Note

Structure of 3-infix–outfix maximal codes

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Abstract

This paper proves that a 3-infix–outfix code is a maximal code if and only if it is a group code, or equivalently, if and only if, it is a full uniform code. Thus, the structure for 3-infix–outfix code being a maximal code can be completely determined.

\textbf{Keywords:} n-prefix-suffix; n-infix–outfix; Group codes; Uniform codes

1. Introduction

Which characteristics does a code $C$ possess when the syntactic monoid $\text{syn}(C^*)$ of the star closure $C^*$ of $C$ is a group? This is an unsolved open problem proposed by Schützenberger in [12]. We call a code $C$ a group code if the $\text{syn}(C^*)$ is a group. Schützenberger characterized the structure of finite group codes and proved that $C$ is a group code if and only if $C$ is a full uniform code [1, 12]. Few properties of infinite group codes are known so far. In [7, 8, 10], the structure of certain infinite codes which are group codes was characterized. By introducing the concept of $n$-infix–outfix codes, additional properties of some group codes different from that in [7, 8] are given in [9]. It also shows that any finite 3-infix–outfix code is maximal if and only if it is a group code, or equivalently, if and only if it is a full uniform code. The proof of this result relies on the finiteness of the code. Therefore, the following problem is proposed in [9]: Is there an infinite 3-infix–outfix code which is a maximal code? This paper is a continuation of [9]. We answer the above problem and completely determine the structure of a 3-infix–outfix code which is a maximal code. Since the class of infix or

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outfix codes is properly contained in the class of 3-infix–outfix codes [9], the result of this paper is a generalization of main results in [7].

This paper first introduces the relevant concepts and notations. Detailed definitions can be found in [1, 5, 4, 11].

Let \( A \) be a finite alphabet. \( A^* \) denotes the free monoid generated by \( A \) and \( A^+ = A^* \setminus \{1\} \) where 1 is the empty word over \( A \). An element and a subset of \( A^* \) is said to be a word and a language over \( A \), respectively. For \( x \in A^* \), \( |x| \) denotes the length of \( A \).

A language \( L \subseteq A^* \), associates its principal congruence \( P_L \) and its syntactic monoid \( \text{syn}(L) = A^*/P_L \), where

\[
L \ni [w] \iff ((\forall x, y \in A^*) xy y \in L \iff xy \in L).
\]

By \([w]\) we denote the \( P_L \)-class of \( w \), i.e., \([w] = \{ x \in A^* \mid x \equiv w(P_L) \} \).

A language \( C \subseteq A^* \) is said to be a code over \( A \) if the submonoid \( C^* \) of \( A^* \) is freely generated by \( C \). A language \( C \subseteq A^* \) is said to be a prefix code if \( CA^+ \cap C = \emptyset \). A language \( C \subseteq A^* \) is said to be a suffix code if \( A^+ C \cap C = \emptyset \). A language \( C \subseteq A^* \) is called a bifix (or biprefix) if it is both a prefix code and a suffix code. If we use the notation "\( \Theta \)" to represent some class of codes, we say that the code \( C \) is a maximal \( \Theta \) code if and only if for any \( \Theta \) code \( D \) over \( A \), \( D \supseteq C \) implies \( D = C \). We say that the code \( C \) is a \( \Theta \) maximal code if and only if \( C \) is both \( \Theta \) and maximal in the class of all codes over \( A \). A code \( C \) is said to be an infix code if \( (\forall u, v, x \in A^*) xu v \in C \iff uxv \in C \). A code \( C \) is said to be an outfix code if \( (\forall u, v, x \in A^*) uv, u x v \in C \iff x = 1 \). A language \( L \) over \( A \) is said to be reflective if \( (\forall x, y \in A^*) xw y \in L \iff x y \in L \). A language \( L \subseteq A^* \) is called an \( n \)-infix–outfix code if every subset of at most \( n \) elements of \( L \) is an infix code or an outfix code. By \( \preceq_p, \preceq_s, \preceq_i, \preceq_o \), we denote the prefix, suffix, infix, and outfix relations on \( A^* \), respectively [4]. By \( P(A), S(A), B(A), I(A), O(A), \) and \( IO_0(A) \) we denote the classes of the prefix, suffix, bifix, infix, outfix, and \( n \)-infix–outfix codes over \( A \), respectively. From [9, 6], we have \( I(A) \subset IO_2(A) \subset B(A), O(A) \subset IO_2(A) \subset B(A) \), \( I(A) \cup O(A) = \cdots = IO_5(A) = IO_3(A) \subset IO_3 \subset IO_2(A) \). For simplicity, this paper assumes that \( A \) is the least alphabet for a code, i.e., let \( C \) be a code over \( A \), then \( A^* a A^* \cap C \neq \emptyset \) for every \( a \in A \).

