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Algorithm for Solving a Linear Exchange Model with Rationing

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Abstract—An exchange model with fixed prices, linear utility functions and functions that construct the constraints of rationing is considered. A finite algorithm for finding a solution of the model is proposed.

1. INTRODUCTION

In exchange models, the equality between supply and demand is achieved by a choice of a set of equilibrium prices. Sometimes it makes sense to restrict the demand of consumers so, that in the process of solution, no demand for a product can become excessively large. An exchange model with rationing is sufficient to consider with fixed prices. Such kind of problems are discussed in [1–3]. The reasons for studying exchange models with fixed prices and rationing are considered in [4].

The characteristics of the model are as follows.

Consumers' preferences are described by linear utility functions. Their demand is restricted by linear functions that depend on rationing parameters, the same for all consumers but different for all products.

If it turns out that fixed prices are above the corresponding equilibrium prices, then the model has no solution.

An algorithm for this exchange model is built in the present paper. It is structured as follows: In Section 1, the exchange model is formulated. In Section 2, we describe a finite algorithm for the model. The properties of the algorithms are considered in Section 3. In Section 4, a simple numerical example is considered.

2. FORMULATION OF THE MODEL

We study an exchange model similar to those considered by Polterovich [5] and Movshovich [6]. The model consists of m consumers and n products, with $\delta_j > 0$ units of product j, $j = \overline{1, n},$ in the market.

The behavior of consumer i is determined by his utility function $\varphi_i(x_i) \equiv \varphi_i(\xi_{i1}, \dots, \xi_{in})$, by a fixed budget $\beta_i > 0$, by functions $\Psi_{ij}(\tau_j)$ that determine the limits of demand for product j depending on the unknown parameters τ_j, and by a fixed vector p of prices $\pi_1, \dots, \pi_n$.

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Denote $d := (\delta_1, \dots, \delta_n)$. We have to find vectors $x_i^* \ge 0$ and scalars $\tau_j^* \ge 0$ such that

$$x_i^* = \text{argmax } \{\varphi_i(x_i) \mid \langle p, x_i \rangle \le \beta_i;\ 0 \le \xi_{ij} \le \Psi_{ij}(\tau_j^*),\ j = \overline{1,n}\},\ i = \overline{1,m};$$
$$\sum_{i=1}^{m} x_i^* = d. \tag{1}$$

Suppose that

$$\langle p, d \rangle \le \sum_{i=1}^{m} \beta_i.$$

Let us denote

$$t = (\tau_1, \dots, \tau_n)$$

and

$$\Phi_i(t) := \max_{x_i} \{\varphi_i(x_i) \mid \langle p, x_i \rangle \le \beta_i;\ 0 \le \xi_{ij} \le \Psi_{ij}(\tau_j)\}.$$

If functions $\varphi_i(x_i)$ and $\Psi_{ij}(\tau_j)$, $i = \overline{1,m}$, $j = \overline{1,n}$ are concave, then functions $\Phi_i(t)$ are concave, too.

It is not difficult to prove that each solution of the following system

$$\varphi_i(x_i) \ge \Phi_i(t);\ \langle p, x_i \rangle \le \beta_i;$$
$$0 \le \xi_{ij} \le \Psi_{ij}(\tau_j);\ \sum_{i=1}^{m} x_i = d,\ i = \overline{1,m};\ j = \overline{1,n} \tag{2}$$

is a solution of the exchange model.

But, unfortunately, system (2) defines a nonconvex set and it is difficult to construct a method based on solving this system. That is why we have to look for other methods.

Below we propose a method that is based on an algorithm for a linear programming problem with restricted nonnegative variables and only one common constraint.

3. ALGORITHM FOR THE EXCHANGE MODEL

We assume that functions $\varphi_i(x_i)$ and $\Psi_{ij}(\tau_j)$ are linear. Let $\varphi_i(x_i) = \langle a_i, x_i \rangle = \sum_{j=1}^{n} \alpha_{ij}\xi_{ij}$, $\Psi_{ij}(\tau_j)$ be increasing functions and $\Psi_{ij}(0) \ge 0$.

