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Projection-proximal methods for general variational inequalities

Muhammad Aslam Noor

Mathematics Department, COMSATS Institute of Information Technology, Islamabad, Pakistan

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Abstract

In this paper, we consider and analyze some new projection-proximal methods for solving general variational inequalities. The modified methods converge for pseudomonotone operators which is a weaker condition than monotonicity. The proposed methods include several new and known methods as special cases. Our results can be considered as a novel and important extension of the previously known results. Since the general variational inequalities include the quasi-variational inequalities and implicit complementarity problems as special cases, results proved in this paper continue to hold for these problems.

Keywords: Variational inequalities; Projection methods; Pseudomonotone operators; Fixed-point; Convergence

1. Introduction

General variational inequalities introduced and studied by Noor [1] have appeared as a novel and useful generalization of the variational inequalities. It turned out that a class of odd-order and nonsymmetric obstacle, free, moving, unilateral and equilibrium problems arising in financial, economics, transportation, elasticity, optimization, pure and applied sciences can be studied via the general variational inequalities; see [2–8]. This field is

E-mail address: noormaslam@hotmail.com.

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dynamic and is experiencing an explosive growth in both theory and applications: as consequence, several numerical techniques including projection, the Wiener–Hopf equations, auxiliary principle, decomposition and descent are being developed for solving various classes of variational inequalities and related optimization problems. Projection methods and its variants forms including the Wiener–Hopf equations represent important tools for finding the approximate solution of variational and quasi-variational inequalities. The main idea in this technique is to establish the equivalence between the variational inequalities and the fixed-point problem by using the concept of projection. This alternative formulation has played a significant part in developing various projection-type methods for solving variational inequalities. It is well known that the convergence of the projection methods requires that the operator must be strongly monotone and Lipschitz continuous. Unfortunately these strict conditions rule out many applications of this method. This fact motivated to modify the projection method or to develop other methods. The extragradient method [4,5] overcome this difficulty by performing an additional forward step and a projection at each iteration according to the double projection. This method can be viewed as predictor–corrector method. Its convergence requires only that a solution exists and the monotone operator is Lipschitz continuous. When the operator is not Lipschitz continuous or when the Lipschitz continuous constant is not known, the extragradient method and its variant forms require an Armijo-like line search procedure to compute the step size with a new projection need for each trial, which leads to expansive computation. To overcome these difficulties, several modified projection and extragradient-type methods have been suggested and developed for solving variational inequalities. One of these projection-type methods is called the proximal point algorithm. Recently several modified proximal point algorithms have been suggested and analyzed for solving monotone variational inequalities. Inspired and motivated by the research going on in this direction, we suggest and analyze a new modified projection method, which includes the classical proximal, extragradient and modified double projection as special cases. Using essentially the idea and technique of He, Yang and Yuan [9], we prove that the convergence of the new projection method requires only pseudomonotonicity, which is a weaker condition than monotonicity. In this respect, our results represent an improvement of the previous methods. The comparison of this new method with the known methods is an interesting problem for the future research. We would like to emphasize that the projection-type method suggested in this paper can be considered as predictor–corrector-type method. Our results can be viewed as significant and novel extension of the results of Solodov and Svaiter [10], He, Yang and Yuan [9] and Noor [3–8,11]. The comparison of this new method with the existing ones is an interesting problem for future research work.

2. Preliminaries

Let H be a real Hilbert space, whose inner product and norm are denoted by $\langle\cdot,\cdot\rangle$ and $\|\cdot\|$, respectively. Let K be a closed convex set in H and $T, g: H \to H$ be continuous (nonlinear) operators. We now consider the problem of finding $u \in H$: $g(u) \in K$ such that

$$\big\langle Tu, g(v) - g(u)\big\rangle \geqslant 0, \quad \forall v \in H\colon\ g(v) \in K. \tag{1}$$

Problem (1) is called the *general variational inequality* which was introduced and studied by Noor [1] in 1988. It has been shown that a large class of unrelated odd-order and non-symmetric obstacle, unilateral, contact, free, moving, and equilibrium problems arising in regional, physical, mathematical, engineering and applied sciences can be studied in the unified and general framework of the general variational inequalities; see [1–8,11].