2. A structure theorem

The two following lemmas directly follow from the definitions.

**Lemma 1.** Let \( C \subseteq A^* \) be a prefix code. Then \( C \) is a maximal prefix code if and only if \( A^* = CA^* \cup C(A^+)^{-1} \) where \( C(A^+)^{-1} = \{ x \in A^* \mid C \cap x A^+ \neq \emptyset \} \). Let \( C \subseteq A^* \) be a suffix code. Then \( C \) is a maximal suffix code if and only if \( A^* - CA^* \cup (A^+)^{-1} C \) where \( (A^+)^{-1} C = \{ x \in A^* \mid C \cap A^+ x \neq \emptyset \} \).

**Lemma 2.** Let \( C \subseteq A^* \) be a 3-infix–outfix code and a maximal code. Then \( C \) must be a maximal prefix code and a maximal suffix code.
Theorem 1. Let $\mathcal{C} \subseteq A^*$ be a 3-infix-outfix code and a maximal code. If there exist $c_1, c_2 (\neq c_1) \in \mathcal{C}$ such that $c_1 \leq c_2$, i.e., $c_2 = uc_1 v$, for some $u, v \in A^+$, then $A^{\mid c_1 \mid} C \subseteq C$.

Proof. Since a 3-infix-outfix code is a bifix code, it suffices to show that $uc_1 A^{\mid c_1 \mid} C \subseteq C$.

Let $v = a_m a_{m-1} \ldots a_1, a_i \in A, 1 \leq i \leq m$. The claim can be proved by induction on $m$ which is the length of the word $v$.

(i) First we verify that $uc_1 a_m a_{m-1} \ldots a_2 a_1 \subseteq C$. Suppose there is $b \subseteq A$, such that,

$$uc_1 a_m a_{m-1} \ldots a_2 b \notin C.$$ 

We show that this implies the existence of infinite sequences

$$y_1, y_2, \ldots \in A^*, \ d_1, d_2, \ldots \in A \ \text{and} \ b_0, b_1, b_2, \ldots \in A$$

such that, for $i = 0, 1, \ldots$,

$$w_i = uc_1 a_m a_{m-1} \ldots a_2 b y_i d_1 y_2 d_2 \ldots y_i d_i b_i \in C,$$

no prefix of

$$w'_i = uc_1 a_m a_{m-1} \ldots a_2 b y_i d_1 y_2 d_2 \ldots y_i d_i a_1$$

is in $C$, and

$$b_i \notin \{b_0, b_1, \ldots, b_{i-1}\}.$$ 

As $A$ is finite, this last statement results in a contradiction. To construct the sequences, we proceed by recursion.

For $i = 0$, let $b_0 = a_1$. Clearly,

$$b_i \notin \{b_j \mid 0 \leq j < i\} = \emptyset.$$ 

Moreover, $w_0 = uc_1 a_m a_{m-1} \ldots a_2 a_1 = c_2 \in C$ and

$$w'_0 = uc_1 a_m a_{m-1} \ldots a_2 b a_1.$$ 

Since $c_1 \leq c_2 \omega_n w'_0$, it follows that $w'_0 \notin C$. As $C$ is a prefix code, no proper prefix of $w_0$ is in $C$. According to the assumption, it follows that no prefix of $w'_0$ is in $C$.