To simplify our consideration, suppose that $a_i > 0$. The quotients α_{ij}/π_j are important for the algorithms, since if $\alpha_{ij_1}/\pi_{j_1} \ge \alpha_{ij}/\pi_j, \forall j \ne j_1$, then for the variable ξ_{ij_1} we have

$$\xi_{ij_1} = \min\left\{\Psi_{ij_1}(\tau_{j_1}), \frac{\beta_i}{\pi_{j_1}}\right\}.$$

This property is used for successive determination of the values of the model's variables in accordance with the preferences revealed by the quotients. For this purpose we will use an operator $XT1$ described below.

We say that positive variables $\xi_{ij_1}, \dots, \xi_{ij_k}$ satisfy the complementary property if

$$\frac{\alpha_{ij_1}}{\pi_{j_1}} \ge \frac{\alpha_{ij_2}}{\pi_{j_2}} \ge \dots \ge \frac{\alpha_{ij_k}}{\pi_{j_k}}, \quad \xi_{ij_q} = \Psi_{ij_q}(\tau_{j_q}), \quad q = \overline{1, k-1}; \tag{3}$$

and

$$\xi_{ij_k} = \min\left\{\Psi_{ij_k}(\tau_{j_k}); \frac{\beta_i - \sum_{q=1}^{k-1} \pi_{j_q}\xi_{ij_q}}{\pi_{j_k}}\right\}. \tag{4}$$

The sense of the operator $XT1$ is to find for the product $\bar{j}$, that is currently preferred by some consumers, the current values $\tau_{\bar{j}}$ and $\xi_{i\bar{j}}$ according to the budgets β_i, supply $\delta_{\bar{j}}$, functions $\Psi_{i\bar{j}}(\tau_{\bar{j}})$ and conditions (3),(4).

To describe the operator $XT1$, let vectors $\tilde{x}_i \geq 0$ and $\tilde{t} \geq 0$ satisfy the following inequalities

$$\langle p, \tilde{x}_i \rangle \leq \beta_i, \quad i = \overline{1,m}; \quad 0 \leq \tilde{\xi}_{ij} \leq \Psi_{ij}(\tilde{\tau}_j), \quad i = \overline{1,m}, \quad j = \overline{1,n},$$

$\bar{j}$ be a fixed product, L be a set of consumers, that currently prefer the product $\bar{j}$ and $\bar{x}_i$ be vectors with components

$$\bar{\xi}_{ij} = \tilde{\xi}_{ij}, \quad \forall j \neq \bar{j}, \quad \bar{\xi}_{i\bar{j}} = 0.$$

According to the complementary property $\xi_{i\bar{j}}$ must be equal to $\kappa_{i\bar{j}}(\tau_{\bar{j}})$ defined below

$$\kappa_{i\bar{j}}(\tau_{\bar{j}}) := \min\left\{\frac{\beta_i - \langle p, \bar{x}_i \rangle}{\pi_{\bar{j}}}, \Psi_{i\bar{j}}(\tau_{\bar{j}})\right\}, \quad i \in L.$$

The operator $XT1$ consists of calculating new values $\tilde{\tau}_{\bar{j}}$ and $\tilde{\xi}_{i\bar{j}}$ according to the following rules. If $\sum_{i \in L}(\beta_i - \langle p, \bar{x}_i \rangle)/\pi_{\bar{j}} < \delta_{\bar{j}}$, then we can assign

$$\tilde{\xi}_{i\bar{j}} := \frac{\beta_i - \langle p, \bar{x}_i \rangle}{\pi_{\bar{j}}}, \quad i \in L,$$

and

$$\tilde{\tau}_{\bar{j}} := \min\left\{\tau_{\bar{j}} \mid \frac{\beta_i - \langle p, \bar{x}_i \rangle}{\pi_{\bar{j}}} \leq \Psi_{i\bar{j}}(\tau_{\bar{j}}), \ i \in L\right\}.$$

It is clear that $\tilde{\xi}_{i\bar{j}} = \kappa(\tilde{\tau}_{\bar{j}}), i \in L$.