For $g \equiv I$, where I is the identity operator, problem (1) is equivalent to finding $u \in K$ such that

$$\langle Tu, v-u\rangle \geqslant 0, \quad \forall v \in K, \tag{2}$$

which is known as the classical variational inequality introduced and studied by Stampacchia [12] in 1964. For the recent state of the art in variational inequalities, see [1–18] and the references therein.

From now onward, we assume that g is onto K unless otherwise specified.

If $N(u) = \{w \in H\colon \langle w, v-u\rangle \leqslant 0,\ \forall v \in K\}$ is a normal cone to the convex set K at u, then the general variational inequality (1) is equivalent to finding $u \in H$, $g(u) \in K$ such that

$$-T(u) \in N\big(g(u)\big),$$

which are known as the *general nonlinear equations*.

If $K^* = \{u \in H\colon \langle u, v\rangle \geqslant 0,\ \forall v \in K\}$ is a polar (dual) cone of a convex cone K in H, then problem (1) is equivalent to finding $u \in H$ such that

$$g(u) \in K, \quad Tu \in K^* \quad \text{and} \quad \big\langle Tu, g(u)\big\rangle = 0, \tag{3}$$

which is known as the general complementarity problem. For $g(u) = m(u) + K$, where m is a point-to-point mapping, problem (3) is called the implicit (quasi) complementarity problem. If $g \equiv I$, then problem (3) is known as the generalized complementarity problem. Such problems have been studied extensively in the literature, see the references.

For suitable and appropriate choice of the operators and spaces, one can obtain several classes of variational inequalities and related optimization problems.

We now recall the following well-known result and concepts.

Lemma 2.1. *For a given $z \in H$, $u \in K$ satisfies the inequality*

$$\langle u-z, v-u\rangle \geqslant 0, \quad \forall v \in K, \tag{4}$$

if and only if

$$u = P_K[z],$$

where P_K is the projection of H onto K. Also the projection operator P_K is nonexpansive and satisfies the inequality

$$\big\| P_K[z] - u\big\|^2 \leqslant \|z-u\|^2 - \big\|z - P_K[z]\big\|^2. \tag{5}$$

Definition 2.1. $\forall u, v \in H$, the operator $T : H \to H$ is said to be

(i) *g-monotone*, if

$$\big\langle Tu - Tv, g(u) - g(v)\big\rangle \geqslant 0;$$

(ii) *g-pseudomonotone*, if

$$\langle Tu, g(v) - g(u)\rangle \geqslant 0 \quad \text{implies} \quad \langle Tv, g(v) - g(u)\rangle \geqslant 0.$$

For $g \equiv I$, Definition 2.1 reduces to the usual definition of monotonicity, and pseudomonotonicity of the operator T. Note that monotonicity implies pseudomonotonicity but the converse is not true; see [17].

3. Main results

In this section, we use the projection technique to suggest and analyze a class of new projection methods for solving general variational inequalities (1). For this purpose, we need the following result, which can be proved by invoking Lemma 2.1.

Lemma 3.1. *The function $u \in H$, $g(u) \in K$ is a solution of* (1) *if and only if $u \in H$ satisfies the relation*

$$g(u) = P_K\big[g(u) - \rho Tu\big], \tag{6}$$

where $\rho > 0$ is a constant and g is onto K.

Lemma 3.1 implies that problems (1) and (6) are equivalent. This alternative formulation is very important from the numerical analysis point of view.