Now consider $i > 0$ and suppose the sequences have been constructed up to step $i-1$. No prefix of $w'_{i-1}$ is in $C$ by construction. As $C$ is maximal, by Lemmas 1 and 2, there exist $x_i \in A^*$ and $b_i \in A$ such that $w_i = w'_{i-1} x_i b_i$; let $y_i \in A^*$ and $d_i \in A$, such that $y_i d_i = a_1 x_i$. The $w_i$ has the required form. Suppose, $b_i = b_j$ for some $j < i$, we have $c_1 \leq_i w'_j \omega_n w'_0$, it contradicts the assumption. Thus,

$$b_j \notin \{b_0, \ldots, b_{j-1}\}.$$ 

In particular, for $j = 0$, one finds that $w'_j \notin C$. As $C$ is a prefix code, no proper prefix of $w'_j$ is in $C$. This implies that no prefix of $w'_j$ is in $C$.

It is a contradiction since $i$ is unbounded and $A$ is finite. Therefore, $uc_1 a_m a_{m-1} \ldots a_2 a_1 \subseteq C$. 

(ii) Next, we verify that 

\[ uca_1a_2a_3\ldots a_k+1 \subseteq C, \text{ for } 0 \leq k \leq m - 1. \]

For \( k = 0 \), by (i), clearly, 

\[ uca_1a_2a_3\ldots a_k+1 \subseteq C. \]

Now consider \( k \) and suppose the conclusion has been true for \( k - 1 \). Suppose there are \( b, e_1, e_2, \ldots, e_k \in A \) such that, 

\[ uca_1a_2a_3\ldots a_k+1 \subseteq C. \]

Now consider 

\[ uca_1a_2a_3\ldots a_k+1b_k+1a_k\ldots a_1. \]

As \( c_1 \leq c_2 = uca_1a_2a_3\ldots a_k+2ba_k+1a_k\ldots a_1, \) thus 

\[ uca_1a_2a_3\ldots a_k+1b_k+1a_k\ldots a_1 \notin C. \]

By the assumption, it is easy to see that no prefix of 

\[ uca_1a_2a_3\ldots a_k+1b_k+1a_k\ldots a_1 \]

is in \( C \). Otherwise, it is a proper prefix of the words in 

\[ uca_1a_2a_3\ldots a_k+1a_k^{k+1} \subseteq C \]

or we deduce that 

\[ uca_1a_2a_3\ldots a_k+2bAk^{k+1} \subseteq C. \]

Therefore, 

\[ uca_1a_2a_3\ldots a_k+2ba_k+1a_k\ldots a_1 \]

Let \( a_k \ldots a_1x_1 = y_1e_1^1 \ldots e_k^1 \) for some \( y_1 \in A^*, e_1^1, \ldots, e_k^1 \in A \), then 

\[ uca_1a_2a_3\ldots a_k+2ba_k+1y_1e_1^1 \ldots e_k^1b_1 \in C, \]

moreover, \( e_k^1 \neq a_{k+1} \). Otherwise, by 

\[ uca_1a_2a_3\ldots a_k+2ba_k+1y_1a_{k+1}e_1^1 \ldots e_k^1b_1 \in C \]

and the assumption, 

\[ uca_1a_2a_3\ldots a_k+2ba_k+1y_1a_{k+1}a_k^{k+1} \subseteq C. \]

Since 

\[ c_1 \leq; uca_1a_2a_3\ldots a_k+1a_1 \omega_2uca_1a_2a_3\ldots a_k+2ba_k+1y_1a_{k+1}a_k^{k+1} \]

this is a contradiction with \( C \) being a 3-infix–outfix code. Again consider 

\[ uca_1a_2a_3\ldots a_k+2ba_k+1y_1a_{k+1}a_k^{k+1} \]

By the assumption and 

\[ uca_1a_2a_3\ldots a_k+2ba_k+1y_1e_1^1 \ldots e_k^1b_1 \in C, \]

we obtain that no proper prefix of 

\[ uca_1a_2a_3\ldots a_k+2ba_k+1y_1a_{k+1}a_k^{k+1} \]

is in \( C \). Therefore, by Lemmas 1 and 2, there exist \( y_2 \in A^*, b_2, e_1^2, \ldots, e_k^2 \in A \) such that, 

\[ uca_1a_2a_3\ldots a_k+2ba_k+1y_1a_{k+1}y_2e_1^2 \ldots e_k^2b_2 \in C, \]

moreover, \( e_k^2 \neq a_{k+1}, e_1^2 \). Repeating the above procedures, there are 