If $\sum_{i \in L'}(\beta_i - \langle p, \bar{x}_i \rangle)/\pi_{\bar{j}} \geq \delta_{\bar{j}}$, then there is a possibility to achieve an equality of the demand and supply. To do that it is sufficient to find the solution $\tilde{\tau}_{\bar{j}}$ of the following equation

$$\sum_{i \in L} \kappa(\tau_{\bar{j}}) = \delta_{\bar{j}}.$$

Then we assign

$$\tilde{\xi}_{i\bar{j}} := \kappa(\tilde{\tau}_{\bar{j}}), \quad i \in L.$$

Since the operator $XT1$ is used for a product that is currently preferred, the condition (3) is always satisfied. If the operator $XT1$ is used for a product $\bar{j}$ such that the demand less than the supply before performing $XT1$, then no violation of the condition (4) for other products can happen. But if the demand equals the supply such a violation can appear. Since no violation of the conditions (3),(4) is permitted we have to correct our decisions to achieve the complementary property. To do that for products with equality of the demand and supply we use an operator $XT2$. To achieve the complementary property for products with a current violation of this equality the operator $XT1$ can be used for each such product separately.

To describe the operator $XT2$ we have to form the set E of products with a current violation of the condition (4) and the corresponding set I of consumers with at least one positive $\tilde{\xi}_{ij}, j \in E$. To do that let us define

$$Q := \{i : \langle p, \tilde{x}_i \rangle < \beta_i\}; \quad G := \left\{j : \sum_{i=1}^{m} \tilde{\xi}_{ij} < \delta_j\right\},$$

and then

$$E := \{j \bar{\in} G : \exists_i (i \in Q, 0 < \tilde{\xi}_{ij} < \Psi_{ij}(\tilde{\tau}_j))\};$$
$$I := \{i : \exists_j (j \in E, \tilde{\xi}_{ij} > 0)\}.$$

In addition, let us define numbers:

$$\mu_{ij} = \begin{cases} 1, & \text{if } 0 < \tilde{\xi}_{ij} = \Psi_{ij}(\tilde{\tau}_j), \\ 0, & \text{if } \tilde{\xi}_{ij} = 0 \text{ or } \tilde{\xi}_{ij} < \Psi_{ij}(\tilde{\tau}_j), \end{cases}$$
$$\nu_{ij} = \begin{cases} 0, & \text{if } \tilde{\xi}_{ij} = 0 \text{ or } 0 < \tilde{\xi}_{ij} = \Psi_{ij}(\tilde{\tau}_j), \\ 1, & \text{if } 0 < \tilde{\xi}_{ij} < \Psi_{ij}(\tilde{\tau}_j), \quad i \in I, j = \overline{1,n}. \end{cases}$$

Further, denote $\hat{\xi}_{ij} := \tilde{\xi}_{ij}, j \bar{\in} E; \hat{\xi}_{ij} := 0, j \in E; \hat{x}_i = (\hat{\xi}_{i1}, \cdots, \hat{\xi}_{in})$. Let τ_j^o and ζ_{ij}^o be a solution of the following linear programming problem:

$$\min \Bigg\{ \sum_{j \in E} \tau_j \mid \sum_{j \in E} \mu_{ij} \pi_j \Psi_{ij}(\tau_j) + \sum_{j \in E} \nu_{ij} \pi_j \zeta_{ij} \le \beta_i - \langle p, \hat{x}_i \rangle,$$
$$i \in I;\ \sum_{i \in I} \mu_{ij} \Psi_{ij}(\tau_j) + \sum_{i \in I} \nu_{ij} \zeta_{ij} = \delta_j,\ \tau_j \ge 0,\ j \in E; \tag{5}$$
$$0 \le \zeta_{ij} \le \Psi_{ij}(\tau_j),\ \forall_{i,j} \left(i \in I, j \in E, 0 < \tilde{\xi}_{ij} < \Psi_{ij}(\tilde{\tau}_j) \right) \Bigg\}.$$

To complete the description of the operator $XT2$ it is sufficient to calculate

$$\tilde{\xi}_{ij} = \mu_{ij} \Psi_{ij}(\tau_j^o) + \nu_{ij} \zeta_{ij}^o.$$

Let us define sets $J_i(x_i) := \operatorname{Arg\,max}_{\xi_{ij}=0} \alpha_{ij}/\pi_j$, $i = \overline{1,m}$. To describe the algorithm for the exchange model we let $\tilde{x}_i = 0$, $i = \overline{1,m}$, $\tilde{\tau}_j = 0$, $j = \overline{1,n}$; $G := \{1, \dots, n\}$; form sets Q and $M := \cup_{i \in Q} J_i(\tilde{x}_i)$.