We now define the residue vector $R(u)$ by the relation

$$R(u) = g(u) - P_K\big[g(u) - \rho Tu\big]. \tag{7}$$

From Lemma 3.1, it follows that $u \in H$ is a solution of (1) if and only if $u \in H$ is a zero of the equation

$$R(u) = 0. \tag{8}$$

For a positive constant γ, we can rewrite Eq. (8) as

$$g(u) = g(u) - \gamma R(u) := g(u) - \gamma\big\{g(u) - P_K\big[g(u) - \rho Tu\big]\big\}. \tag{9}$$

This fixed-point formulation can be used to suggest the following iterative method.

Algorithm 3.1. For a given $u_0 \in H$, compute the approximate solution u_{n+1} by the iterative scheme

$$\begin{aligned} g(u_{n+1}) &= P_K\big[g(u_n) - \gamma_n R(u_{n+1})\big] \\ &= P_K\big[g(u_n) - \gamma_n\big\{g(u_n) - P_K\big[g(u_n) - \rho T u_{n+1}\big]\big\}\big], \quad n = 0, 1, 2, \ldots, \end{aligned}$$

or equivalently

$$\begin{aligned} g(y_n) &= P_K\big[g(u_n) - \rho T u_{n+1}\big], \\ g(u_{n+1}) &= P_K\big[g(u_n) - \gamma_n\big\{g(u_n) - g(y_n)\big\}\big], \quad n = 0, 1, 2, \ldots, \end{aligned}$$

which can be considered as a proximal point method and appears to be a new one. Note that for $\gamma_n = 1$, Algorithm 3.1 reduces to:

Algorithm 3.2. For a given $u_0 \in H$, compute the approximate solution u_{n+1} by the iterative scheme

$$g(u_{n+1}) = P_K\big[g(u_n) - \rho T u_{n+1}\big], \quad n = 0, 1, 2 \ldots,$$

which is known as the proximal method. In recent years, proximal methods have been considered and studied extensively. Several conditions have been studied which are easy to implement and to accelerate the convergence; see [9–11,13,16].

We use the technique of updating the solution, which has been used to suggest and analyze various two-step and three-step forward–backward splitting type methods for solving variational inequalities and related optimization problems. This technique is mainly due to Noor [4]. To this end, we can rewrite Eq. (6) in the form

$$g(y) = P_K\big[g(u) - \rho T u\big], \tag{10}$$

$$g(u) = P_K\big[g(y) - \rho T y\big]. \tag{11}$$

These equivalent formulations have been used to suggest and analyze the following iterative methods for solving the general variational inequalities (1).

Algorithm 3.3. For a given $u_0 \in H$, calculate the approximate solution u_{n+1} by the iterative schemes

$$g(y_n) = P_K\big[g(u_n) - \rho T u_{n+1}\big],$$
$$g(u_{n+1}) = P_K\big[g(y_n) - \rho T y_n\big], \quad n = 0, 1, 2, 3, \ldots.$$

Algorithm 3.3 can be viewed as a two-step projection-proximal method. Algorithm 3.3 is quite different from the extragradient method.

Now we look at Algorithm 3.3 from a different angle. Consider $g(y)$ defined by (10) as an approximate solution of the general variational inequality (1) and define

$$g(w) = P_K\big[g(y) - \rho T y\big], \tag{12}$$

$$g(z) = P_K\big[g(u) - \rho T w\big]. \tag{13}$$

We use this formulation to suggest the following iterative method

Algorithm 3.4. For a given $u_0 \in H$, calculate the approximate solution u_{n+1} by the iterative schemes

$$g(y_n) = P_K\big[g(u_n) - \rho T u_{n+1}\big],$$
$$g(w_n) = P_K\big[g(y_n) - \rho T y_n\big],$$
$$g(u_{n+1}) := g(z_n) = P_K\big[g(u_n) - \rho T w_n\big], \quad n = 0, 1, 2, \ldots,$$

which is called the modified proximal-extragradient method and appears to be a new one. Note that for $w_n = y_n$, Algorithm 3.4 reduces to

Algorithm 3.5. For a given $u_0 \in H$, compute the approximate solution u_{n+1} by the iterative schemes

$$\begin{aligned} g(y_n) &= P_K\big[g(u_n) - \rho T u_{n+1}\big], \\ g(u_{n+1}) &= P_K\big[g(u_n) - \rho T y_n\big], \quad n = 0, 1, 2, \ldots. \end{aligned}$$

For a positive constant α, consider

$$g(u) = g(u) - \alpha\big(g(u) - g(z)\big). \tag{14}$$

Here the positive constant α can be viewed as a step length along the direction $-(g(u) - g(z))$.