\[ y_1, y_2, \ldots \in A^*, e_1^i, e_2^i, \ldots, e_k^i \in A, i \geq 1, \text{ and } b_0, b_1, b_2, \ldots \in A \]
such that
\[ uc_1 a_m a_{m-1} \ldots a_k b a_{k+1} y_1 e_1 \ldots e_k b_j \in C, \]
\[ uc_1 a_m a_{m-1} \ldots a_k b a_{k+1} y_1 a_{k+1} y_2 e_2 \ldots e_k b_2 \in C, \]
\[ uc_1 a_m a_{m-1} \ldots a_k b a_{k+1} y_1 a_{k+1} y_2 a_{k+1} y_3 e_3 \ldots e_k b_3 \in C, \]
\[ \vdots \]
\[ uc_1 a_m a_{m-1} \ldots a_k b a_{k+1} y_1 a_{k+1} y_2 a_{k+1} y_3 e_3 \ldots e_k b_j \in C. \]

Moreover, \( e'_i \notin \{ a_{k+i}, e_1, \ldots, e_i^{(-1)} \}, \ i \geq 1. \)

It is a contradiction since \( i \) is unbounded and \( A \) is finite. Therefore,
\[ uc_1 a_m a_{m-1} \ldots a_k b a_{k+1} \in C. \]

In particular, \( k = m - 1, \ uc_1 A^m \subseteq C, \ uc_1 A^{|r|} \subseteq C. \) By the duality and \( uc_1 A^{|r|} \subseteq C. \) we get
\[ A^{|n|} \subseteq c_1 A^{|r|} \subseteq C. \]

This completes the proof of Theorem 1. \( \Box \)

**Theorem 2.** Let \( C \subseteq A^* \) be a 3-infix–outfix code and a maximal code. Then there exists a positive integer \( n \) such that \( a^n \in C \) for all \( a \in A \).

**Proof.** Assume that \( C \) is an outfix code, by \([4]\) or \([7]\), then \( C = A^n \) for some \( n \). Thus, the conclusion is obvious. Now, we suppose that \( C \) is not an outfix code, there exist \( c_1, c_2, \ldots, c_2 \neq c_1 \) such that \( c_1 c_2 c_2 \in C \), i.e., there exist \( u, v, x \in A^+ \) such that \( c_1 = uv, c_2 = uxv \).

We first prove that, for every \( a \in A \), there exists \( m \) such that \( a^m \in C \). Suppose that \( a^l \notin C \) for all \( l \), then we obtain the following contradiction. As \( C \) is a 3-infix–outfix code and \( c_1 c_2 c_1 A^\ast \cap C = \emptyset \). Now we consider \( c_1 c_2 \). Since \( c_1 c_2 \notin C \cup C(A^+) \), by Lemma 1, \( c_1 c_2 \in C A^+ \). Therefore, there exists \( x_1 \in A^+ \) which is a proper prefix of \( c_1 \), such that \( ax_1 \in C \). Similarly, consider \( a^2 c_1, a^3 c_1, \ldots, a^n c_1, m \geq 2 \). We get that \( a^2 c_1, a^3 c_1, \ldots, a^n c_1 \in C A^+ \). Hence, there exist \( x_2, x_3, \ldots, x_m \), they are the proper prefixes of \( c_1 \), such that \( a^i x_2, a^i x_3, \ldots, a^i x_m \in C \). Since \( |c_1| \) is finite, there exist \( x_i, x_j \) such that \( a^i x_i, a^i x_j \in C \) and \( i \neq j \). This is a contradiction with \( C \) being a bifix code. Therefore, for every \( a \in A \), there is \( m \) such that \( a^m \in C \). Now let \( a \neq b \in A \), \( a^p, b^q \in C \). We show that \( p = q \). If \( p > q \) then we deduce a contradiction. Consider \( a^{p-1} b^q \). As \( C \) is a bifix code, \( a^{p-1} b^q \in C \). By Lemmas 1 and 2, \( a^{p-1} b^q \in C A^+ \cup C(A^+)^{-1} \). If \( a^{p-1} b^q \in C(A^+)^{-1} \), there exists \( y \in A^+ \) such that \( a^{p-1} b^q y \notin C \). According to Theorem 1, \( b^q \notin a^{p-1} b^q y \). But \( b^q \) is the proper suffix of a word in \( a^{p-1} b^q A^{|r|} \), this is impossible. Thus, \( a^{p-1} b^q \notin C(A^+)^{-1} \). Similarly, consider \( a^{p-2} b^q, a^{p-3} b^q, \ldots, a^2 b^q, a b^q \). Then \( a^{p-2} b^q, a^{p-3} b^q, \ldots, a^2 b^q, a b^q \in C A^+ \). There exist \( l \leq l_{p-1}, l_{p-2}, \ldots, l_2, l_1 \leq q - 1 \) such that
\[ a^{p-1} b^{l_{p-1}}, a^{p-2} b^{l_{p-2}}, a^{p-3} b^{l_{p-3}}, \ldots, a^2 b^{l_2}, a b^{l_1} \in C. \]
As $C$ is a bifix code, $l_s \neq l_t$ for $1 \leq s \neq t \leq p - 1$. Therefore, $p - 1 \leq q - 1$, $p \leq q$, a contradiction. Hence, there exists a common $n$ such that $a^n \in C$. □

