The Autocontinuity of Set Function
and the Fuzzy Integral

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The autocontinuity and some other concepts of a set function are introduced and Sugeno’s fuzzy measure with some annexed conditions is studied. On a fuzzy measure space, Egoroff’s theorem is proved. It is also proved that F-mean convergence is equivalent to convergence in measure for a sequence of measurable functions. Finally, some convergence theorems of sequence of fuzzy integrals are proved, especially, using the concept of autocontinuity. A necessary and sufficient condition for the convergence in measure theorem is given.

1. INTRODUCTION

Sugeno [1] introduced the concepts of fuzzy measure and of fuzzy integral. They represent a fuzziness which is different from the one described by Zadeh [2]. The monotone convergence theorem of the sequence of fuzzy integrals was proposed by Sugeno [1], but his proof does not seem correct. Ralescu and Adams [4] gave another equivalent definition of a fuzzy integral in a more general context and proved the monotone convergence theorem. However, in the example given at the end of [4], the set function \( \mu \) is not continuous from above and, consequently, is not a fuzzy measure. Indeed, the condition that a fuzzy measure \( \mu \) is subadditive is too strong for the convergence theorem. By the way, we must pay attention to the fact that the continuity does not miss in the examples of fuzzy measure. Just as Ralescu [5, 6] pointed out, a possibility measure was considered in [1, 10, 11] as a fuzzy measure by mistake, but, in fact, the possibility measure does not always hold continuity. Even then in [4, p. 563], examples (c) and (d) of fuzzy measure are erroneous, because the set functions \( \mu \) given in these examples do not hold continuity from above.

In Section 2 of this paper, a new concept—autocontinuity of a set function—will be introduced. It will play an important role in the theories of fuzzy measure and possibility measure. Indeed, we shall find that, although

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the possibility measure does not hold continuity in general, it always holds autocontinuity. In Section 3, we shall discuss the convergence of the sequence of measurable functions on a fuzzy measure space, and Egoroff's theorem will be proved. In Sections 4 and 5, some properties of fuzzy integral will be presented and some interesting results concerning the convergence theorems will be proved. We shall especially give a necessary and sufficient condition for the following property: whenever a sequence of measurable functions converges in measure, then the corresponding sequence of fuzzy integrals converges.

Throughout this paper, let $X$ be a set, $\mathcal{F}$ be a $\sigma$-algebra of subsets of $X$, $\mathcal{P}(X)$ be the class of all subsets of $X$, an extended real-valued set function $\mu$ is defined on $\mathcal{F}$, and we make the following convention: $\sup\{i : i \in \phi\} = 0$, $\infty - \infty = 0$, $0 \cdot \infty = 0$, and if the index set $I$ is empty, then $\sum_I \cdot = 0$.

2. The Autocontinuity and the Fuzzy Measure

A fuzzy measure is a nonnegative extended real-valued set function $\mu: \mathcal{F} \to [0, \infty]$ with the properties:

(FM1) $\mu(\emptyset) = 0$.

(FM2) $A \subset B \Rightarrow \mu(A) \leq \mu(B)$.

(FM3) $A_1 \subset A_2 \subset \cdots, A_n \in \mathcal{F} \Rightarrow \mu(\bigcup_{n=1}^{\infty} A_n) = \lim_{n \to \infty} \mu(A_n)$.

(FM4) $A_1 \supset A_2 \supset \cdots, A_n \in \mathcal{F}$, and there exists $n_0$, such that $\mu(A_{n_0}) < \infty$, $\Rightarrow \mu(\bigcap_{n=1}^{\infty} A_n) = \lim_{n \to \infty} \mu(A_n)$.

If $\mu$ is a fuzzy measure, we call $(X, \mathcal{F}, \mu)$ a fuzzy measure space.

**Definition 1.** A set function $\mu$ is called null-additive, if we have $\mu(E \cup F) = \mu(E)$ whenever $E \in \mathcal{F}$, $F \in \mathcal{F}$, $E \cap F = \emptyset$ and $\mu(F) = 0$.

**Proposition 1.** Let $\mu$ be a set function. If we have $\mu(E) \neq 0$ whenever $E \in \mathcal{F}$, $E \neq \emptyset$, then $\mu$ is null-additive.

**Proposition 2.** Let $(X, \mathcal{F}, \mu)$ be a fuzzy measure space, then the following statements are equivalent:

1. $\mu$ is null-additive.