The sequence of computation is as follows:

(1) if $M \cap G \neq \emptyset$, choose $\bar{j} \in M \cap G$; if $M \cap G = \emptyset$, then choose $\bar{j} \in M$;
(2) form sets $N := \{i : \tilde{\xi}_{i\bar{j}} > 0\}; D := \{i \in Q : \bar{j} \in J_i(\tilde{x}_i)\}$; and $L := D \cup N$;
(3) perform $XT1$;
(4) if $M \cap G \neq \emptyset$, correct set G and go to 10; if $M \cap G = \emptyset$, go to 5;
(5) form the set Q;
(6) if $E = \{j \bar{\in} G : \exists_i (i \in Q, 0 < \tilde{\xi}_{ij} < \Psi_{ij}(\tilde{\tau}_j)\} \neq \emptyset$, go to 7; if $E = \emptyset$, then go to 8;
(7) perform $XT2$;
(8) form the set $E_o = \{j \in G : \exists_i (i \in Q, \tilde{\xi}_{ij} > 0)\}$; if $E_o \neq \emptyset$, go to 9; if $E_o = \emptyset$, then go to 10;
(9) for each $\bar{j} \in E_o$ use $XT1$ with $L = \{i : \bar{j}_{i\bar{j}} > 0\}$ and correct G;
(10) form sets Q and $M := \cup_{i \in Q} J_i(\tilde{x}_i)$;
(11) if $M = \emptyset$, go to 12; if $M \neq \emptyset$, then go to 1;
(12) end.

4. PROPERTIES OF THE ALGORITHM

To begin with, we establish

LEMMA 1. *If*

$$\sum_{i=1}^{m} \Psi_{ij}(0) < \delta_j, \quad j = \overline{1,n}, \tag{6}$$

then $XT1$ can be executed.

PROOF. It is clear that if

$$\sum_{i \in L} \frac{\beta_i - \langle p, \bar{x}_i \rangle}{\pi_{\bar{j}}} < \delta_{\bar{j}}, \tag{7}$$

then $XT1$ can be executed.

If (5) does not hold, there is $\bar{\tau}_j$ such that

$$\sum_{i\in L} \kappa_{i\bar{j}}(\bar{\tau}_{\bar{j}}) \geq \delta_{\bar{j}}.$$

According to (6),

$$\sum_{i\in L} \kappa_{i\bar{j}}(0) < \delta_{\bar{j}}.$$

Since $\kappa_{i\bar{j}}(\tau_{\bar{j}})$ are continuous functions, there is a solution of the equation

$$\sum_{i\in L} \kappa_{i\bar{j}}(\tau_{\bar{j}}) = \delta_{\bar{j}}.$$

This proves that $XT1$ can be performed. ∎

LEMMA 2. *There exists a solution τ_j^o and ζ_{ij}^o of the linear programming problem (5), and variables τ_j, ξ_{ij} obtained by $XT2$ satisfy the complementary property.*

PROOF. The current values $\tilde{\tau}_j$ and $\tilde{\xi}_{ij}$ form a feasible solution $\tau_j = \tilde{\tau}_j$; $\zeta_{ij} = \tilde{\xi}_{ij}$ for (5). Since the objective function of this problem is nonnegative for each feasible solution, the optimal solution exists.

Let us imagine that the solution generated by $XT2$ does not satisfy the complimentary property. It means that there are $\tilde{\imath}$ and $\tilde{\jmath}$ such that

$$\begin{aligned} &0 < \tilde{\xi}_{\tilde{\imath}\tilde{\jmath}} < \Psi_{\tilde{\imath}\tilde{\jmath}}(\tilde{\tau}_{\tilde{\jmath}}), \\ &\langle p, \tilde{x}_{\tilde{\imath}} \rangle < \beta_{\tilde{\imath}}. \end{aligned} \tag{8}$$

Let us carry out $XT1$. According to the definition of $XT1$ the new variable $\tilde{\tau}_j$ will be less then $\tilde{\tau}_j$. Further, the constraints of problem (5) are not violated. Thus, there is a feasible solution that is better than the optimal of solution of (5). This contradiction proves that (8) holds true.

