On the symmetric hyperspace of the circle

Naotsugu Chinen\textsuperscript{a}, Akira Koyama\textsuperscript{b,}\textsuperscript{*}

\textsuperscript{a} Hiroshima Institute of Technology, Saeki-ku, Hiroshima 731-5193, Japan
\textsuperscript{b} Department of Mathematics, Faculty of Science, Shizuoka University, Suruga-ku, Shizuoka 422-8529, Japan

\section*{A R T I C L E   I N F O}

Article history:
Received 17 January 2008
Received in revised form 19 October 2009
Accepted 19 October 2009

Keywords:
Symmetric hyperspace
Symmetric product
Compactification
Dunce hat

\section*{A B S T R A C T}

By \(X(n), n \geq 1\), we denote the \(n\)-th symmetric hyperspace of a metric space \(X\) as the space of non-empty finite subsets of \(X\) with at most \(n\) elements endowed with the Hausdorff metric. In this paper we shall describe the \(n\)-th symmetric hyperspace \(S^1(n)\) as a compactification of an open cone over \(\Sigma D^{n-2}\), here \(D^{n-2}\) is the higher-dimensional dunce hat introduced by Andersen, Marjanović and Schori (1993) \cite{2} if \(n\) is even, and \(D^{n-2}\) has the homotopy type of \(S^{n-2}\) if \(n\) is odd (see Andersen et al. (1993) \cite{2}). Then we can determine the homotopy type of \(S^1(n)\) and detect several topological properties of \(S^1(n)\).

\(©\) 2010 Elsevier B.V. All rights reserved.

\section{1. Introduction}

As an interesting construction in topology, Borsuk and Ulam \cite{3} introduced the \(n\)-th symmetric hyperspace of a metric space \(X\), denoted by \(X(n)\). Namely \(X(n)\) is the space of non-empty finite subsets of \(X\) with at most \(n\) elements endowed with the Hausdorff metric. They showed that for \(n = 1, 2, \text{ or } 3\), \(I(n)\) is homeomorphic to \(I^n\), denoted by \(I^n \approx \mathbb{I}^n\), and for \(n \geq 4\), \(I(n)\) cannot be embedded into \(\mathbb{R}^n\). Following them, many authors have considered topological structures of \(X(n)\).

For example, Molski \cite{10} showed that \(\mathbb{I}^2(2) \approx \mathbb{I}^2\) and for \(n \geq 3\) neither \(I^n(2)\) nor \(I^n(2)\) can be embedded into \(\mathbb{R}^{2n}\). In this direction Schori \cite{12} investigated a characterization of spaces of the type \(\mathbb{I}^n(n)\) by using suitable equivalence relations. In particular, he gave a description of \(I(n)\) as the product space \(c(D^{n-2}) \times \mathbb{I}\), where \(D^{n-2} = \{ A \in I(n) : 0, 1 \in A\}\) and \(c(Z)\) is the cone over a topological space \(Z\). Moreover Andersen, Marjanović and Schori \cite{2} studied the spaces \(D^n\) and showed that \(D^2\) is homeomorphic to the dunce hat \cite{14}, in general, \(D^{2n}\) is contractible but not collapsible, and \(D^{2n+1}\) has the same homotopy type of \(S^{2n+1}\). They call the spaces \(D^{2n}, n \geq 2\), higher-dimensional dunce hats.

Turning toward the spaces \(S^1(n)\), Wu \cite{13} proved that \(S^1(2n+1)\) has the same homology groups as the \((2n + 1)\)-sphere \(S^{2n+1}\) and \(H^0(S^1(2n)) = H^{2n-1}(S^1(2n)) = \mathbb{Z}\) and \(H^1(S^1(2n)) = 0\) if \(n \neq 0, 2n - 1\). In \cite{5} Bott corrected Borsuk’s statement \cite{4} and showed that \(S^1(3) \approx S^3\). In this paper we shall give a description of the spaces \(S^1(n)\) by using the identification spaces \(D^m\). Thus,

\textbf{Theorem 1.1.} For \(n \geq 2\) there exists a closed subset \(R\) of the \(n\)-th symmetric hyperspace \(S^1(n)\) which is homeomorphic to \(D^{n-1}\) such that \(S^1(n) \setminus R\) is homeomorphic to the open cone over \(\Sigma D^{n-2}\).

As its consequences we shall give an alternative proof of Bott’s theorem and show several interesting properties of \(S^1(m)\).

\textbf{Corollary 1.2.} For each \(n = 1, 2, \ldots, S^1(2n+1)\) has the same homotopy type of \(S^{2n+1}\) and \(S^1(2n)\) has the same homotopy type of \(S^{2n-1}\).

\textsuperscript{*} Corresponding author.

E-mail addresses: naochin@cc-it-hiroshima.ac.jp (N. Chinen), sakoyam@ipc.shizuoka.ac.jp (A. Koyama).

0166-8641/$ – see front matter \(©\) 2010 Elsevier B.V. All rights reserved.
doi:10.1016/j.topol.2010.07.012
Corollary 1.3. For \( n \geq 4 \), \( S^1(n) \) is not an \( n \)-manifold. Moreover there exists no embedding of any orientable closed \( n \)-dimensional manifold into \( S^1(n) \).

We note that the space \( X(n) \) is often called symmetric product or symmetric potency of a space \( X \), for example, by Borsuk–Ulam, Schori, and many people or by Borsuk and Bott, respectively. However, we have another type of symmetric products and the notation is widely known. Thus, given a topological space to avoid any confusion we introduce a new name to the space \( SP_n \). Hereby in order to avoid any confusion we introduce a new name to the space \( X(n) \) “\( n \)-th symmetric hyperspace”. We shall mention some relations between two notations.

By definitions it is clear that \( X(2) \) is homeomorphic to \( SP^2(X) \) for any space \( X \). It is a folklore that for every \( n \geq 2 \), \( SP^n(S^2) \) is homeomorphic to the complex projective space \( PC^n \) (compare with Molski [10]). Morton [11] gave a topological description of \( SP^n(S^1) \). Namely, if \( n \) is odd, \( SP^n(S^1) \) is homeomorphic to the product space \( S^1 \times \mathbb{R}^{n-1} \), and if \( n \) is even, \( SP^n(S^1) \) is the non-orientable \( \mathbb{R}^{n-1} \)-bundle over \( S^1 \). Hence \( SP^n(S^1) \) has the same homotopy type of \( S^1 \). Thereby we can see that, if \( n \geq 3 \), Corollary 1.2 shows that \( SP^n(S^1) \) is even homotopically different from \( S^1(n) \). On the other hand, Illanes and Nadler [8, Question 83-14] posed the problem: \textbf{Is the 2-sphere embeddable in the second symmetric product of a curve?} Recently Koyama, Krasinkiewicz and Spie\'{z} [9] showed that the \( n \)-sphere, \( n \geq 2 \), cannot be embedded into the \( n \)-th symmetric product \( SP^n(X) \) of any curve (\( = \) one-dimensional continuum) \( X \). Here we pose the following problem:

Problem 1.4. Is the \( n \)-sphere, \( n \geq 4 \), embeddable into the \( n \)-th symmetric hyperspace \( X(n) \) of a curve \( X \)?