2. Whenever $E \in \mathcal{F}$, $F \in \mathcal{F}$, $\mu(F) = 0$, we have $\mu(E \cup F) = \mu(E)$.

3. Whenever $E \in \mathcal{F}$, $F \in \mathcal{F}$, $F \subseteq E$ and $\mu(F) - 0$, we have $\mu(E - F) = \mu(E)$.

4. Whenever $E \in \mathcal{F}$, $F \in \mathcal{F}$, $\mu(F) = 0$, we have $\mu(E - F) = \mu(E)$.

5. Whenever $E \in \mathcal{F}$, $F \in \mathcal{F}$, $\mu(F) = 0$, we have $\mu(E \triangle F) = \mu(E)$.
The simplest example of fuzzy measure which is not null-additive is given as
\[ X = \{ a, b \}, \quad \mathcal{F} \in \mathcal{P}(X) \quad \text{and} \quad \mu(E) = 1, \quad E = X, \]
\[ = 0, \quad E \neq X. \]

**Theorem 1.** Let \( \mu \) be a null-additive fuzzy measure, and \( A \in \mathcal{F} \). We have \( \mu(A \cup B_n) \to \mu(A) \) for any decreasing sequence \( \{B_n\} \subset \mathcal{F} \) for which \( \mu(B_n) \to 0 \) and there exists at least one \( n_0 \) such that \( \mu(A \cup B_{n_0}) < \infty \) as \( \mu(A) < \infty \).

Proof. It is sufficient to prove the theorem for \( \mu(A) < \infty \). Write \( B = \bigcap_{n=1}^\infty B_n \), we have \( \mu(B) = \lim_{n \to \infty} \mu(B_n) = 0 \). Since \( A \cup B_n \searrow A \cup B \), it follows, from the continuity and the null-additivity of \( \mu \), that
\[ \mu(A \cup B_n) \to \mu(A \cup B) = \mu(A). \]

**Theorem 2.** Let \( \mu \) be a null-additive fuzzy measure, and \( A \in \mathcal{F} \). We have \( \mu(A - B_n) \to \mu(A) \) for any decreasing sequence \( \{B_n\} \subset \mathcal{F} \) for which \( \mu(B_n) \to 0 \).

Proof. Since \( A - B_n \not\subseteq A - (\bigcap_{n=1}^\infty B_n) \), Proposition 2 and \( \mu(\bigcap_{n=1}^\infty B_n) = 0 \) imply that
\[ \mu(A - B_n) \to \mu\left( A - \left( \bigcap_{n=1}^\infty B_n \right) \right) = \mu(A). \]

The following example indicates that Theorem 1 is not true without the finiteness condition described in its statement.

**Example 1.** Let \( X = \{0, 1, 2, \ldots\}, \quad \mathcal{F} \in \mathcal{P}(X) \).
\[ \mu(E) = \frac{1}{2^{i+1}}, \quad 0 \notin E, \]
\[ = \infty, \quad 0 \in E \text{ and } E - \{0\} \neq \emptyset, \]
\[ = 1, \quad E = \{0\}. \]

By Proposition 1, \( \mu \) is null-additive. We take \( A = \{0\}, \quad B_n = \{n, n+1, \ldots\}, \quad n = 1, 2, \ldots \) then \( \mu(A \cup B_n) = \infty \), \( n = 1, 2, \ldots \), but \( \mu(A) = 1 \) that is to say,
\[ \mu(A \cup B_n) \not\to \mu(A). \]
DEFINITION 2. A set function \( \mu \) is called autocontinuous from above (resp. from below), if we have

\[
\mu(A \cup B_n) \to \mu(A) \quad \text{(resp. } \mu(A - B_n) \to \mu(A) \text{)}
\]

whenever \( A \in \mathcal{F}, B_n \in \mathcal{F}, A \cap B_n = \emptyset \) (resp. \( B_n \subseteq A \)), \( n = 1, 2, \ldots, \mu(B_n) \to 0 \); \( \mu \) is called autocontinuous, if it is both autocontinuous from above and autocontinuous from below.

Obviously, if \( \mu \) is a fuzzy measure, then "\( A \cap B_n = \emptyset \)" and "\( B_n \subseteq A \)" may be omitted from the statement of the above definition.