We use this fixed-point formulation to suggest the following iterative method.

Algorithm 3.6. For a given $u_0 \in H$, compute the following iterative schemes:

$$\begin{aligned} g(y_n) &= P_K\big[g(u_n) - \rho_n T u_{n+1}\big], \\ g(w_n) &= P_K\big[g(y_n) - \rho_n T y_n\big], \\ g(z_n) &= P_K\big[g(u_n) - \rho_n T w_n\big], \\ g(u_{n+1}) &= P_K\big[g(u_n) - \alpha\big(g(u_n) - g(z_n)\big)\big], \quad n = 0, 1, 2, \ldots, \end{aligned} \tag{15}$$

$$\alpha = \frac{\|g(z_n) - g(w_n)\|^2 + \|g(u_n) - g(z_n)\|^2 - \Delta(w_n)}{2\|g(u_n) - g(w_n)\|^2}, \tag{16}$$

where

$$\begin{aligned} \Delta(w_n) &\leqslant \nu\big(\big\|g(z_n) - g(w_n)\big\|^2 + \big\|g(u_n) - g(z_n)\big\|^2\big), \quad \nu < 1 \\ &= \nu\big\{2\big\langle g(w_n) - g(z_n), g(w_n) - g(u_n) + \rho_n T w_n\big\rangle \\ &\quad - \big\|g(w_n) - g(z_n)\big\|^2\big\} \end{aligned} \tag{17}$$

and

$$\Delta(w_n) = 2\big\langle g(w_n) - g(z_n), g(w_n) - g(z_n) + \rho_n T w_n\big\rangle - \big\|g(w_n) - g(z_n)\big\|^2, \tag{18}$$

which is known as the inexactness criteria.

For $\alpha = 1$ and $z_n = w_n$, Algorithm 3.6 is exactly Algorithm 3.4. If $y = w$, then Algorithm 3.6 reduces to:

Algorithm 3.7. For a given $u \in H$, compute the approximate solution u_{n+1} by the iterative schemes

$$\begin{aligned} g(y_n) &= P_K\big[g(u_n) - \rho_n T u_{n+1}\big], \\ g(w_n) &= P_K\big[g(u_n) - \rho_n T y_n\big], \\ g(u_{n+1}) &:= g(z_n) = P_K\big[g(u_n) - \alpha\big(g(u_n) - g(w_n)\big)\big], \quad n = 0, 1, 2, \ldots, \\ \alpha &= \frac{\|g(u_n) - g(y_n)\|^2 + \|g(u_n) - g(w_n)\|^2 - \Delta(y_n)}{2\|g(u_n) - g(w_n)\|^2}, \end{aligned}$$

which is an approximate proximal-extragradient projection method for solving (1). In particular, for $g = I$, the identity operator, Algorithm 3.7 appears to be a new one. In a similar way, one can obtain various new and known algorithms as special cases of Algorithm 3.6. This shows that Algorithm 3.6 unifies several recently proposed (implicit or explicit) algorithms for solving variational inequalities.

We now study the convergence analysis of Algorithm 3.6. The analysis is in the spirit of He, Yang and Yuan [9]. To convey the idea and for the sake of completeness, we include the details.