THEOREM. *If the model satisfies property (6), then the algorithm terminates, a solution for the model exists and can be found by the constructed algorithm.*

PROOF. At each iteration of the described algorithm at least one variable that was equal to 0 at the previous iteration becomes positive and will remain positive. Hence, after a finite number of iterations, the set M becomes empty. To prove that G is empty, let us suppose that $G \neq \emptyset$. It means that the inequality

$$\sum_{i=1}^{m} \tilde{x}_i \leq d$$

holds. The last inequality leads to the following

$$\sum_{i=1}^{m} \langle p, \tilde{x}_i \rangle < \langle p, d \rangle \leq \sum_{i=1}^{m} \beta_i$$

that shows that $Q \neq \emptyset$.

In accordance with the procedure for calculation of values $\tilde{\xi}_{ij}$ for $j \in G$ and $i \in Q$, we have $\tilde{\xi}_{ij} = 0$. But in this case, $M \neq \emptyset$. Obtained contradiction proves that $G = \emptyset$. It means that calculated values $\tilde{x}_i$ satisfy the equation

$$\sum_{i=1}^{m} x_i = d.$$

According to Lemma 2 steps (6),(7) and (8),(9) form a solution that satisfies the complementary property. It, therefore, follows that vectors $\tilde{x}_i$ are optimal for consumers.

We conclude that the solution of the model exists, and it can be calculated by the described algorithm.

5. AN EXAMPLE

Consider an example with $\varphi_1(x_1) = 3\xi_{11} + \xi_{12}, \varphi_2(x_2) = 2\xi_{21} + 3\xi_{22}; \beta_1 = 2; \beta_2 = 1.2; d = (1,2); p = (1,1); \Psi_{11}(\tau_1) = 0.2+\tau_1;\quad \Psi_{21}(\tau_1) = 0.3+2\tau_1; \Psi_{12}(\tau_2) = 0.5+\tau_2;\quad \Psi_{22}(\tau_2) = 0.2+\tau_2$.
The starting values are: $\tilde{x}_1 = 0; \tilde{x}_2 = 0; G = \{1,2\}; Q = \{1,2\}$.

ITERATION 1.

(1) $M \cap G = \{1,2\}, \tilde{\jmath} := 1;$
(2) $N = \emptyset; D = \{1\}; L = \{1\};$
(3) $\tilde{\xi}_{11} = 1; \tilde{\tau}_1 = 0.8;$
(4) $G = \{2\};$
(10) $Q = \{1,2\}; M = \{1,2\},$
(11) $M \neq \emptyset$, go to 1.

ITERATION 2.

(1) $M \cap G = \{2\}, \tilde{\jmath} = 2;$
(2) $N = \emptyset, D = \{1,2\}; L = \{1,2\}$
(3) $\tilde{\xi}_{12} = 1; \tilde{\xi}_{22} = 1; \tilde{\tau}_2 = 0.8;$
(4) $G = \emptyset;$
(10) $Q = \{2\}; M = \{1\};$
(11) $M \neq \emptyset$, go to 1.

ITERATION 3.

(1) $M \cap G = \emptyset, \tilde{\jmath} = 1;$
(2) $N = \{1\}; D = \{2\}; L = \{1,2\};$
(3) $\tilde{\xi}_{11} = 0.8; \tilde{\xi}_{21} = 0.2; \tilde{\tau}_1 = 0.6;$
(4) $M \cap G = \emptyset,$
(5) $Q = \{1\};$
(6) $E = \{2\};$
(7) $\tilde{\xi}_{11} = \frac{13}{20}; \tilde{\xi}_{21} = \frac{7}{20}; \tilde{\xi}_{12} = \frac{23}{20}; \tilde{\xi}_{22} = \frac{17}{20}; \tilde{\tau}_1 = \frac{9}{20}; \tilde{\tau}_2 = \frac{13}{20};$
(8) $E_o = \emptyset;$
(10) $Q = \emptyset; M = \emptyset;$
(11) go to 12;
(12) end.

Thus,

$$\tilde{\tau}_1 = \frac{9}{20}; \quad \tilde{\tau}_2 = \frac{13}{20}.$$

Notice that when $|E| = 1$ or $|E| > 1$ but the corresponding positive variables are weakly connected on step (7), we can use operator $XT1$, instead of $XT2$. But, if a correction of a variable by $XT1$ violates the complementary property we must use $XT2$.

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