PROPOSITION 3. If a set function \( \mu \) is autocontinuous from above or autocontinuous from below, then it is null-additive.

PROPOSITION 4. A fuzzy measure \( \mu \) is autocontinuous, if and only if we have \( \mu(A \triangle B_n) \to \mu(A) \) whenever \( A \in \mathcal{F}, B_n \in \mathcal{F}, n = 1, 2, \ldots, \mu(B_n) \to 0 \).

Example 2 indicates that, for a fuzzy measure, the autocontinuity from above is not equivalent to the autocontinuity from below.

EXAMPLE 2. Let \( X = \{1, 2, \ldots\}, \mathcal{F} = \mathcal{P}(X), \mu(E) = k \cdot \sum_{i \in E} \left(1/2^n\right) \) for \( E \in \mathcal{F} \), where \( k \) is the number of points in \( E \). Then, \( \mu \) is autocontinuous from below (of course, it is also null-additive). But, it is not autocontinuous from above. In fact, we take \( A = \{1\}, B_n = \{n\}, n = 1, 2, \ldots, \) then \( \mu(B_n) = 1/2^n \to 0 \) and \( \mu(A \cup B_n) = 2 \cdot \left(\frac{1}{2} + 1/2^n\right) \to 1 \); however, \( \mu(A) = \frac{1}{2} \).

PROPOSITION 5. If a finite set function \( \mu \) is continuous from above at 0 (refer to [7]) and autocontinuous from above (resp. from below), then it is continuous from above (resp. from below).

PROPOSITION 6. If a nonnegative monotone set function \( \mu \) is continuous from above at 0 and autocontinuous from above, then it is continuous from above.

DEFINITION 3. A set function \( \mu \) is called uniformly autocontinuous from above (resp. from below), if for every \( \varepsilon > 0 \), there exists \( \delta = \delta(\varepsilon) > 0 \), such that, whenever \( A \in \mathcal{F}, B \in \mathcal{F}, A \cap B = \emptyset \) (resp. \( B \subseteq A \)), \( |\mu(B)| \leq \delta \), then

\[
\mu(A) - \varepsilon \leq \mu(A \cup B) \leq \mu(A) + \varepsilon
\]

(resp. \( \mu(A) - \varepsilon \leq \mu(A - B) \leq \mu(A) + \varepsilon \))

holds; \( \mu \) is called uniformly autocontinuous, if it is both uniformly autocontinuous from above and from below.
Similarly, if $\mu$ is a fuzzy measure, then "$A \cap B = \emptyset$" and "$B \subset A$" may be omitted in the statement of Definition 3.

**Theorem 3.** If $\mu$ is a fuzzy measure, then the following statements are equivalent:

1. $\mu$ is uniformly autocontinuous;
2. $\mu$ is uniformly autocontinuous from above;
3. $\mu$ is uniformly autocontinuous from below;
4. for every $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon) > 0$, such that, whenever $A \in \mathcal{F}$, $B \in \mathcal{F}$, $\mu(B) \leq \delta$, then

\[ \mu(A \triangle B) - \varepsilon \leq \mu(A) \leq \mu(A \triangle B) + \varepsilon \]

holds.

**Proof.** (1) $\Rightarrow$ (2) Obvious.

(2) $\Rightarrow$ (3) Since $\mu(B \cap A) \leq \mu(B) \leq \delta$, the desired conclusion follows from $\mu(A) = \mu[(A - B) \cup (B \cap A)] \leq \mu(A - B) + \varepsilon$.

(3) $\Rightarrow$ (4) Since $\mu(A \cap B) \leq \mu(B) \leq \delta$, we have

\[ \mu(A \triangle B) = \mu[(A \cup B) - (B \cap A)] \geq \mu(A \cup B) - \varepsilon \geq \mu(A) - \varepsilon. \]

On the other hand, since $\mu(B - A) \leq \mu(B) \leq \delta$, we have

\[ \mu(A) \geq \mu(A - B) = \mu[(A \triangle B) - (B - A)] \geq \mu(A \triangle B) - \varepsilon. \]

(4) $\Rightarrow$ (1) Obvious.

The proof of the theorem is complete. \[ \square \]

**Definition 4.** A class $\mathcal{C}$ of sets in $\mathcal{F}$ is called a chain, if, whenever $c_1 \in \mathcal{C}$, $c_2 \in \mathcal{C}$, then either $c_1 \subseteq c_2$, or $c_2 \subseteq c_1$.

