \( \mathcal{H} f \cdot v = 0 \) for some non-isotropic \( v \). Then \( f \) is orthogonally degenerate.

**Proof.** Replacing \( v \) by \( v/\langle v, v \rangle \), we may assume that \( \langle v, v \rangle^{1/2} = 1 \). Then, using the Gram–Schmidt process, we can find an orthogonal \( T \in GL_n(K) \) such that \( v \) is the last column of \( T \), i.e.

\[
v = T \cdot e_n.
\]

So \( \mathcal{H} f \cdot T \cdot e_n = \mathcal{H} f \cdot v = 0 \). From (1), it follows that

\[
\mathcal{H}(f \circ T) \cdot e_n = 0.
\]

So the last column of \( \mathcal{H}(f \circ T) \) equals zero. Now apply proposition 1.2, (iv) \( \Rightarrow \) (v). \( \square \)

The following lemma, which gives a class of \( f \in K[x] \) that are isotropically degenerate, is harder to prove than the above lemma. A problem is that (iv) \( \Rightarrow \) (v) of Proposition 1.2 cannot be applied directly.

**Lemma 1.5.** Let \( f \in K[x_1, x_2, \ldots, x_n] \) be reduced such that \( \mathcal{H} f \cdot v = 0 \) for some isotropic \( v \neq 0 \). Then \( f \) is isotropically degenerate.

**Proof.** Since permutation matrices are orthogonal, we may assume that \( v_1 \neq 0 \). Replacing \( v \) by \( v/v_1 \), we may assume that \( v_1 = 1 \). The first standard basis vector \( e_1 \) and the
vector \( \tilde{v} = -i(0, v_2, v_3, \ldots, v_n) \) satisfy \( \langle e_1, \tilde{v} \rangle = 0, \langle e_1, e_1 \rangle = 1 \), and \( \langle \tilde{v}, \tilde{v} \rangle = \langle e_1, e_1 \rangle - \langle v, v \rangle = 1 \).

By the Gram–Schmidt process there exists an orthogonal \( T \in GL_n(K) \) such that the last two columns of \( T \) are \( e_1 \) and \( \tilde{v} \), in this order. So

\[
T \cdot (e_{n-1} + ie_n) = (Te_{n-1} + iTe_n) = (e_1 + i\tilde{v}) = v.
\]

and therefore, \( \mathcal{H} f \cdot T \cdot (e_{n-1} + ie_n) = \mathcal{H} f \cdot v = 0 \).

In order to prove this lemma, we define \( S \in GL_n(K) \) by

\[
S(x) = (x_1, x_2, \ldots, x_{n-2}, x_{n-1} + ix_n, -ix_n).
\]

Then \( S^{-1}(x) = (x_1, x_2, \ldots, x_{n-2}, x_{n-1} + x_n, iix_n) \). In particular, \( S^{-1} \cdot e_n = e_{n-1} + ie_n \).

Therefore, it follows from (3) that

\[
T \cdot S^{-1} \cdot e_n = T \cdot (e_{n-1} + ie_n) = v.
\]

Consequently, \( \mathcal{H} f \cdot T \cdot S^{-1} \cdot e_n = \mathcal{H} f \cdot v = 0 \). From (1), it follows that \( g = f \circ T \circ S^{-1} \) satisfies \( \mathcal{H}g \cdot e_n = 0 \) and therefore

\[
g \in K[x_1, x_2, \ldots, x_{n-1}].
\]

Consequently, \( f \circ T = g \circ S \) is a polynomial in the first \( n - 1 \) coordinates of \( Sx \), i.e. \( f \circ T \in K[x_1, x_2, \ldots, x_{n-2}, x_{n-1} + ix_n] \). \( \square \)

The following proposition claims that a degenerate \( f \) is either orthogonally or isotropically degenerate.

**Proposition 1.6.** Let \( f \in K[x_1, x_2, \ldots, x_n] \) be degenerate (not necessarily reduced). Say that \( f \circ T \in K[x_1, x_2, \ldots, x_{n-1}] \) with \( T \in GL_n(K) \). Let \( v = Te_n \) be the last column of \( T \).