Theorem 3.1. *Let the operator T be g-pseudomonotone. If $u \in H$: $g(u) \in K$ be a solution of the general variational inequality* (1) *and u_{n+1} be the approximate solution obtained from Algorithm* 3.6*, then*

$$\begin{aligned} &\|g(u_{n+1}(\alpha)) - g(u)\|^2 \\ &\quad \leqslant \|g(u_n) - g(u)\|^2 \frac{(1-\nu)^2}{4}\left\{\|g(z_n) - g(w_n)\|^2 + \|g(u_n) - g(z_n)\|^2\right\}. \end{aligned} \tag{19}$$

Proof. Let $u \in H$: $g(u) \in K$ be a solution of (1). Then

$$\langle Tu, g(v) - g(u)\rangle \geqslant 0, \quad \forall v \in H\colon\ g(v) \in K,$$

implies that

$$\langle Tv, g(v) - g(u)\rangle \geqslant 0, \tag{20}$$

since T is g-monotone.

Taking $v = w_n$ in (20), we have

$$\langle Tw_n, g(w_n) - g(u)\rangle \geqslant 0,$$

which can be written as

$$\langle Tw_n, g(z_n) - g(u)\rangle \geqslant \langle Tw_n, g(z_n) - g(w_n)\rangle. \tag{21}$$

Taking $z = g(u_n) - \rho_n Tw_n$, $u = g(z_n)$ and $v = g(u)$ in (4), we have

$$\langle g(u_n) - \rho_n Tw_n - g(z_n), g(z_n) - g(u)\rangle \geqslant 0,$$

from which we have

$$\langle g(u_n) - g(z_n), g(z_n) - g(u)\rangle \geqslant \langle g(z_n) - g(u), \rho_n Tw_n\rangle. \tag{22}$$

From (21) and (22), we have

$$\langle g(u_n) - g(z_n), g(z_n) - g(u)\rangle \geqslant \langle \rho_n Tw_n, g(z_n) - g(w_n)\rangle. \tag{23}$$

Consider

$$\begin{aligned} &\|g(u_n) - g(u)\|^2 - \|g(u_{n+1}(\alpha)) - g(u)\|^2 \\ &\quad = \|g(u_n) - g(u)\|^2 - \|P_K[g(u_n) - \alpha(g(u_n) - g(z_n))] - P_K[g(u)]\|^2 \\ &\quad \geqslant \|g(u_n) - g(u)\|^2 - \|g(u_n) - g(u) - \alpha(g(u_n) - g(z_n))\|^2 \end{aligned}$$

$$\begin{aligned}
&= 2\alpha\langle g(u_n) - g(u), g(u_n) - g(z_n)\rangle - \alpha^2\|g(u_n) - g(z_n)\|^2 \\
&= 2\alpha\|g(u_n) - g(z_n)\|^2 + 2\alpha\langle g(z_n) - g(u), g(u_n) - g(z_n)\rangle \\
&\quad - \alpha^2\|g(u_n) - g(z_n)\|^2.
\end{aligned} \tag{24}$$

Combining (16)–(18), (23) and (24), we obtain

$$\begin{aligned}
&\|g(u_n) - g(u)\|^2 - \|g(u_{n+1}(\alpha)) - g(u)\|^2 \\
&\quad \geqslant \alpha\{\|g(z_n) - g(w_n)\|^2 + \|g(u_n) - g(z_n)\|^2 - \Delta(w_n)\} \\
&\qquad - \alpha^2\|g(u_n) - g(z_n)\|^2,
\end{aligned} \tag{25}$$

which is a quadratic in α and has a maximum at

$$\alpha^* = \frac{\|g(z_n) - g(w_n)\|^2 + \|g(u_n) - g(z_n)\|^2 - \Delta(w_n)}{2\|g(u_n) - g(z_n)\|^2}. \tag{26}$$

From (17), (18), (25) and (26), we have the required result (19). □

Theorem 3.2. *Let H be a finite dimensional subspace and g be an injective and continuous. Let $u \in H$: $g(u) \in K$ be a solution of* (1) *and u_{n+1} be the approximate solution obtained from Algorithm* 3.6*, then* $\lim_{n\longrightarrow\infty}(u_n) = u$.