The concept of chain coincides with the one given in [8].

**Definition 5.** A chain $\mathcal{C}$ is called $\mu$-bounded, if there exists $M > 0$, such that $|\mu(c)| \leq M$ for every $c \in \mathcal{C}$.

**Definition 6.** A set function $\mu$ is called local-uniformly autocontinuous from above (resp. from below), if it is autocontinuous from above (resp. from below), and for every $\mu$-bounded chain $\mathcal{C} \subset \mathcal{F}$ and every $\varepsilon > 0$, there exists $\delta = \delta(\mathcal{C}, \varepsilon) > 0$, such that

\[ \mu(A) - \varepsilon \leq \mu(A \cup B) \leq \mu(A) + \varepsilon \]

(resp. $\mu(A) - \varepsilon \leq \mu(A - B) \leq \mu(A) + \varepsilon$),
whenever \( A \in \mathcal{C}, \ B \in \mathcal{F} \), \( A \cap B = \emptyset \) (resp. \( B \subseteq A \)), \( |\mu(B)| \leq \delta \); \( \mu \) is called local-uniformly autocontinuous, if it is both local-uniformly autocontinuous from above and from below.

It is clear that uniform autocontinuity implies local-uniform autocontinuity, and if \( \mu \) is a fuzzy measure, then "\( A \cap B = \emptyset \)" and "\( B \subseteq A \)" may be omitted in the statement of Definition 6.

**Lemma 1.** If \( \mathcal{C} \) is an infinite chain, then there exists a monotone sequence of sets of \( \mathcal{C} \).

**Proof.** We arbitrarily take \( c_1 \in \mathcal{C} \). It is clear that, there exists an infinite subclass \( \mathcal{C}_1 \) of \( \mathcal{C} \) satisfying the property: either \( c \subseteq c_1 \) for every \( c \in \mathcal{C}_1 \) or \( c \supseteq c_1 \) for every \( c \in \mathcal{C} \). We arbitrarily take \( c_2 \in \mathcal{C}_1 \). Similarly, there exists an infinite subclass \( \mathcal{C}_2 \) of \( \mathcal{C}_1 \) satisfying the property: either \( c \subseteq c_2 \) for every \( c \in \mathcal{C}_2 \) or \( c \supseteq c_2 \) for every \( c \in \mathcal{C}_2 \). We arbitrarily take \( c_3 \in \mathcal{C}_2 \), ... . Finally, we obtain a sequence \( \mathcal{C}^* = \{c_i\} \) satisfying the property: for every \( c_i \in \mathcal{C}^* \), if \( c_i \rightarrow c_{i+1} \), then \( c_i \rightarrow c_j \) for every \( j > i \), and if \( c_i \leftarrow c_{i+1} \), then \( c_i \leftarrow c_j \) for every \( j > i \). Write \( \mathcal{C}^* = \mathcal{C}^+ \cup \mathcal{C}^- \), where \( \mathcal{C}^+ = \{c_i; c_i \subset c_{i+1}\} \) and \( \mathcal{C}^- = \{c_i; c_i \supseteq c_{i+1}\} \), then, between \( \mathcal{C}^+ \) and \( \mathcal{C}^- \), there exists at least an infinite subclass. Obviously, if \( \mathcal{C}^+ \) is infinite, then \( \mathcal{C}^+ = \{c_i\} \) is an increasing sequence of sets of \( \mathcal{C} \); if \( \mathcal{C}^- \) is infinite, then \( \mathcal{C}^- = \{c_i\} \) is a decreasing sequence of sets of \( \mathcal{C} \).

**Theorem 4.** If \( \mu \) is a fuzzy measure, then, \( \mu \) is autocontinuous (resp. from above; from below) \( \Leftrightarrow \mu \) is local-uniformly autocontinuous (resp. from above; from below).

**Proof.** It is sufficient to prove that autocontinuity from above (resp. from below) implies local-uniform autocontinuity from above (resp. from below).