1. If \( v \) is not isotropic, then \( f \) is orthogonally degenerate.
2. If \( v \) is isotropic, then \( f \) is isotropically degenerate.

**Proof.** Write \( \tilde{f} \) for the reduced part of \( f \). Suppose first that \( v \) is not isotropic. Assume without loss of generality that \( \langle v, v \rangle = 1 \). From Lemma 1.4, it follows that there is an orthogonal \( \tilde{T} \in GL_n(K) \) such that \( \tilde{f} \circ \tilde{T} \in K[x_1, x_2, \ldots, x_{n-1}] \). Without loss of generality, we may assume that \( f(0) = 0 \). Since \( f - \tilde{f} \) is linear, it follows that

\[
(f \circ \tilde{T}) = ((f - \tilde{f}) \circ \tilde{T}) + (\tilde{f} \circ \tilde{T})
\]

\[
= ((f - \tilde{f}) \circ (\tilde{T} - T)) + ((f - \tilde{f}) \circ T) + (\tilde{f} \circ \tilde{T})
\]

\[
= ((f - \tilde{f}) \circ (\tilde{T} - T)) + (f \circ T) - (\tilde{f} \circ T) + (\tilde{f} \circ \tilde{T}).
\]

According to (2), we may assume that the last column of \( \tilde{T} - T \) equals zero. So \( \tilde{T} - T \in K[x_1, x_2, \ldots, x_{n-1}]^\ell \). Since \( f \circ T \in K[x_1, x_2, \ldots, x_{n-1}] \) by assumption and \( \tilde{f} \circ T \) is the reduced part of \( f \circ T \), it follows from (7) that \( f \circ \tilde{T} \in K[x_1, x_2, \ldots, x_{n-1}] \).
Suppose next that $v$ is isotropic. Assume without loss of generality that $v_1 = 1$. From Lemma 1.5, it follows that there is an orthogonal $\tilde{T} \in GL_n(K)$ such that $\hat{f} \circ \tilde{T} \in K[x_1, x_2, \ldots, x_{n-1} + ix_n]$. Take $S$ as in the proof of Lemma 1.5. Similar to the case that $v$ is orthogonal, it follows from (6) that $g = f \circ \tilde{T} \circ S^{-1} \in K[x_1, x_2, \ldots, x_{n-1}]$ and therefore $f \circ \tilde{T} = g \circ S \in K[x_1, x_2, \ldots, x_{n-1} + ix_n]$. \qed

To describe the next results, it is convenient to introduce some new notation. Let $x = x_1, x_2, \ldots, x_n, M \in Mat_n(K[x])$, and $f \in K[x]$. Then write $x_*$ for $x_1, x_2, \ldots, x_{n-1}$, $M_*$ for $(M_{ij})_{1 \leq i, j \leq n-1} \in Mat_{n-1}(K[x])$, and $f_*$ for $f$, viewed as polynomial in $x_*$ over $K[x_*]$. Similarly, we define $x_{**} = x_1, x_2, \ldots, x_{n-2}$, $M_{**} = (M_{ij})_{1 \leq i, j \leq n-2}$, etc. Also, we define $\mathcal{H}_*, \mathcal{H}_{**}, \mathcal{H}_*, \mathcal{H}_{**}$ in a similar way. So we have for example $\mathcal{H}_* f_* = (\mathcal{H} f)_*$.

Suppose that $f \in K[x]$ such that $\mathcal{H} f \cdot v = 0$. Take $T$ as in Proposition 1.6 and write $g = f \circ T$. If $v$ is not isotropic, then $g \in K[x_*]$ and therefore

$$
\mathcal{H} \ g = \begin{pmatrix}
0 \\
\mathcal{H}_* g_* \\
\vdots \\
0 \\
\cdots \\
0
\end{pmatrix}.
$$

Consequently

$\mathcal{H} g$ is nilpotent, if and only if $\mathcal{H}_* g_*$ is nilpotent

(8)
in case $v$ is not isotropic.