2. Preliminaries

Notation 2.1. Let denote the set of all natural numbers, integers and real numbers by \( \mathbb{N}, \mathbb{Z} \) and \( \mathbb{R} \), respectively. Write \( I = [0, 1], \mathbb{R}_+ = [0, \infty), S^1_+ = \{ (x, y) \in \mathbb{R}^2: x^2 + y^2 = 1 \} \) and the \( n \)-th product of \( S^1 \) by \( T^n \). Define \( q : I \rightarrow S^1_+ \) by \( q(t) = (\cos 2\pi t, \sin 2\pi t) \) and \( q_n : I^n \rightarrow T^n \) by \( q_n(x_1, \ldots, x_0) = (q(x_1), \ldots, q(x_0)) \).

For convenience we specify the following symbols: For \( 0 \leq s < t \leq 1 \)

\[
\begin{align*}
\Delta^n &= \{ (x_1, \ldots, x_n) \in \mathbb{R}^n: x_1 \leq \cdots \leq x_n \}, \\
\widehat{\Delta}^n &= \{ x \in \Delta^n: 0 < x_1 < \cdots < x_n < 1 \}, \\
\Delta_1^n(s, t) &= \{ x \in \Delta^n: s \leq x_1, x_n \leq t \}, \\
\Delta_0^n(s, t) &= \{ x \in \Delta^n(s, t): x_1 = s \}, \\
\Delta_1^n(s, t) &= \{ x \in \Delta^n(s, t): x_n = t \}, \\
\Delta_{0,1}^n(s, t) &= \Delta_0^n(s, t) \cap \Delta_1^n(s, t) = \{ x \in \Delta^n(s, t): x_1 = s, x_n = t \}.
\end{align*}
\]

For \( 0 \leq t \leq 1/2, \alpha_t = 1/2 - t, \beta_t = 1/2 + t \in I \),

\[
\begin{align*}
a_0(t) &= (\alpha_t, \ldots, \alpha_t), & a_1(t) &= (\beta_t, \ldots, \beta_t) \in \Delta^n, \\
\Delta_0^n(t) &= \Delta_0^n(\alpha_t, \beta_t), & \Delta_1^n(t) &= \Delta_1^n(\alpha_t, \beta_t), \\
\Delta_1^n(t) &= \Delta_1^n(t) \cup \Delta_0^n(t) & \text{and } \Delta_{0,1}^n(t) &= \Delta_0^n(t) \cap \Delta_1^n(t).
\end{align*}
\]

(See Figs. 2.1.1 and 2.1.2.)
Definition 2.2. Define an equivalence relation \( \equiv \) in \( \Delta^2 \) generated by the followings: \((x_1, x_2) \equiv (x_1', x_2')\) if and only if \(x_1 = x_2 = 1\) and \(x_2 = x_1'\) or \(x_1 = 0\) and \(x_2 = x_1 = x_2' = x_1' = 0\). We say \(D^2 = \Delta^2 / \equiv\) the dunce hat which is a contractible 2-polyhedron. It is known that \(D^2\) is not collapsible but \(D^2 \times \mathbb{I}\) is collapsible (see Fig. 2.2.1). See \cite{14} for details.

Definition 2.3. Let \((X, d)\) be a metric space, \(2^X = \{A \subset X: A\) is a closed subset of \(X\}\) with the Hausdorff metric and \(n \in \mathbb{N}\). For \(n = 1, 2, \ldots\), the \(n\)-th symmetric hyperspace of \(X\) is defined by

\[
X(n) = \{ A \subset X: 1 \leq |A| \leq n \} \subset 2^X.
\]

Let \(p_{X(n)}: X^n \to X(n)\) be the projection. We write

\[
r_n = p_{S(0)} \circ (q_{n, \Delta^n}) : \Delta^n \to S^1(n), \quad r'_n = p_{I(n)} : \Delta^n \to \mathbb{I}(n),
\]

\[
D^n = r_{n+2}(\Delta^n_{0,1}(1/2)) \quad \text{and} \quad I^n_0(n+2) = r'_{n+2}(\Delta^n_{0,1}(1/2)).
\]

Lemma 2.4.

(1) Both \(r_n : \Delta^n \to S^1(n)\) and \(r'_n : \Delta^n \to I(n)\) are surjective.

(2) For every \(z \in \Delta^n\), \(r_n^{-1}(r_n(z))\) and \(r'_n^{-1}(r'_n(z))\) are degenerate.

(3) Let \(0 < s < t < 1\). Then, \(x \in \Delta^n_{0,1}(s, t)\) implies \(r_n^{-1}(r_n(x)) \subset \Delta^n_{0,1}(s, t)\) and \(r'_n^{-1}(r'_n(x)) \subset \Delta^n_{0,1}(s, t)\).

Lemma 2.5. For every \(n \in \mathbb{N}\), \(D^n\) is homeomorphic to \(I^n_0(n+2)\).

Proof. It suffices to show that for each point \(x = (0, x_2, \ldots, x_{n+1}, 1) \in \Delta^{n+2}_{0,1}(1/2)\)

\[
r^{-1}_{n+2}(r_{n+2}(x)) \cap \Delta^{n+2}_{0,1}(1/2) = r^{-1}_{n+2}(r'_{n+2}(x)) \cap \Delta^{n+2}_{0,1}(1/2).
\]

Take any \(x' = (0, x'_2, \ldots, x'_{n+1}, 1) \in r^{-1}_{n+2}(r_{n+2}(x)) \cap \Delta^{n+2}_{0,1}(1/2)\), and set \(A = \{0, x_2, \ldots, x_{n+1}, 1\}\), \(A' = \{0, x'_2, \ldots, x'_{n+1}, 1\}\) and \(k = |A| \geq 2\). Then, \(A = A'\). Thus, there exists \(y_i \in \mathbb{I} (i = 1, \ldots, k)\) such that \(0 = y_1 < y_2 < \cdots < y_{k-1} < y_k = 1\) and \(A = \{y_1, \ldots, y_k\}\). By definition \(r_n(x'') = [y_1, \ldots, y_k] = r_{n+2}(x)\), hence, \(r^{-1}_{n+2}(r_{n+2}(x)) \cap \Delta^{n+2}_{0,1}(1/2) \subset r^{-1}_{n+2}(r'_{n+2}(x)) \cap \Delta^{n+2}_{0,1}(1/2)\).