We now give the structure theorem for 3-infix–outfix codes that are maximal codes. The three following Lemmas were given in [7].

**Lemma 3** (Long, [7, Lemma 3]). Let $C$ be a code over $A$. If $\text{syn}(C^*)$ is a group, then $C$ is a maximal prefix code and a maximal suffix code.

**Lemma 4** (Long, [7, Corollary 5]). Let $C$ be a code over $A$. If $C^*$ is reflective, then $\text{syn}(C^*)$ is a group.

**Lemma 5** (Long, [7, Lemma 6]). Let $C$ be a code over $A$. Then $C$ is a full uniform code if and only if $C$ and $C^*$ are reflective.

**Theorem 3.** Let $C \subseteq A^*$ be a 3-infix–outfix code. Then the following conditions are equivalent:

1. $C$ is a maximal code;
2. $C$ is a maximal prefix code;
3. $C$ is a full uniform code, $C = A^n$ for some $n$;
4. $C$ is a group code, $\text{syn}(C^*)$ is a group;
5. $\text{syn}(C^*)$ is a cyclic group of order $n$ for some $n$;
6. $C^*$ is reflective, $uv \in C^*$ implies $vu \in C^*$.

**Proof.** By the definitions, we get $(1) \Rightarrow (2)$, $(3) \Rightarrow (5)$, $(5) \Rightarrow (4)$; By Lemmas 3, 4, and 5, we obtain $(4) \Rightarrow (1)$, $(6) \Rightarrow (4)$, $(3) \Rightarrow (6)$. Therefore, it suffices to prove that $(2) \Rightarrow (3)$. Then we can get $(4) \Rightarrow (1) \Rightarrow (2) \Rightarrow (3) \Rightarrow (5) \Rightarrow (4)$, $(6) \Rightarrow (4) \Rightarrow (3) \Rightarrow (6)$.

By Theorem 2, there exists a positive integer $n$ such that $a^n \in C$ for every $a \in A$. By induction on $i$, we shall prove that: For all $a, a_1, a_2, \ldots, a_i \in A$, $1 \leq i \leq n - 1$, the following words are in $C$:

\[
\begin{align*}
a^{n-i}a_1a_2\ldots a_{i-1}a_i, \\
a^{n-(i+1)}a_1a_2\ldots a_{i-1}a_i^2, a^{n-(i+1)}a_1a_2\ldots a_{i-1}a_i, \\
a^{n-(i+2)}a_1a_2\ldots a_{i-1}a_i^3, a^{n-(i+2)}a_1a_2\ldots a_{i-1}a_i^2, a^{n-(i+2)}a_1a_2\ldots a_{i-1}a_i, \\
\vdots \\
a^n_i a_1a_2\ldots a_{i-1}a_i^{n-i}, a^n_i a_1a_2\ldots a_{i-1}a_i^{n-(i+2)}, \ldots, a^n_i a_1\ldots a_{i-1}a_i^{n-(i+1)}, \\
a_1a_2\ldots a_{i-1}a_i^{n-i}, a_1a_2\ldots a_{i-1}a_i^{n-(i+2)}, \ldots, a_1\ldots a_{i-1}a_i^{n-i}.
\end{align*}
\]