If $v$ is isotropic, then $g \in K[x_{**}, x_{n-1} + ix_n]$ and therefore

$$
\mathcal{H} \ g = \begin{pmatrix}
\mathcal{H}_{**} g_{**} & w & i w \\
w^t & a & i a \\
i w^t & i a & -a
\end{pmatrix}
$$

for some $w \in K[x]^{n-2}$ and $a \in K[x]$. In order to obtain the ‘isotropic analogon’

$\mathcal{H} g$ is nilpotent, if and only if $\mathcal{H}_{**} g_{**}$ is nilpotent

(9)
of (8), we need the following lemma.

**Lemma 1.7.** Let $R$ be a commutative ring with $i = \sqrt{-1}$ and $M \in Mat_n(R)$ symmetric and of the form

$$
M = \begin{pmatrix}
M_{**} & w & i w \\
w^t & a & i a \\
i w^t & i a & -a
\end{pmatrix}
$$

with $w \in R^{n-2}$ and $a \in R$. Then $M$ is nilpotent, if and only if $M_{**}$ is nilpotent.
Proof. Suppose by induction that \( M^r \) is of the form

\[
M^r = \begin{pmatrix}
(M_{ss})^r & u & iu \\
u^t & b & ib \\
iu^t & ib & -b
\end{pmatrix}.
\]

(10)

Then,

\[
M^{r+1} = M \cdot M^r = \begin{pmatrix}
(M_{ss})^{r+1} & M_{ss}u & iM_{ss}u \\
(M_{ss}u)^t & \langle w, u \rangle & i\langle w, u \rangle \\
i(M_{ss}u)^t & i\langle w, u \rangle & -\langle w, u \rangle
\end{pmatrix}.
\]

So \( M^r \) is of the form (10) for all \( r \).

If \( M \) is nilpotent, then \( M_{ss} \) is nilpotent as well, since \( (M_{ss})^r \) is a submatrix of \( M^r \) for all \( r \). Hence assume that \( M_{ss} \) is nilpotent. Take \( r \) such that \( (M_{ss})^r = 0 \). Then

\[
M^r = \begin{pmatrix}
\emptyset & u & iu \\
u^t & b & ib \\
iu^t & ib & -b
\end{pmatrix}
\]

and one can easily verify that \( M^{2r} = (M^r)^2 = 0 \). So \( M \) is nilpotent. □

2. The main result

The following result is the main theorem of this paper. Again, we assume that \( K \) is a field of characteristic zero.

Theorem 2.1. Let \( n \geq 1 \) and suppose that \( H \in K[x]^n \) such that \( JH \) is nilpotent and symmetric.

1. If \( SDP(p) \) has an affirmative answer for all \( p \leq n \), then \( x + H \) is invertible.
2. If \( H \) is homogeneous, \( SDP(p) \) has an affirmative answer for all \( p \leq n - 2 \), and also \( HSDP(n-1) \) and \( HSDP(n) \) have an affirmative answer, then \( x + H \) is invertible.

Proof. In case \( n = 1 \), \( JH = 0 \) and therefore \( H \) is constant and \( x - H \) is the inverse of \( x + H \). So assume that \( n \geq 2 \). Since \( JH \) is symmetric, we have \( JH = Hh \) for some reduced \( h \). If \( H(0) = 0 \), then \( (JH)^t = H \) as well. Since translations are invertible, we assume that \( H(0) = 0 \).

We shall show the following assertions.

(i) If \( h \) is degenerate and (homogeneous and \( H \)) \( SDP(n-1) \) has an affirmative answer, then \( h \) is isotropically degenerate.
(ii) If \( h \in K[x_{ss}x_{n-1} + iu_n] \), then \( x + H \) is invertible over \( K \), if and only if \( x_{ss} + H_{ss} \) is invertible over \( K(x_{n-1} + iu_n) \).
Suppose first that these assertions hold. Since $SDP(n)$ resp. $HSDP(n)$ is assumed to have an affirmative answer, it follows from (i) and Proposition 1.2 that $h$ is isotropically degenerate. Suppose that conclusion 1. resp. 2. of this theorem does not hold. Take $n$ minimal such that $H$ satisfies the conditions of this theorem, but $x + H$ is not invertible. Since $h$ is isotropically degenerate, there is an orthogonal $T \in GL_n(K)$ such that $h \circ T \in K[x_{ss}, x_{n-1} + ix_n]$.