Take any \(x' = (0, x'_2, \ldots, x'_{n+1}, 1) \in r'^{-1}_{n+2}(r'_{n+2}(x)) \cap \Delta^{n+2}_{0,1}(1/2)\), and set \(A = \{0, x_2, \ldots, x_{n+1}, 1\}\), \(A' = \{0, x'_2, \ldots, x'_{n+1}, 1\}\) and \(k = |A| \geq 2\). Then, \(A = A'\). Thus, there exists \(y_i \in \mathbb{I} (i = 1, \ldots, k)\) such that \(0 = y_1 < y_2 < \cdots < y_{k-1} < y_k = 1\) and \(A = \{y_1, \ldots, y_k\}\). Then by definition \(r_{n+2}(x') = q(y_1), \ldots, q(y_{k-1})\) = \(r_{n+2}(x)\), hence, \(r'^{-1}_{n+2}(r'_{n+2}(x)) \cap \Delta^{n+2}_{0,1}(1/2) \supset r^{-1}_{n+2}(r'_{n+2}(x)) \cap \Delta^{n+2}_{0,1}(1/2)\). \(\square\)

Thus, \(D^0\) is a point, \(D^1\) is homeomorphic to \(S^1\) by \([2, p. 9]\) and \(D^2\) is homeomorphic to the dunce hat by \([2, Theorem \, 2.1]\). Moreover, by \([2, Theorem \, 3.4]\), for every \(n \in \mathbb{N} \cup \{0\}\) \(D^{2n}\) is contractible and \(D^{2n+1}\) has the same homotopy type of the \((2n+1)\)-sphere \(S^{2n+1}\).

Lemma 2.6. For every \(n \in \mathbb{N} \) with \(n \geq 2\), \(r_n(\Delta^n_{0,1}(1/2))\) is homeomorphic to \(D^{n-1}\).

Proof. Recall \(\Delta^n_{0,1}(1/2) = \{x \in \Delta^n: 0 = x_1 < x_2 < \cdots < x_n < 1\}\) and \(\Delta^{n+1}_{0,1}(1/2) = \{x \in \Delta^{n+1}: 0 = x_1 < x_2 < \cdots < x_n < x_{n+1} = 1\}\). Define \(f : \Delta^n_{0,1}(1/2) \to \Delta^{n+1}_{0,1}(1/2)\) by \(f(0, x_2, \ldots, x_n) = (0, x_2, \ldots, x_n, 1)\). We note that for every \(x \in \Delta^n_{0,1}(1/2)\), \(f(r_n^{-1}(r_n(x))) = r^{-1}_{n+1}(r_{n+1}(f(x))) \cap \Delta^{n+1}_{0,1}(1/2)\). Hence, \(f\) induces the homeomorphism \(F : r_n(\Delta^n_0(t)) \to r_{n+1}(\Delta^{n+1}_{0,1}(1/2)) = D^{n-1}\). \(\square\)

Lemma 2.7. Let \(0 < t \leq 1/2\) and \(n \in \mathbb{N}\) with \(n \geq 2\). Then, \(r_n(\Delta^n_{0,1}(t))\) is homeomorphic to \(D^{n-2}\).
Proof. We may assume $0 < t < 1/2$. Recall $\Delta^n_{0,1}(t) = \{ x \in \Delta^n : x_1 = x_2 = \cdots = x_{n-1} = 0 \}$ and $\Delta^n_{0,1}(1/2) = \{ x \in \Delta^n : x_1 \leq x_2 \leq \cdots \leq x_n = 1 \}$. Define $\psi : [\alpha_t, \beta_t] \to \Delta^{n-1}$ by $\psi((1 - s)\alpha_t + s \beta_t) = x_i$ for each $s \in [0, 1]$ and $f : \Delta^n_{0,1}(t) \to \Delta^n_{0,1}(1/2)$ by $f(\alpha_t, x_2, \ldots, x_{n-1}, \beta_t) = (0, \varphi(x_2), \ldots, \varphi(x_{n-1}), 1)$. We note that for every $x \in \Delta^n_{0,1}(t)$, $f(r_n(t x)) = r_n(f(t x)) \in \Delta^n_{0,1}(1/2)$. Hence, $f$ induces the homeomorphism $F : r_n(\Delta^n_{0,1}(t)) \to r_n(\Delta^n_{0,1}(1/2)) = \mathbb{D}^{n-2}$. □

3. $S^1(n)$ is a compactification of the open cone of $\Sigma \Delta^n_{0,1}(t)$

The cone of a metric space $X$ is the identification space $X \times [0, 1]$. Let $\varphi : X \times [0, 1] \to X \times [0, 1]$ be the quotient map. Then we call the image $\varphi(X \times \{ 0 \})$ the cone point. The open cone of $X$ is the image $\varphi(X \times (0, 1))$ and we denote by $C(X)$. Suppose that a subset $X$ of $\mathbb{R}^M$ and a point $v_0 \in \mathbb{R}^N$ are in general position. Then we call the subset $X \cup \{ s v_0 + s x \in \mathbb{R}^N : x \in X \}, 0 \leq s \leq 1$ the geometric cone of $X$ with the vertex $v_0$. The suspension $\Sigma X$ of $X$ is the quotient space of $X \times \{ -1, 1 \}$ by the equivalence relation induced by $(x, -1) \sim (x', -1)$ and $(x, 1) \sim (x', 1)$. Namely, $\Sigma X = X \times \{ -1, 1 \}/(X \times \{ -1, 1 \}, X \times \{ 1 \})$. We call the identified points of $X \times \{ -1, 1 \}$ and $X \times \{ 1 \}$ the suspension points of $X$.

Lemma 3.1. Let $0 < t < 1/2$ and $j = 0, 1$. Then, $r_n(\Delta^n_{0,1}(t))$ is homeomorphic to the cone $c(r_n(\Delta^n_{0,1}(t)))$ of $r_n(\Delta^n_{0,1}(t))$ with the cone point $r_n(\alpha_t(j))$.

Proof. Recall $\Delta^n_{0,1}(t) = \{ x \in \Delta^n : x_1 = x_2 = \cdots = x_{n-1} = 0 \}$ and $\Delta^n_{0,1}(t) = \{ x \in \Delta^n : x_n = 1 \}$. Here we may identify the cone $c(\Delta^n_{0,1}(t))$ with the geometric cone of $\Delta^n_{0,1}(t)$ with the vertex $\alpha_t(0)$. Write $c(\Delta^n_{0,1}(t)) = \{ (1 - s)\alpha_t(0) + s x \in \Delta^n : x \in \Delta^n_{0,1}(t), 0 \leq s \leq 1 \}$.

We note that $c(\Delta^n_{0,1}(t)) = \Delta^n_{0,1}(t)$: For any point $x = (\alpha_t, x_2, \ldots, x_{n-1}, \beta_t) \in \Delta^n_{0,1}(t)$ and $0 \leq s \leq 1$. Since $\alpha_t \leq x_2 \leq \cdots \leq x_{n-1} \leq \beta_t$,

$$\alpha_t \leq (1 - s)\alpha_t + s x_2 \leq \cdots \leq (1 - s)\alpha_t + s x_{n-1} \leq (1 - s)\alpha_t + s \beta_t \leq \beta_t.$$ Thus,

$$(1 - s)\alpha_t(0) + s x \leq (\alpha_t, (1 - s)\alpha_t + s x_2, \ldots, (1 - s)\alpha_t + s x_{n-1}, (1 - s)\alpha_t + s \beta_t) \leq \Delta^n_{0,1}(t).$$

Hence, $c(\Delta^n_{0,1}(t)) \subset \Delta^n_{0,1}(t)$. Take any $x = (\alpha_t, x_2, \ldots, x_{n-1}, \beta_t) \in \Delta^n_{0,1}(t)$. Since $\alpha_t \leq x_n \leq \beta_t$, there exists $s \in [0, 1]$ such that $x_n = (1 - s)\alpha_t + s \beta_t$. If $s = 0, x_1 = \alpha_t$ for each $i = 1, \ldots, n$. We may assume that $s > 0$. For $i = 1, \ldots, n$ set $x_i' = s^{-1}[x_i - (1 - s)\alpha_t]$. Since $\alpha_t \leq x_2 \leq \cdots \leq x_{n-1} \leq (1 - s)\alpha_t + s \beta_t \leq \beta_t$,

$$(\alpha_t, x_2, \ldots, x_{n-1}, \beta_t) = (\alpha_t, (1 - s)\alpha_t + s x_2, \ldots, (1 - s)\alpha_t + s x_{n-1}, (1 - s)\alpha_t + s \beta_t) \leq (\alpha_t, x_2, \ldots, x_{n-1}, \beta_t) \in \Delta^n_{0,1}(t).$$

Hence $x_n = (1 - s)\alpha_t + s x_n' \leq (1 - s)\alpha_t + s \beta_t$.