(i) For $i = 1$, we verify that $a^{n-1}a_1, a^{n-2}a_1^2, \ldots, a_1^{n-1}$ are in $C$. By the assumption, $a^n_i a_1^i \in C$. Consider $a^{n-1}a_1^i, a^{n-2}a_1^2, \ldots, a_1^{n-1}$. According to Theorem 1, $a^n_i a_1^i, a^{n-2}a_1^2, \ldots, a_1^{n-1} \notin C \cup C(A^+)^{-1}$, thus they are in $CA^+$. Therefore, there exist $1_{n-1}^1, 1_{n-2}^1, \ldots,$
\( l_1, l_2 \) such that
\[
a^{n-1}a_{l_1-1}^{l_1}, a_{l_1}^{n-2}a_{l_1-2}, \ldots, a_{l_1}^{l_1} \in C.
\]

As \( C \) is a bifix code, \( 1 \leq l_1 \neq l_2 \leq n-1 \), for \( 1 \leq s \neq t \leq n-1 \). Again \( C \) is a 3-infix–outfix code, then \( l_1 \neq 1 \). Otherwise, we have \( a_{l_1}^{l_1} \leq \ a_{l_1}^{l_1}a_{l_1}^{l_1-n}a_{l_1}^{l_1-n}a_{l_1}^{l_1-n} \). This is impossible. Similarly, we have \( l_1, l_2, \ldots, l_{n-2} \neq 1 \). Hence, \( l_{n-1} = 1 \). If \( l_1 = 2 \), then \( a_{l_1}^{l_1} \leq \ a_{l_1}^{l_1-n}a_{l_1}^{l_1-n}a_{l_1}^{l_1-n}a_{l_1}^{l_1-n} \); this contradicts with \( C \) being a 3-infix–outfix code. Therefore, \( l_1 \neq 2 \). Similarly, \( l_2 \neq 2, \ldots, l_{n-3} \neq 2, l_{n-2} = 2 \). Repeating the above procedures, we have that \( l_{n-3} = 3, \ldots, l_2 = n-2, l_1 = n-1 \), therefore,
\[
a^{n-1}a_{1}^{n-2}a_{1}^{2}, \ldots, a_{1}^{n-2}a_{1}^{2}, a_{1}^{n-1} \in C.
\]

(ii) Suppose that the conclusion is true for \( i = k \). For all \( a, a_1, a_2, \ldots, a_k \in A \), the following words are in \( C \):
\[
a^{n-k}a_1a_2 \ldots a_{k-1}a_k,
\]
\[
a^{n-(k+1)}a_1a_2 \ldots a_{k-1}a_k^{k+1}, a_{n-(k+1)}a_1a_2 \ldots a_{k-1}a_k^{2},
\]
\[
a^{n-(k+2)}a_1a_2 \ldots a_{k-1}a_k^{2}, a_{n-(k+2)}a_1a_2 \ldots a_{k-1}a_k^{3}, a_{n-(k+2)}a_1a_2 \ldots a_{k-1}a_k.
\]
\[\vdots\]
\[
a^{2}a_1a_2 \ldots a_{k-1}a_k^{n-k+1}, a_{n-k+1}a_1a_2 \ldots a_{n-k+1}a_k^{n-k+2}, \ldots, a_{n-k+1}a_1a_2 \ldots a_{n-k+1}a_k.
\]
\[
a_{1}a_1 \ldots a_{k-1}a_{k-1}^{n-k}, a_1a_2 \ldots a_{k-1}a_{k-1}^{n-k}a_{k-1}a_1 \ldots a_{k-1}a_{k-1}^{n-k}a_{k-1}a_{k-1}.
\]