Proof. Let $u \in H$ be a solution of (1). From (19), it follows that the sequence $\{\|g(u) - g(u_n)\|\}$ is nonincreasing and consequently $\{g(u_n)\}$ is bounded. Thus it follows the sequence $\{u_n\}$ is bounded under the assumption of g. Furthermore, we have

$$\sum_{n=1}^{\infty} \frac{(1-\nu)^2}{4}\{\|g(z_n) - g(w_n)\|^2 + \|g(u_n) - g(z_n)\|^2\} \leqslant \|g(u_0) - g(u)\|^2,$$

which implies that

$$\lim_{n\to\infty}\|g(z_n) - g(w_n)\| = 0, \tag{27}$$

$$\lim_{n\to\infty}\|g(u_n) - g(z_n)\| = 0, \tag{28}$$

from which, we have

$$\lim_{n\to\infty}\|z_n - w_n\| = 0, \tag{29}$$

$$\lim_{n\to\infty}\|u_n - z_n\| = 0, \tag{30}$$

since g is injective.

Thus we see that the sequences $\{w_n\}$ and $\{z_n\}$ are also bounded. Also from (27) and (28), we have

$$\begin{aligned}
\|R(w_n)\| &= \|g(w_n) - P_K[g(w_n) - \rho T w_n]\| \\
&= \|g(w_n) - g(z_n) + g(z_n) - P_K[g(w_n) - \rho T w_n]\| \\
&\leqslant \|g(w_n) - g(z_n)\| + \|P_K[g(u_n) - \rho T w_n] - P_K[g(w_n) - \rho T w_n]\|
\end{aligned}$$

$$\leqslant \|g(w_n) - g(z_n)\| + \|g(u_n) - g(w_n)\|$$
$$= 0.$$

Thus

$$\lim_{n\to\infty} R(w_n) = 0. \tag{31}$$

Let $\hat{u}$ be a cluster point of $\{w_n\}$ and the subsequence $\{w_{n_i}\}$ converges to $\hat{u}$. Since $R(u)$ is a continuous function of u, it follows that

$$\lim_{n\to\infty} R(w_{n_i}) = R(\hat{u}) = 0,$$

which shows that $\hat{u}$ is a solution of the general variational inequality (1). From (29) and (30), we know that $\lim_{n\to\infty}(w_{n_i}) = \hat{u} = \lim_{n\to\infty}(z_{n_i})$. Hence from (19), we have

$$\|u_{n+1} - \hat{u}\|^2 \leqslant \|u_n - \hat{u}\|^2, \quad \forall n \geqslant 0,$$

which shows that the sequence $\{u_n\}$ converges to $\hat{u}$, the required result. □

Remark 3.1. We now show that the results derived in the paper can be extended for a class of quasi-variational inequalities. If the convex set K depends upon the solution explicitly or implicitly, then variational inequality problem is known as the quasi-variational inequality. For a given operator $T : H \to H$, and a point-to-set mapping $K : u \to K(u)$, which associates a closed convex-valued set $K(u)$ with any element u of H, we consider the problem of finding $u \in K(u)$ such that

$$\langle Tu, v - u\rangle \geqslant 0, \quad \text{for all } v \in K(u). \tag{32}$$

Inequality of type (32) is called the quasi-variational inequality. For the formulation, applications, numerical methods and sensitivity analysis of the quasi-variational inequalities; see [11,14,17,18].

Using Lemma 2.1, one can show that the quasi-variational inequality (32) is equivalent to finding $u \in K(u)$ such that

$$u = P_{K(u)}[u - \rho Tu]. \tag{33}$$

In many important applications, the convex-valued set $K(u)$ is of the form

$$K(u) = m(u) + K, \tag{34}$$

where m is a point-to-point mapping and K is a closed convex set.