Therefore $c(\Delta^n_{0,1}(t)) = \Delta^n_{0,1}(t)$. By (3.1), note that every $s$-level of $c(\Delta^n_{0,1}(t))$ is equal to $\Delta^n_{0,1}(\alpha_t, (1 - s)\alpha_t + s \beta_t) \subset \Delta^n_{0,1}(t)$.

Let take points $x = (\alpha_t, x_2, \ldots, x_{n-1}, \beta_t), x' = (\alpha_t, x_2', \ldots, x_{n-1}', \beta_t) \in \Delta^n_{0,1}(t)$. If $r_n(x) = r_n(x')$, by (3.1),

$$r_n((1 - s)\alpha_t(0) + s x) = (1 - s)\alpha_t(0) + s x'$$

for each $s \in [0, 1]$. If $r_n((1 - s)\alpha_t(0) + s x) = r_n((1 - s)\alpha_t(0) + s x')$ for some $s \in [0, 1]$, by (3.1) and Lemma 2.4(3), we similarly have that $x \in r_n(\alpha_t(0))$. Hence, the homeomorphism $f_0 : c(\Delta^n_{0,1}(t)) \to \Delta^n_{0,1}(t)$ given by $f_0((1 - s)\alpha_t(0) + s x) = (1 - s)\alpha_t(0) + s x$ induces the homeomorphism $F_0 : r_n(\alpha_t(0)) \to r_n(\alpha_t(0))$ and $F_0(\alpha_t(0)) = r_n(\alpha_t(0))$.

Similarly we have that $r_n(\alpha_t(0))$ is homeomorphic to the cone $c(r_n(\alpha_t(0)))$ with the cone point $r_n(\alpha_t(0))$. □

Lemma 3.2. Let $0 < t < 1/2$. Then $r_n(\Delta^n_{0,1}(t))$ is homeomorphic to the suspension $\Sigma r_n(\Delta^n_{0,1}(t))$ of $r_n(\Delta^n_{0,1}(t))$ with suspension points $r_n(\alpha_t(0))$ and $r_n(\alpha_t(1))$.

Proof. By Lemma 3.1 it suffices to see that $r_n(\alpha_t(0)) \cap r_n(\alpha_t(1)) = r_n(\alpha_t(0))$. Take any $x = (x_1, x_2, \ldots, x_{n-1}, \beta_t) \in \Delta^n_{0,1}(t)$ such that $r_n(x) \in r_n(\alpha_t(0)) \cap r_n(\alpha_t(1))$. Since $r_n(x) \in r_n(\alpha_t(0))$, $0 < \alpha_t \leq x_1$ and $\Delta^n_{0,1}(t) = \{ z \in \Delta^t : \alpha_t = 1 \leq z_2 \leq \cdots \leq z_n \leq \beta_t \}, x_1 = \alpha_t$. Thus, $x = (\alpha_t, x_2, \ldots, x_{n-1}, \beta_t) \in \Delta^n_{0,1}(t)$. Hence, $r_n(\alpha_t(0)) \cap r_n(\alpha_t(1)) \subset r_n(\alpha_t(0))$. As the converse relation is clear, we have that $r_n(\alpha_t(0)) \cap r_n(\alpha_t(1)) = r_n(\alpha_t(0))$. □

Lemma 3.3. $r_n(\alpha_{1/2}) = r_n(\Delta^n_{0,1}(1/2)) = r_n(\Delta^n_{0,1}(1/2))$.

Proof. We show only $r_n(\alpha_{1/2}) = r_n(\Delta^n_{0,1}(1/2))$. For any point $x = (0, x_2, \ldots, x_{n-1}, 1) \in \Delta^n_{0,1}(1/2)$, we define the point $x' = (x_2, \ldots, x_{n-1}, 1) \in \Delta^n_{0,1}(1/2)$. Then since $q(0) = q(1), r_n(x) = r_n(x')$. Hence, $r_n(\Delta^n_{0,1}(1/2)) \supset r_n(\Delta^n_{0,1}(1/2))$. Similarly, we can show that $r_n(\Delta^n_{0,1}(1/2)) \subset r_n(\Delta^n_{0,1}(1/2))$, which proves the desired equation. □
Theorem 3.4. Fix a given $t \in (0, 1/2)$. Then $S^1(n) \setminus r_n(\Delta^n(1/2))$ is homeomorphic to the open cone $\tilde{c}(\Sigma r_n(\Delta^n_0,1(t)))$ of $\Sigma r_n(\Delta^n_0,1(t))$ with the cone point $r_n(a_0(0)) = r_n(a_1(0))$. Namely $S^1(n)$ is a compactification of $\tilde{c}(\Sigma r_n(\Delta^n_0,1(t)))$ whose remainder is homeomorphic to $r_n(\Delta^n_G(1/2))$.

**Proof.** We consider the subset of $\mathbb{R}^n$

$$\{(1-s)c + sx: x \in \Delta^n(t), 0 \leq s < \gamma_1\},$$

where $c = a_0(0) = (1/2, 1/2, \ldots, 1/2)$ and $\gamma_1 = (1 - 2\alpha t)^{-1} = (2t)^{-1}$.