By the assumption, we prove that the conclusion is true for \( i = k + 1 \). Consider the following words:
\[
a^{n-(k+1)}a_1a_2 \ldots a_{k-1}a_k^{n-k},
\]
\[
a^{n-(k+2)}a_1a_2 \ldots a_{k-1}a_k^{n-k+1}a_{n-(k+2)}a_1a_2 \ldots a_{k-1}a_k^{2},
\]
\[
a^{n-(k+3)}a_1 \ldots a_{k-1}a_k^{n-k}, a_{n-(k+3)}a_1 \ldots a_{k-1}a_k^{2}a_{k-1}^{n-k}, a_{n-(k+3)}a_1 \ldots a_{k-1}a_k^{3}a_{k-1}^{n-k+1},
\]
\[\vdots\]
\[
a^{2}a_1 \ldots a_{k-1}a_k^{n-k+1}, a^{2}a_1 \ldots a_{k-1}a_k^{n-k}a_{k-1}^{2}a_{k-1}^{n-k+1}, \ldots, a^{2}a_1a_2 \ldots a_{k-1}a_k^{n-k+2}a_{k-1}^{2}a_{k-1}^{n-k+1},
\]
\[
a_{1}a_1 \ldots a_{k-1}a_{k-1}^{n-k+1}, a_1a_2 \ldots a_{k-1}a_{k-1}^{n-k}a_{k-1}a_1 \ldots a_{k-1}a_{k-1}^{n-k}a_{k-1}^{2}a_{k-1}^{n-k+1}.
\]

By the assumption and Theorem 1, \( a^{n-(k+1)}a_1 \ldots a_{k-1}a_k^{n-k+1}a^{n-k+1} \cap C = \emptyset \). Otherwise, \( a_{k+1}^{n-k+1} \) and \( a^{n-(k+1)}a_1 \ldots a_{k-1}a_k^{n-k+1}y \in C, a_{k+1}^{n-k+1} \) must be a proper suffix of a word in \( C \). This is impossible. Again by the assumption, no proper prefix of \( a^{n-(k+1)}a_1 \ldots a_{k-1}a_k^{n-k+1} \) is in \( C \), thus, \( a^{n-(k+1)}a_1 \ldots a_{k-1}a_k^{n-k+1} \notin C \cup C(A^{+})^{-1}, a^{n-(k+1)}a_1 \ldots a_{k-1}a_k^{n-k+1} \in CA^{+} \).
Therefore, there exist some positive integers:

\[ l_{n-(k+1)}, l_{n-(k+2)}, \ldots, l_{2}, l_{1}, \]
\[ l_{n-(k+2)}, l_{n-(k+3)}, \ldots, l_{2}, l_{1}, \]
\[ l_{n-(k+3)}, l_{n-(k+4)}, \ldots, l_{2}, l_{1}, \]
\[ \vdots \]
\[ l_{n-(k+3)}, l_{n-(k+3)}, l_{1}, \]
\[ l_{2}, l_{n-(k+2)}, l_{1}, \]
\[ l_{n-(k+1)}, l_{1}, \]

such that the following words are in C:

\[ a^{n-(k+1)}a_{1}a_{2} \ldots a_{k-1}a_{k}a_{k+1}, \]
\[ a^{n-(k+2)}a_{1}a_{2}a_{3} \ldots a_{k-1}a_{k}a_{k+1}, \]
\[ a^{n-(k+3)}a_{1}a_{2} \ldots a_{k-1}a_{k}a_{k+1}, \]
\[ \vdots \]
\[ a_{1}a_{2} \ldots a_{k-1}a_{k}a_{k+1}, a_{1}a_{2}a_{3} \ldots a_{k-1}a_{k}a_{k+1}, \]

According to the choice of the above words, \( l_{p} \neq l_{q}, 1 \leq p \neq q \leq n - (k + 1) \). By the assumption, for all \( a, a_{1}, a_{2}, \ldots, a_{k} \in A \),

\[ a_{1} \ldots a_{k-1}a_{k}^{n-k}, a_{1}a_{2} \ldots a_{k-2}a_{k-1}^{n-(k+1)}, a_{1}a_{2}a_{3} \ldots a_{k-3}a_{k-2}^{n-(k+2)}, \ldots, a_{1}a_{2} \ldots a_{k-1}a_{k}^{n-k} \in C. \]

Then,

\[ 1 \leq l_{n-(k+1)}, l_{n-(k+2)}, \ldots, l_{2}, l_{1} \leq n - (k + 1). \]