We assume that, \( \mu \) is not local-uniformly autocontinuous from above, but is autocontinuous from above. Then, there exist a \( \mu \)-bounded chain \( \mathcal{C} \), \( \varepsilon > 0 \) and two sequences of sets \( \{A_n\} \in \mathcal{C} \) and \( \{B_n\} \in \mathcal{C} \), such that \( \mu(B_n) \rightarrow 0 \) and \( \mu(A_n \cup B_n) > \mu(A_n) + \varepsilon \) for all \( n \). Since \( \mathcal{C} \) is a chain, \( \{A_n\} \) is also a chain. By Lemma 1, there exists a monotone subsequence \( \{A_{n_i}\} \) of \( \{A_n\} \). Now, there are two cases: (1) If \( \{A_{n_i}\} \) is decreasing, since \( \mu(A_{n_i}) < \infty \), \( i = 1, 2, \ldots \), we have

\[
\mu(A_{n_i}) \searrow \mu \left( \bigcap_{i=1}^{\infty} A_{n_i} \right)
\]

when \( i \rightarrow \infty \). It follows that, there exists \( i_0 \), such that

\[
\mu(A_{n_{i_0}}) \geq \mu(A_{n_{i_0}}) - \frac{\varepsilon}{2}
\]
when \( i \geq i_0 \). Consequently,

\[
\mu(A_{n_0} \cup B_n) \geq \mu(A_{n_1} \cup B_n) > \mu(A_{n_1}) + \varepsilon \geq \mu(A_{n_0}) + \frac{\varepsilon}{2}
\]

for all \( i \geq i_0 \), which implies that \( \mu \) is not autocontinuous from above. (2) If \( \{A_n\} \) is increasing, write \( A = \bigcup_{i=0}^{\infty} A_n \), then

\[
\mu(A_{n_0}) \not\rightarrow \mu(A) < \infty
\]

when \( i \to \infty \). It follows that there exists \( i_0 \), such that

\[
\mu(A) \leq \mu(A_{n_0}) + \frac{\varepsilon}{2}
\]

when \( i \geq i_0 \). Consequently,

\[
\mu(A \cup B_n) \geq \mu(A_{n_1} \cup B_n) > \mu(A_{n_1}) + \varepsilon \geq \mu(A) + \frac{\varepsilon}{2}
\]

for all \( i \geq i_0 \). Thus, as obtained in the first case, \( \mu \) is not autocontinuous from above. This is a contradiction.

Similarly, we can prove that, if \( \mu \) is autocontinuous from below, then it is local-uniformly autocontinuous from below.

In Example 3, we give a fuzzy measure \( \mu \) which is local-uniformly autocontinuous, but is not uniformly autocontinuous.

**Example 3.** Let \( X = X^- \cup X^+ \), where \( X^- = \{-1, -2, \ldots\} \), \( X^+ = \{1, 2, \ldots\} \) and let \( \mathcal{F} = \mathcal{P}(X) \). A set function \( \mu \) is defined on \( \mathcal{F} \) as

\[
\mu(E) = k + \sum_{i \in X^-} \frac{1}{2^{i - 1}}.
\]

where \( k \) is the number of points in the set

\[
A^* = \{i: i \subset X^-, |i| \leq \sup \{j: j \subset E \cap X^+ \} \} \cup (E \cap X^+).
\]

It is easy to see that \( \mu \) is autocontinuous, therefore, by Theorem 4, it is local-uniformly autocontinuous. But, it is not uniformly autocontinuous. In fact, for any \( \delta > 0 \) and \( 0 < \varepsilon < 1 \), there exist \( i \in X^- \) and \( j = -i \in X^+ \), such that

\[
\mu(|i|) = 1/2^{-i} < \delta \quad \text{and} \quad \mu(|j|) = 2 - 1 = 1 > \varepsilon.
\]

Definition 7 and Proposition 7 will be related to the concepts of \( T \)-function and quasi-measure defined in [13].
Definition 7. A nonnegative set function $\mu$ is called quasi-additive if there exists a $T$-function, such that whenever $E \in \mathcal{F}, F \in \mathcal{F}, E \cap F = \emptyset$, then $\mu(E \cup F) = T^{-1}[T\mu(E) + T\mu(F)]$ holds. A nonnegative monotone set function $D$ is called sub-quasi-additive if there exists a $T$-function, such that whenever $E \in \mathcal{F}, F \in \mathcal{F}, E \cap F = \emptyset$, then $\mu(E \cup F) < T^{-1}[T\mu(E) + T\mu(F)]$ holds.