Suppose that for points $x = (x_1, x_2, \ldots, x_n)$. Then, $(1-s)/2 + sx_i = (1-s)/2 + s'x_i$ for all $i = 1, \ldots, n$. In particular, $(1-s)/2 + sx_1, (1-s)/2 + sx_n) = ((1-s)/2 + s'x_1, (1-s)/2 + s'x_n).$ If $s = 0$, $s' = 0$ or $x'_1 = x'_n = 1/2$. If $x'_1 = x'_n = 1/2$, then $x'_i = x_i = \ldots = x'_{n-1} = x'_n = 1/2$. It follows that $x' \notin \Delta^n(t)$, a contradiction. Suppose that $s \neq 0$ and $s' \neq 0$. By Fig. 3.4.1, we have $(x_1, x_n) = (x'_1, x'_n).$ Then we have $s = s'$. Hence $x_i = x'_i, i = 1, 2, \ldots, n$. Thus, $x = x'$. This shows that our set is homeomorphic to the open cone $\tilde{c}(\Delta^n(t))$ of $\Delta^n(t)$. Therefore we can denote our set as follows

$$\tilde{c}(\Delta^n(t)) = \{(1-s)c + sx: x \in \Delta^n(t), 0 \leq s < \gamma_1\}.$$

Next we show that $\Delta^n \setminus \Delta^n(1/2) = \tilde{c}(\Delta^n(t))$. Take any $(1-s)c + sx \in \tilde{c}(\Delta^n(t))$ for $0 \leq s < \gamma_t$ and $x \in \Delta^n(t)$. Then

$$(1-s)c + sx = (1-s)2sx_1, 1/2, \ldots, (1-s)2sx_n)/2.$$

Since $\alpha t \leq x_1$, thus, $\gamma_2^{-1} = 1 - 2\alpha t \geq 1 - 2x_1$, we have that $1 - s + 2sx_1 = 1 - s(1 - 2x_1) > 1 - \gamma_2(1 - 2x_1) > 0$. Since $1 - 2\alpha t = 2\beta t - 1$ and $x_n \leq \beta t$, $1 - (1-s)2sx_1)/2 = (1-s)(1-2sx_1)/2 \geq (1-s)(1-2\beta t)/2 = (1-s)/\gamma_2)/2 > 0$. Since $\alpha t \leq x_1 \leq \ldots \leq x_n \leq \beta t$, $0 < (1-s)2sx_1)/2 \leq \ldots \leq (1-s)2sx_n)/2 < 1$.

Hence $\Delta^n \setminus \Delta^n(1/2) = \tilde{c}(\Delta^n(t))$.

Take any $x \in \Delta^n \setminus \Delta^n(1/2)$ with $x \neq c$. There exist $\lambda \in (0, \infty)$ and $(x'_1, x'_n) \in \Delta^n(t)$ such that $(x'_1, x'_n) = ((1-\lambda)/2 + \lambda x_1, (1-\lambda)/2 + \lambda x_n)$ (see Fig. 3.4.2). Then we define $x'_i = (1-\lambda)/2 + \lambda x_i$ for each $i = 1, \ldots, n$ and consider the point $x' = (x'_1, \ldots, x'_n) = (1-\lambda)c + \lambda x \in \mathbb{R}^n$. Since we clearly see that $\alpha t \leq x'_1 \leq \ldots \leq x'_n \leq \beta t$, and $x'_i = \alpha t$ or $x'_n = \beta t, x' \in \Delta^n(t).$

Defining $\mu = \lambda^{-1} > 0$, we have

$$(1-\mu)c + \mu x' = (1-\mu)c + \mu(1-\lambda)c + \mu\lambda x = (1-\lambda^{-1})c + \lambda^{-1}(1-\lambda)c + x = x.$$

Suppose that $\mu \geq \gamma_1$. If $x'_1 = \alpha t$, then

$$(1-\mu)/2 + \mu\alpha t = 1/2 + \mu(\alpha t - 1/2) = 1/2 - \mu t \leq 0,$$

a contradiction. If $x'_n = \beta t$,

$$(1-\mu)/2 + \mu\beta t = 1/2 + \mu(\beta t - 1/2) = 1/2 + \mu t \geq 1,$$

a contradiction. Thus, we see that $0 < \mu < \gamma_1$. Hence $x \in \tilde{c}(\Delta^n(t))$, and $\Delta^n \setminus \Delta^n(1/2) \subset \tilde{c}(\Delta^n(t))$. Therefore $\Delta^n \setminus \Delta^n(1/2) = \tilde{c}(\Delta^n(t))$. 

Fig. 3.4.1. Fig. 3.4.2.
Let $0 \leq s < \gamma t$ and $x \in \Delta^n(t)$. Then

$$(1 - s)c + sx = ((1 - s + 2sx_1)/2, \ldots, (1 - s + 2sx_n)/2).$$

For any $x' \in r^{-1}_n(r_n(x))$, since

$$(1 - s)c + sx' = ((1 - s + 2sx'_1)/2, \ldots, (1 - s + 2sx'_n)/2),$$

$r_n((1 - s)c + sx) = r_n((1 - s)c + sx')$, $(1 - s)c + sx' \in r^{-1}_n(r_n((1 - s)c + sx))$. Next we consider the case when there exist points $y, z \in \Delta^n \setminus \Delta^n(1/2)$ with $r_n(y) = r_n(z) = r_n(c)$. Then as seen as in the above, there exist $y', z' \in \Delta^n(t)$ and $\lambda, \lambda' \in [0, \gamma t)$ such that $y' = (1 - \lambda)c + \lambda y$ and $z' = (1 - \lambda')c + \lambda' z$. Since $r_n(y) = r_n(z)$, $y_1 = z_1$ and $y_n = z_n$. Thus, $(y'_1, y'_n) = (z'_1, z'_n)$. Then as seen as in the above, $\lambda = \lambda'$. Hence, if $y_1 = z_1$ for some $i$, $y'_i = z'_j$. Therefore $r_n(y') = r_n(z')$. Moreover, we note that $\lambda = \lambda'$ implies $y, z \in \Delta^n(t')$ for some $t' \in [0, 1/2)$. Hence the identity map $\tilde{\phi} : \tilde{\varphi}(\Delta^n(t)) \to \Delta^n \setminus \Delta^n(1/2)$ induces the homeomorphism $\tilde{\varphi}_n : \tilde{\varphi}(\Delta^n(t)) \to r_n(\Delta^n \setminus \Delta^n(1/2))$. Therefore, by Lemmas 3.2 and 3.3, we have Theorem 3.4.

**Corollary 3.5.** Let $n \in \mathbb{N}$ with $n \geq 2$. Then there exists a closed subset $R$ of $S^1(n)$ which is homeomorphic to $D^{n-1}$ such that $S^1(n) \setminus R$ is homeomorphic to the open cone $\tilde{\varphi}(\Sigma D^{n-2})$ over $\Sigma D^{n-2}$.

**Theorem 3.6.** Let $0 < t_0 < 1/2$, $n \in \mathbb{N}$ and $K = \bigcup_{t_0 \leq t \leq 1/2} r_n(\Delta^n(t))$. Then there exists a deformation retraction from $K$ to $r_n(\Delta^n(1/2))$.

**Proof.** For each $x \in \Delta^n(t_0)$ we define

$$x' = s_0(x - c) + c,$$

where $s_0 = 1/(2t_0)$ and $c = a_0(0) = (1/2, \ldots, 1/2) \in \Delta^n$. Since $s_0(\alpha(t_0 - 1/2) + 1/2 = 0$ and $s_0(\beta(t_0 - 1/2) + 1/2 = 1$, $x \in \Delta^n(t_0)$ implies $x' \in \Delta^n(1/2)$ for $j = 0, 1$.

Let us consider the set

$$Z = \{(1 - \mu)x + \mu x' : \mu \in \mathbb{I}, x \in \Delta^n(t_0)\}.$$

For points $x, y \in \Delta^n(t_0)$ suppose that there exist $\mu, \nu \in \mathbb{I}$ such that $(1 - \mu)x + \mu x' = (1 - \nu)y + \nu y'$. Then, since $(x'_1, x'_n) = (y'_1, y'_n)$, $(x_1, x_n) = (y_1, y_n)$ and $\mu = \nu$. Hence we have that $x = y$. This shows that $Z$ is homeomorphic to $\Delta^n(1/2) \times I$ and $\Delta^n(t_0) \times I$ (see Fig. 3.6.1).