Otherwise, if \( l_{n-(k+i)} \geq n - k \) then \( a^{n-(k+i)}a_{1}a_{2} \ldots a_{k-1}a_{k}a_{k+1}^{l_{n-(k+i)}} \in C \) and a proper suffix of \( a^{n-(k+i)}a_{1}a_{2} \ldots a_{k-1}a_{k}a_{k+1}^{l_{n-(k+i)}} \) with the \( n \) length is in \( C \). It is a contradiction. Similarly,

\[ 1 \leq l_{n-(k+2)}, l_{n-(k+3)}, \ldots, l_{2}, l_{1} \leq n - (k + 2), \]
\[ 1 \leq l_{n-(k+3)}, l_{n-(k+4)}, \ldots, l_{2}, l_{1} \leq n - (k + 3), \]
\[ \vdots \]
If \( l_1^1 = 1 \), then
\[
\begin{align*}
aa_1a_2 \ldots a_{k-1}a_k a_{k+1} &\leq a^{n-(k+1)}a_1a_2 \ldots a_{k-1}a_k a_{k+1}, \\
aa_1a_2 \ldots a_{k-1}a_k a_{k-1}(o_o)aa_1a_2 \ldots a_{k-1}a_k a_{k+1} &\leq a^{n-(k+1)}a_1a_2 \ldots a_{k-1}a_k a_{k+1}.
\end{align*}
\]
This contradicts with \( C \) being a 3-infix–outfix code. Thus, \( l_1^1 \neq 1 \). Similarly, \( l_1^2 \neq 1, l_1^3 \neq 1, \ldots, l_1^{n-(k+2)} \neq 1 \), therefore, \( l_1^{n-(k+1)} = 1 \). Repeating the above procedures, we get
\[
\begin{align*}
l_1^{n-(k+1)} - 1, l_2^{n-(k+2)} - 1, \ldots, l_{n-(k+1)}^{n-(k+1)} &= 1, \\
l_1^{n-(k+2)} - 1, l_2^{n-(k+3)} - 1, \ldots, l_{n-(k-2)}^{n-(k+2)} &= 2, \\
l_1^{n-(k+3)} - 1, l_2^{n-(k+4)} - 1, \ldots, l_{n-(k+3)}^{n-(k+3)} &= 3, \\
&\vdots \\
l_1^n - 1, l_2^n - 1, l_3^n - 1 &= n - (k + 2), \\
l_1^n - 1, l_2^n - 1, l_3^n - 1 &= n - (k + 1).
\end{align*}
\]
Thus, the following words are in \( C \):
\[
\begin{align*}
a^{n-(k+1)}a_1a_2 \ldots a_{k-1}a_k a_{k+1}, \\
a^{n-(k+2)}a_1a_2 \ldots a_{k-1}a_k a_{k+1}^2 a_{k+1}, a^{n-(k+2)}a_1a_2 \ldots a_{k-1}a_k a_{k+1}^2 a_{k+1}, \\
a^{n-(k+3)}a_1a_2 \ldots a_{k-1}a_k a_{k+1}^2 a_{k+1}, a^{n-(k+3)}a_1a_2 \ldots a_{k-1}a_k a_{k+1}^2 a_{k+1}, \\
&\vdots \\
a^{n-(k+2)}a_1a_2 \ldots a_{k-1}a_k a_{k+1}^2 a_{k+1}, a^{n-(k+3)}a_1a_2 \ldots a_{k-1}a_k a_{k+1}^2 a_{k+1}, a^{n-(k+2)}a_1a_2 \ldots a_{k-1}a_k a_{k+1}^2 a_{k+1}, \\
a^{n-(k+1)}a_1a_2 \ldots a_{k-1}a_k a_{k+1}^2 a_{k+1}, a^{n-(k+1)}a_1a_2 \ldots a_{k-1}a_k a_{k+1}^2 a_{k+1}, a^{n-(k+1)}a_1a_2 \ldots a_{k-1}a_k a_{k+1}^2 a_{k+1}.
\end{align*}
\]
This proves that the conclusion is true for \( i = k + 1 \).

Combining (i) and (ii), we obtain that for all \( a, a_1, a_2, \ldots, a_l \in A, 1 \leq i \leq n - 1, a^{n-i} a_1 \ldots a_i \in C \). In particular, taking \( i = n - 1 \), one has \( aa_1 \ldots a_{n-1} \in C \). \( A^n \subseteq C \). As \( C \) is a maximal code, \( C = A^n \). This completes the proof of Theorem 3. \( \square \)

According to Theorem 3, we have

**Corollary 1.** There exist no infinite maximal codes which are 3-infix–outfix.
Corollary 2. There exists no maximal code which is 3-infix–outfix but neither infix nor outfix.

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