From (33) and (34), we see that problem (32) is equivalent to

$$u = P_{K(u)}[u - \rho Tu] = P_{m(u)+K}[u - \rho Tu] = m(u) + P_K\big[u - m(u) - \rho Tu\big],$$

which implies that

$$g(u) = P_K\big[g(u) - \rho Tu\big] \quad \text{with } g(u) = u - m(u),$$

which is equivalent to the general variational inequality (1) by an application of Lemma 3.1. We have shown that the quasi-variational inequalities (32) with the convex-valued set $K(u)$ defined by (34) are equivalent to the general variational inequalities (1). Thus all the results obtained in this paper continue to hold for quasi-variational inequalities (32) with $K(u)$ defined by (34).

Remark 3.2. In this paper, we have suggested and analyzed several projection-proximal methods for solving the general variational inequalities. These methods are more general and flexible even for the classical variational inequalities (2). We remark that a special of Algorithm 3.6 coincides with the approximate proximal-extragradient algorithm of He, Yang and Yuan [9]. It has been shown in [9] that these proximal-extragradient methods are very efficient and are reasonably easy to use for computations as compared with the method of Solodov and Svaitor [10] for solving the network equilibrium problems. The comparison of these new methods with other methods is an interesting problem for further research in this area.

References

[1] M.A. Noor, General variational inequalities, Appl. Math. Lett. 1 (1988) 119–121.
[2] M.A. Noor, K.I. Noor, Self-adaptive projection algorithms for general variational inequalities, J. Appl. Math. Comput. 151 (2004) 659–670.
[3] M.A. Noor, New extragradient-type methods for general variational inequalities, J. Math. Anal. Appl. 277 (2003) 379–395.
[4] M.A. Noor, Some algorithms for general monotone mixed variational inequalities, Math. Comput. Modelling 29 (1999) 19–24.
[5] M.A. Noor, Some developments in general variational inequalities, Appl. Math. Comput. 152 (2004) 199–277.
[6] M.A. Noor, New approximation schemes for general variational inequalities, J. Math. Anal. Appl. 251 (2000) 217–229.
[7] M.A. Noor, K.I. Noor, T.M. Rassias, Some aspects of variational inequalities, J. Comput. Appl. Math. 47 (1993) 285–312.
[8] N. Xiu, J. Zhang, M.A. Noor, Tangent projection equations and general variational equalities, J. Math. Anal. Appl. 258 (2001) 755–762.
[9] B.S. He, Z. Yang, X.M. Yuan, Approximate proximal-extragradient type method for monotone variational inequalities, J. Math. Anal. Appl. 300 (2004) 362–374.
[10] M.V. Solodov, B.F. Svaiter, Error bounds for proximal point subproblems and associated inexact proximal point algorithms, Math. Programming 88 (2000) 371–389.
[11] M.A. Noor, Theory of general variational inequalities, preprint, Etisalat College of Engineering, Sharjah, United Arab Emirates, 2004.
[12] G. Stampacchia, Formes bilineaires coercitives sur les ensembles convexes, C. R. Acad. Sci. Paris 258 (1964) 4413–4416.
[13] B.S. He, L.Z. Liao, Improvement of some projection methods for monotone nonlinear variational inequalities, J. Optim. Theory Appl. 112 (2002) 111–128.
[14] F. Giannessi, A. Maugeri, P.M. Pardalos, Equilibrium Problems: Nonsmooth Optimization and Variational Inequality Models, Kluwer Academic, Dordrecht, 2001.
[15] D.P. Bertsekas, J. Tsitsiklis, Parallel and Distributed Computation: Numerical Methods, Prentice Hall, Englewood Cliffs, NJ, 1989.
[16] R.T. Rockafellar, Monotone operators and the proximal point algorithm, SIAM J. Control Optim. 14 (1976) 877–898.
[17] F. Giannessi, A. Maugeri, Variational Inequalities and Network Equilibrium Problems, Plenum, New York, 1995.
[18] M. Patriksson, Nonlinear Programming and Variational Inequalities: A Unified Approach, Kluwer Academic, Dordrecht, 1999.