Next we show that $Z \subseteq \bigcup_{t_0 \leq t \leq 1/2} \Delta^n(t)$. It is clear that $Z \subseteq \Delta^n$. For $\mu \in \mathbb{I}$ $(1 - \mu)\alpha(t_0) = \alpha(t_0)$ for some $t_0 \leq t \leq 1/2$ if and only if $(1 - \mu)\beta(t_0) + \mu = \beta(t_0)$. Hence for any $x \in \Delta^n(t_0)$ or $\Delta^n(t)$, $(1 - \mu)x + \mu x' \in \Delta^n(t)$ or $\Delta^n(t)$ for some $t_0 < t < 1/2$, respectively. Therefore $Z \subseteq \bigcup_{t_0 \leq t \leq 1/2} \Delta^n(t)$. Conversely take any $y \in \bigcup_{t_0 \leq t \leq 1/2} \Delta^n(t)$. Since $(y_1, y_n) \in \bigcup_{t_0 \leq t \leq 1/2} \Delta^n(t)$, we can easily find $\lambda \in \mathbb{I}$ and $(x_1, x_n) \in \Delta^n(t_0)$ such that

$$(x_1, x_n) = \left((1 - \lambda)/2 + \lambda y_1, (1 - \lambda)/2 + \lambda y_n\right).$$

Defining $x_i = (1 - \lambda)/2 + \lambda y_i$ for each $i = 1, \ldots, n$, we have the points

$$x = (x_1, \ldots, x_n) \in \Delta^n(t_0) \quad \text{and} \quad x' = s_0(x - c) + c \in \Delta^n(1/2).$$

Define $\mu = (1 - \lambda)/(s_0 - 1) \geq 0$. Since $1 - \mu + \mu s_0 = \lambda^{-1}$ and $\mu(s_0 - 1) = (1 - \lambda)/\lambda$, for every $i = 1, 2, \ldots, n$,
(1 − µ)x₁ + µx′₁ = (1 − µ + µs₀)x₁ + µ(1 − s₀)/2
= (1 − µ + µs₀)(1 − λ)/2 + λ(1 − µ + µs₀)y₁ + µ(1 − s₀)/2
= (1 − λ)/2λ + y₁ − (1 − λ)/2λ.

This shows that µ ∈ I and Y ⊂ Z. Hence \( \bigcup_{0 \leq t \leq 1/2} \Delta^n(t) = Z \) and \( r_n(Z) = K \). Moreover, we essentially show that \( \Delta^n(1/2) = \{ x' : x \in \Delta^n(t₀) \} \) and there exists the natural homeomorphism \( F : \Delta^n(1/2) \times I \to \bigcup_{0 \leq t \leq 1/2} \Delta^n(t) \) defined by \( F(x, µ) = (1 − µ)x + µx' \).

Define \( H : \bigcup_{0 \leq t \leq 1/2} \Delta^n(t) \times I \to \bigcup_{0 \leq t \leq 1/2} \Delta^n(t) \) by

\[ H((1 − µ)x + µx', u) = (1 − (µ + (1 − µ)u)x + (µ + (1 − µ)u)x' \] (3.4)

for each \( x \in \Delta^n(t₀) \) and each \( u \in I \). It is clear that \( H_0 = id_Z \), \( H_u|_{\Delta^n(1/2)} = id_{\Delta^n(1/2)} \) and \( H_1(Z) = \Delta^n(1/2) \). For \( x \in \Delta^n(t₀) \), \( u \in I \), by (3.3) and (3.4),

\[ H((1 − µ)x + µx', u) = (1 − ν)x + νx' = (1 − ν + νs₀)x + ν(1 − νs₀)c, \]

where \( ν = µ + (1 − µ)µ \). Hence for any \( y \in r_n^{-1}(r_n(x)) \) and any \( µ \in I \) we can see that \( r_n(H((1 − µ)y + µy', u)) = r_n(H((1 − µ)y + µy', u)) \). Hence, \( H \) induces the map \( \bar{H} : K \times I \to K \) such that \( \bar{H}_0 = id_K \), \( \bar{H}_u|_{r_n^{-1}(\Delta^n(1/2))} = id_{r_n^{-1}(\Delta^n(1/2))} \) and \( \bar{H}_1(K) = r_n(\Delta^n(1/2)) \). Therefore \( \bar{H}_1 : K \to r_n(\Delta^n(1/2)) \) is a deformation retraction, which prove the theorem. □

4. Application: a homotopy type of \( S^1(n) \)

**Theorem 4.1.** Let \( n \in N. \) Then \( S^1(2n + 1) \) has the same homotopy type of the \( (2n + 1) \)-sphere \( S^{2n+1} \).

**Proof.** By Theorem 3.4, Lemmas 2.6 and 2.7, \( S^1(2n + 1) \) is homeomorphic to the open cone \( \tilde{c}(\Sigma D^{2n-1}) \) of \( \Sigma D^{2n-1} \) and \( S^1(2n + 1) \) is a compactification of \( \tilde{c}(\Sigma D^{2n-1}) \) whose remainder is homeomorphic to \( D^{2n} \). Let \( \rho : S^1(2n + 1) \to S^1(2n + 1)/r_{2n+1}(\Delta^{2n+1}(1/2)) \) be the projection. Then \( S^1(2n + 1)/r_{2n+1}(\Delta^{2n+1}(1/2)) \) is homeomorphic to the double suspension \( S^2 D^{2n-1} \) of \( D^{2n-1} \). Hence, by [2, Theorem 3.4], \( S^1(2n + 1)/r_{2n+1}(\Delta^{2n+1}(1/2)) \) has the same homotopy type of the \( (2n + 1) \)-sphere \( S^{2n+1} \) and \( \rho \) is a cell-like map. Therefore \( S^1(2n + 1) \) has the same homotopy type of the \( (2n + 1) \)-sphere \( S^{2n+1} \). □

**Theorem 4.2.** Let \( n \in N. \) Then \( S^1(2n) \) has the same homotopy type of the \( (2n − 1) \)-sphere \( S^{2n-1} \).

**Proof.** By Theorem 3.4, Lemmas 2.6 and 2.7, \( S^1(2n) \) is homeomorphic to the open cone \( \tilde{c}(\Sigma D^{2n-2}) \) of \( \Sigma D^{2n-2} \) and \( S^1(2n) \) is a compactification of \( \tilde{c}(\Sigma D^{2n-2}) \) whose remainder is homeomorphic to \( D^{2n-1} \).