Proposition 7. If a nonnegative monotone set function $\mu$ is sub-quasi-additive, then it is autocontinuous. If $\mu$ is quasi-additive, then it is sub-quasi-additive, and therefore, it is autocontinuous. Further, any quasi-measure is local-uniformly autocontinuous.

Proposition 8. If a fuzzy measure $\mu$ is subadditive, then it is uniformly autocontinuous. Particularly, any measure is uniformly autocontinuous.

In Proposition 9, we shall use the concept of $F$-additivity given in [1, p. 12] and the concept of possibility given in [3, 5].

Proposition 9. For a nonnegative monotone set function, $F$-additivity implies subadditivity, and therefore, implies uniform autocontinuity. Particularly, any possibility measure is $F$-additive, and therefore, is uniformly autocontinuous.

Prade [12] introduced a class of fuzzy measures which issue from triangular norms. The relation between autocontinuity and triangular norms will be studied in another paper.

Finally, we use Scheme I to express some relations between the concepts presented in this section when $\mu$ is a fuzzy measure.

3. THE MEASURABLE FUNCTION

Throughout Section 3–5, let $(X, \mathcal{F}, \mu)$ be a fuzzy measure space. The definition of the measurable function $f: X \to [0, \infty]$ coincides with the definition given in the classical measure theory (refer to [7]). But, we only consider the nonnegative measurable functions. The concepts of convergence everywhere, convergence a.e., uniform convergence, almost uniform convergence, and convergence in (fuzzy) measure of a sequence of measurable functions wholly coincide with the concepts given in [7], too. Though the fuzzy measure has not additivity that the measure has, we can obtain some results resembling the ones obtained in the classical measure theory.

The following theorem was given in [4]:


THEOREM 5. If \( \mu \) is finite, then convergence a.e. implies convergence in measure.

We now give the following theorem (Egoroff's theorem) as a principal result of this section.

THEOREM 6. If \( \mu \) is autocontinuous from above, and a sequence of measurable functions \( \{f_n\} \) converges a.e. to an a.e. finite-valued measurable function \( f \) on \( A \in \mathcal{F}, \mu(A) < \infty \), then for any \( \varepsilon > 0 \), there exists \( E \in \mathcal{F} \) with \( \mu(E) < \varepsilon \), such that \( \{f_n\} \) converges to \( f \) uniformly on \( A - E \).

Proof. Since autocontinuity from above implies null-additivity, we can suppose that, \( f_n \to f \) everywhere and \( f \) is finite on \( A \), without any loss of generality.
Write

$$E^m_n = \bigcap_{i=n}^{\infty} \{ x : |f_i - f| < \frac{1}{m} \}, \quad m = 1, 2, \ldots.$$ 

then $E_1^m \subset E_2^m \subset \cdots$. Since $f_n \to f$ everywhere on $A$, we have $\lim_{n \to \infty} E_n^m \supset A$, therefore

$$\lim_{n \to \infty} (A - E_n^m) = \emptyset.$$ 

It follows, by the continuity of $\mu$, that

$$\lim_{n \to \infty} \mu(A - E_n^m) = 0.$$ 

Then, for every $m = 1, 2, \ldots$ there exists $n_m$, such that

$$\mu(A - E_{n_m}^m) < \frac{1}{m}.$$ 

If $F_m = A - E_n^m$, then we have $\mu(F_m) \to 0$ as $m \to \infty$. Now, we choose a subsequence $\{F_{m_j}\}$ in $\{F_m\}$ as follows: For a given $\varepsilon > 0$, we take $F_{m_1}$ such that $\mu(F_{m_1}) < \varepsilon/2$; Since $\mu$ is autocontinuous from above,

$$\lim_{j \to \infty} \mu(F_{m_1} \cup F_j) = \mu(F_{m_1}),$$

holds, thus, we can choose $F_{m_2}$, such that

$$\mu(F_{m_1} \cup F_{m_2}) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2^2} = \frac{3}{4} \varepsilon.$$ 