If $n \geq 3$, then it follows from (9) and (ii) that there is a $G \in K(y)[x_{ss}]$ such that $JG$ is nilpotent and symmetric, but $x_{ss} + G$ is not invertible over $K(y)$. Since $G$ satisfies the conditions of this theorem as well as $H$, we have a contradiction, so $n \leq 2$. The case $n = 1$ is trivial, so assume that $n = 2$. Take $G = (H_1, H_2, 0)$. Then $x + H$ is invertible, if and only if $(x, x_3) + G$ is invertible. Furthermore, the invertibility of $G$ reduces to the case $n = 1$ of this theorem, which is trivially satisfied.

First, we show assertion (i). Suppose that $h$ is (homogeneous and) not isotropically degenerate. Since $h$ is assumed to be degenerate, $h$ is orthogonally degenerate according to Proposition 1.6. Take $T$ orthogonal such that $h \circ T \in K[x^*]$. Since $g = h \circ T$ is reduced, it follows from (8) and the fact that $(H)SDP(n - 1)$ has an affirmative answer that $g$ is degenerate, so $g$ is either isotropically or orthogonally degenerate according to Proposition 1.6. If $g$ is isotropically degenerate, then $h$ is isotropically degenerate as well. If, on the other hand, $g$ is orthogonally degenerate, then there is an orthogonal $S \in GL_n(K)$ such that $g \circ S \in K[x_{ss}]$. Therefore $h \circ (T \circ S) \in K[x_{ss}] \subseteq K[x_{ss}, x_{n-1} + ix_n]$ and $h$ is isotropically degenerate.

Next, we show assertion (ii). Suppose that $h \in K[x_{ss}, x_{n-1} + ix_n]$, say that $h = g(x_{ss}, x_{n-1} + ix_n)$ with $g \in K[x^*]$. Put $H_{ss} = (J_{ss}h)^\gamma$. Then

$$P_1 := x + H$$

$$= \left( x_{ss} + H_{ss}, x_{n-1} + \left( \frac{\partial}{\partial x_{n-1}} g \right)(x_{ss}, x_{n-1} + ix_n) \right)$$

is invertible, if and only if

$$P_2 := (x_{ss}, x_{n-1} + ix_n, x_n) \circ P_1$$

$$= \left( x_{ss} + H_{ss}, x_{n-1} + ix_n, x_n + i \left( \frac{\partial}{\partial x_{n-1}} g \right)(x_{ss}, x_{n-1} + ix_n) \right)$$

is invertible. Put $G_{ss} = J_{ss}g$, then $G_{ss} = H_{ss}(x_{ss}, x_{n-1} - ix_n, x_n)$. So $P_2$ is invertible, if and only if

$$P_3 := P_2 \circ (x_{ss}, x_{n-1} - ix_n, x_n)$$

$$= \left( x_{ss} + G_{ss}, x_{n-1}, x_n + i \left( \frac{\partial}{\partial x_{n-1}} g \right)(x_{ss}, x_{n-1}) \right)$$
is invertible. $P_3$ is invertible, if and only
\[ P_4 := \left( x^*, x_n - i \left( \frac{\partial}{\partial x_{n-1}} g \right)(x^*) \right) \circ P_3 = (x^* + G^*^*, x_{n-1}, x_n) \]
is invertible. Since $G^*^* \in K[x^*]^{n-2}$, it follows that $P_4$ is invertible, if and only if $x^* + G^*^*$ is invertible over $K[x_{n-1}]$, i.e. if and only if $x^* + H^*^*$ is invertible over $K[x_{n-1} + IX_n]$. By [2, Lemma 1.1.8], this last statement is equivalent to the assertion that $x^* + H^*^*$ is invertible over $K(x_{n-1} + iX_n)$. This gives assertion (ii). □

References