Let us define

\[ K_0 = \bigcup_{1/4 \leq t \leq 1/2} r_{2n}(\Delta^{2n}(t)) \quad \text{and} \quad K_1 = Cl(S^1(2n) \setminus K_0). \]

Then \( S^1(2n) = K_0 \cup K_1 \) and \( K_0 \cap K_1 = r_{2n}(\Delta^{2n}(1/4)) \) is homeomorphic to \( \Sigma D^{2n-2} \). Hence by [2, Theorem 3.4] \( K_0 \cap K_1 \) is contractible, and thereby an AR. Then there exists a retraction \( r_1 : K_1 \to K_0 \cap K_1 \) which is homotopic to the identity map \( K_1 \to K_1 \). Hence, by Theorem 3.6, there exists a retraction \( r : S^1(2n) \to r_{2n}(\Delta^{2n}(1/2)) \) which is homotopic to the identity map \( S^1(2n) \to S^1(2n) \). Therefore \( S^1(2n) \) has the same homotopy type of \( D^{2n-1} \). It follows, by [2, Theorem 3.4], that \( S^1(2n) \) has the homotopy type of the \( (2n − 1) \)-sphere \( S^{2n-1} \).

It is well known that \( S^1(2) \) is homeomorphic to the Möbius band. By Theorem 4.2 we can see that \( S^1(2n), n \geq 2, \) is not a closed \( 2n \)-manifold: Suppose that \( S^1(2n) \) is a \( 2n \)-manifold. Then \( S^1(2n) \) is orientable because \( \pi_1(S^1(2n)) \) is trivial. However \( H_{2n}(S^1(2n)) \) is trivial; a contradiction. Thus, we have the following (see Theorem 6.4 for general case).

**Corollary 4.3.** Any \( S^1(2n) \) is not a closed \( 2n \)-manifold.

5. Application: \( S^1(3) \) is homeomorphic to \( S^3 \)

In case of \( n = 3 \) Theorem 3.4 implies the more precise description as follows:

**Theorem 5.1.** \( S^1(3) \setminus r_{3}(\Delta^{3}(1/2)) \) is homeomorphic to \( \mathbb{R}^3 \) and \( S^1(3) \) is a compactification of \( \mathbb{R}^3 \) whose remainder is homeomorphic to the dunce hat \( D^2 \).
Lemma 5.2. $S^1(3)$ is a 3-manifold.

Proof. By Theorem 5.1, $S^1(3) \setminus r_3(\Delta^3(1/2))$ is homeomorphic to $R^3$. Hence, every $z \in S^1(3) \setminus r_3(\Delta^3(1/2))$ has a neighborhood which is homeomorphic to $R^3$. Choose $z \in \Delta^1(1/2)$. Then, we can write $q_3(z) = (s, t, u)$ such that $x_0 \in [s, u]$, where $x_0 = (1, 0) \in S^1$. There exists a homeomorphism $h : S^1 \to S^1$ such that $p_{S^1(3)}(h(s), h(t), h(u)) \in S^1(3) \setminus r_3(\Delta^3(1/2))$. Let $H : S^1(3) \to S^1(3)$ be the homeomorphism induced by $h \times h \times h : T^3 \to T^3$ such that $H \circ r_3(z) \in S^1(3) \setminus r_3(\Delta^3(1/2))$. Hence, $r_3(z)$ has a neighborhood which is homeomorphic to $R^3$. □

From Theorem 4.1 and Lemma 5.2 we obtain the following (cf. [1, Theorem 3]).

Corollary 5.3. (Bott [5]) $S^3(3)$ is homeomorphic to $S^3$.

Since $r_3(\Delta^3(1/2))$ satisfies the cellularity criterion, we get the following. See [7, pp. 143–147].

Corollary 5.4. The dunce hat $r_3(\Delta^3(1/2))$ is cellular in $S^1(3)$.

6. Application: $S^1(n)$ is not an $n$-manifold for each $n \geq 4$

Notation 6.1. Write $T_m = \{(t \cos 2\pi k/m, t \sin 2\pi k/m) \in R^2 : t \in \mathbb{R}, k = 0, \ldots, m-1\}$ for $m, n \in \mathbb{N}$.

$$\triangle_n^{n+2} = \{(x_1, \ldots, x_{n+2}) \in \triangle_n^{n+2}(1/2) : x_i = x_{i+1}\} \text{ for } i = 1, \ldots, n+1,$$

$$\partial \triangle_n^{n+2}(1/2) = \triangle_n^{n+2} \cup \cdots \cup \triangle_{n+1}^{n+2} \text{ and } \triangle_{n+1}^{n+2} = \triangle_{0,1}^{n+2}(1/2) \setminus \partial \triangle_{0,1}^{n+2}(1/2).$$

Lemma 6.2. For every $n \in \mathbb{N}$ and every $i = 1, \ldots, n+1$, $r_n(\triangle_{1}^{n+2})$ is homeomorphic to $D^{n-1}$ and $r_{n+2}(\partial \triangle_{0,1}^{n+2}(1/2)) = r_{n+2}(\triangle_{1}^{n+2})$.

Proof. For $i = 1, 2, \ldots, n+1$, we define the homeomorphism $f_i : \triangle_i^{n+2} \to \triangle_{0,1}^{n+2}(1/2)$ by $f_i(x_1, x_2, \ldots, x_{i+1}, \ldots, x_n) = (x_1, x_2, \ldots, x_i, x_{i+2}, \ldots, x_n)$. Since $f_i(\triangle_i^{n+2}(r_n(\triangle_{1}^{n+2}(x))) \cap \triangle_{0,1}^{n+2}(1/2)) = \triangle_{0,1}^{n+2}(1/2)$ for each $x \in \triangle_{1}^{n+2}$, $f_i$ induces the homeomorphism $F_i : \triangle_i^{n+2} \to D^{n-1}$.

To prove the latter part, it suffices to show that $r_{n+2}(\triangle_{1}^{n+2}) = r_{n+2}(\triangle_{j}^{n+2})$ for any $i, j = 1, \ldots, n+1$ with $i < j$. Let $x = (x_1, \ldots, x_{n+2}) \in \triangle_{1}^{n+2}$. Define

$$x_k' = \begin{cases} x_k & \text{if } 1 \leq k \leq i, \\ x_{k+1} & \text{if } i+1 \leq k \leq j-1, \\ x_{j+1} & \text{if } k = j, j+1, \\ x_k & \text{if } j+2 \leq k \leq n+2, \end{cases}$$

and $x' = (x_1', \ldots, x_{n+2}') \in \triangle_{j}^{n+2}$. Since $r_{n+2}(x) = r_{n+2}(x')$, $r_{n+2}(\triangle_{1}^{n+2}) \subseteq r_{n+2}(\triangle_{j}^{n+2})$. Similarly, we can show that $r_{n+2}(\triangle_{1}^{n+2}) \subseteq r_{n+2}(\triangle_{j}^{n+2})$. □

By Lemma 6.2, $D^{n-1} = D^n \setminus r_{n+2}(\triangle_{1}^{n+2})$ and $D^0 \subseteq D^1 \subseteq \cdots \subseteq D^n$.

Lemma 6.3. For each point $z \in r_{n+1}(\triangle_{n}^{n+1}) \subseteq D^{n-1} \subseteq D^n$, there exists an arbitrarily small neighborhood in $D^n$ which is homeomorphic to $T_{n+1}^{n+1}$.

Proof. Let $U$ be a neighborhood of $z$ in $D^n$. By Lemma 6.2, for every $i = 1, \ldots, n+1$ there exists $x_i' = (x_1', \ldots, x_{n+2}') \in \triangle_{n+2}$ such that $z = r_{n+2}(x_i')$ for each $i = 1, \ldots, n+1$, $x_j' = x_j$ if and only if $(j, j') = (i, i+1)$. There exists a closed neighborhood $V_i$ of $x_i'$ in $\triangle_{i}^{n+2}$ which is homeomorphic to an $(n-1)$-cell such that for every $y \in V_i$, $y_j = y_j'$ if and only if $(j, j') = (i, i+1)$, and $r_{n+2}(V_i) = r_{n+2}(V_i')$ for any $i, i' = 1, \ldots, n+2$. By Lemma 2.4, there exists a closed neighborhood $W_i$ of $x_i'$ in $\triangle_{i}^{n+2}(1/2)$ which is homeomorphic to an $n$-cell such that $W_i \cap W_i' = \emptyset$ whenever $i \neq i'$, every $r_{n+2}(W_i)$ is homeomorphic to an n-cell in $U$, $\triangle_{i}^{n+2} \cap W_i \subset V_i$, and $r_{n+2}(W_i') \cap W_i = r_{n+2}(W_i') \cap V_i$ is homeomorphic to an $(n-1)$-cell for any $i, i' = 1, \ldots, n+1$. Thus, $W = \bigcup_{i=1}^{n+1} r_{n+2}(W_i)$ is a neighborhood of $z$ in $U$ which is homeomorphic to $T_{n+1}$. □

Followings are obtained by more general results in [6, Lemmas 4.2 and 4.3]. However our setting is simpler, we are giving direct proofs.
Theorem 6.4. $S^3(n)$ is not an $n$-manifold for each $n \in \mathbb{N}$ with $n \geq 4$.

Proof. By Lemma 6.3, some point in $D^{n-2}$ has a small neighborhood which is homeomorphic to $T_n^{n-2}$. Thus, some point in $c(\Sigma D^{n-2})$ has a small neighborhood which is homeomorphic to $T_n^{n-1}$. Since $n \geq 4$, by Theorem 3.4, $S^3(n)$ is not an $n$-manifold. \qed

Let $X$ be an $n$-dimensional compact metric space. By $S(X)$ we denote the set of all points at which $X$ fails to be a topological $n$-manifold. It is clear that $S(S^3(n)) \subset S^3(n) \setminus r_n(\Delta^n)$. In [2, Remark 3.6] it is pointed out that $S(\Sigma D^3)$ is homeomorphic to the 3-sphere $S^3$.

Corollary 6.5.

$S(S^3(4)) = S^3(4) \setminus r_n(\Delta^3) = \{r_4(x_1, x_2, x_3, x_4) : |\{x_1, x_2, x_3, x_4\}| \leq 3\}.$

Proof. Let $0 < t < 1/2$ and $D^3 = \Delta^3_1(t)$ by Theorem 3.4. It is clear that $D^3 \supset D^4 = \{r_4(x_1, x_2, x_3, x_4) : (x_1, x_2, x_3, x_4) \in \Delta^4_1(t), |\{x_1, x_2, x_3, x_4\}| = 2, 3\}$ and $z \in D^3 \subset D^2$ has small neighborhood in $D^2$ which is homeomorphic to $T^2_3$. By Theorem 3.4, $S^3(\Sigma D^3) = S^3(\Sigma D^2) = \{r_4(x_1, x_2, x_3, x_4) : (x_1, x_2, x_3, x_4) \in \Delta^4_1(t), 1 \leq |\{x_1, x_2, x_3, x_4\}| \leq 3\}$. By Theorem 3.4, $S^3(\Sigma D^3) = \{r_4(x_1, x_2, x_3, x_4) : (x_1, x_2, x_3, x_4) \notin \Delta^3(1/2), 1 \leq |\{x_1, x_2, x_3, x_4\}| \leq 3\}$. There exists a homeomorphism $h : S^3 \to S^3$ such that $\beta_3(h(x + x) = h(x + h(q_4(\Delta^3(1/2)))) \subset S^3(4) \setminus r_4(\Delta^3(1/2))$. Let $H : S^3(4) \to S^3(4)$ be the homeomorphism induced by $h \times x \times h \times x : T^4 \to T^4$ such that $H \circ r_4(\Delta^3(1/2)) \subset S^3(4) \setminus r_4(\Delta^3(1/2))$. Hence, $S(S^3(4)) = \{r_4(x_1, x_2, x_3, x_4) : |\{x_1, x_2, x_3, x_4\}| \leq 3\}. \quad \square$

7. Application: On the embedding in $S^3(n)$

Theorem 7.1. For every $n \in \mathbb{N}$ with $n \geq 4$ there exists no embedding from an orientable closed $n$-dimensional topological manifold into $S^3(n)$.

Proof. On the contrary, assume that there exists an embedding $i$ from an orientable closed $n$-dimensional topological manifold $M$ into $S^3(n)$.

Suppose $n$ is even. By Theorem 4.2, the $n$-th cohomology group $H^n(S^3(n))$ of $S^3(n)$ is zero. Since $S^3(n)$ is an $n$-dimensional CW-complex, $H^{n+1}(S^3(n), i(M)) = 0$. From the exact sequence

$$\cdots \to H^n(S^3(n)) \to H^n(i(M)) \to H^{n+1}(S^3(n), i(M)) \to \cdots$$

of the cohomology groups, we have a contradiction.

Suppose $n$ is odd. For every $x \in \Delta^3_n$ there exist a small open neighborhood $U$ of $x$ in $\Delta^3_n$ and a strong deformation retraction $r : \Delta^3_n \times U \to \Delta^3_n \setminus \Delta^3_n$. By Lemma 2.4, for every $x \in \Delta^3_n$ there exist a small open neighborhood $V$ of $r_n(x)$ in $\Delta^3_n$ and a strong deformation retraction $R : S^3(n) \setminus V \to S^3(n) \setminus r_n(\Delta^3_n)$. Suppose $i(M) \neq S^3(n)$. Since $r_n(\Delta^3_n)$ is open dune $S^3(1)$, there exist $x \in \Delta^3_n$, a small open neighborhood $V$ of $r_n(x)$ in $\Delta^3_n$ and $i(M)$ and a strong deformation retraction $R : S^3(n) \setminus V \to S^3(n) \setminus r_n(\Delta^3_n)$. Since $S^3(n) \setminus r_n(\Delta^3_n)$ is an $(n-1)$-dimensional CW-complex, $H^n(S^3(n) \setminus V) = H^{n+1}(S^3(n), i(M)) = 0$. From the exact sequence

$$\cdots \to H^n(S^3(n) \setminus V) \to H^n(i(M)) \to H^{n+1}(S^3(n) \setminus V, i(M)) \to \cdots$$

of the cohomology groups, we see that $i(M) = S^3(n)$, but, by Theorem 6.4, we have a contradiction. \qed

Corollary 7.2. Let $n \in \mathbb{N}$. Then there exists an embedding from the $n$-sphere $S^3(n)$ into $S^3(n)$ if and only if $n = 1, 3$.

References