Similarly, for $F_{m_1} \cup F_{m_2}$, we can choose $F_{m_3}$, such that

$$\mu(F_{m_1} \cup F_{m_2} \cup F_{m_3}) < \frac{3}{4} \varepsilon + \frac{\varepsilon}{2^3} = \frac{7}{8} \varepsilon;$$

and so on. Finally, we obtain a subsequence $\{F_{m_j}\}$ of $\{F_m\}$ with $\mu(\bigcup_{j=1}^{\infty} F_{m_j}) < \varepsilon$. Write $E = \bigcup_{j=1}^{\infty} F_{m_j} \in \mathcal{F}$, then $\mu(E) < \varepsilon$. Now, we come to show that $\{f_n\}$ converges to $f$ uniformly on $A - E$. For $\varepsilon' > 0$, take $k$ such that $k = m_{i_0} > 1/\varepsilon'$. If $x \in A - E$, then $x \in A$, but $x \notin E$, therefore $x \in A$ and $x \notin F_k$. It follows that $x \in E_{k+1}^k$, namely,

$$x \in \bigcap_{i=n_k}^{\infty} \{ x : |f_i - f| < \frac{1}{k} \}.$$
It implies that $|f_j - f| < 1/k < \varepsilon'$ as $j \geq n_k$. The proof of the theorem is complete. ■

Theorem 6 asserts that, if \( \mu \) is autocontinuous from above, then convergence a.e. on a set of finite fuzzy measure implies almost uniform convergence. This result generalizes Egoroff's theorem in the classical measure theory.

The following result goes in the converse direction:

\[ \text{Theorem 7. If } \{f_n\} \text{ is a sequence of measurable functions which converges to } f \text{ almost uniformly, then } \{f_n\} \text{ converges to } f \text{ a.e.} \]

\[ \text{Proof. The classical proof works. ■} \]

\[ \text{Theorem 8. Almost uniform convergence implies convergence in measure.} \]

\[ \text{Proof. The classical proof works, too. ■} \]

### 4. Fuzzy Integral and Its Elementary Properties

Let \( f(x) \) be a nonnegative extended real-valued measurable function on \((X, \mathcal{F}, \mu)\) and \( A \in \mathcal{F} \), the fuzzy integral of \( f \) on \( A \) with respect to \( \mu \) is defined by

\[
\int_A f \, d\mu = \sup_{\alpha \in [0, \infty)} [\alpha \wedge \mu(A \cap F_\alpha)],
\]

where \( F_\alpha = \{x: f \geq \alpha\}, \alpha \in [0, \infty) \).

\[ \text{Proposition 10.} \]

\[
\int_A f \, d\mu = \sup_{\alpha \in [0, \infty)} [\alpha \wedge \mu(A \cap F_\alpha)]
= \sup_{E \in 2^f} [(\inf_{x \in E} f(x)) \wedge \mu(A \cap E)]
= \sup_{E \in 2^f} [(\inf_{x \in E} f(x)) \wedge \mu(A \cap E)].
\]

where \( \mathcal{F}(f) \) is the \( \sigma \)-algebra generated by \( f \).

Ralescu and Adams \cite{4} gave another equivalent definition of a fuzzy integral. Hereafter, we denote \( \int_A f \, d\mu \) by \( \int f \, d\mu \), and \( f, f_n, n = 1, 2 \ldots \).
represent nonnegative extended real-valued measurable functions. It is easy to obtain these elementary properties of fuzzy integral as follows:

**Proposition 11.** \( \int_A f \, d\mu = \int_A f \cdot \chi_A \, d\mu \), where
\[
\chi_A(x) = \begin{cases} 
1, & x \in A, \\
0, & x \notin A.
\end{cases}
\]

**Proposition 12.** For any constant \( a \in [0, \infty) \), \( \int_A a \, d\mu = a \wedge \mu(A) \).

**Proposition 13.** If \( f_1 \leq f_2 \) on \( A \), then \( \int_A f_1 \, d\mu \leq \int_A f_2 \, d\mu \).

**Proposition 14.** If \( \int_A f \, d\mu = 0 \), then \( \mu(A \cap \{f > 0\}) = 0 \).

**Proposition 15.** For any constant \( a \in [0, \infty) \),
\[
\int_A (f + a) \, d\mu \leq \int_A f \, d\mu + a \wedge \mu(A) = \int_A f \, d\mu + \int_A a \, d\mu.
\]

**Theorem 9.** Let \( a \in [0, \infty] \). If \( |f_1 - f_2| \leq a \) on \( A \), then
\[
\left| \int_A f_1 \, d\mu - \int_A f_2 \, d\mu \right| \leq a.
\]

**Proof.** From \( f_1 \leq f_2 + a \) on \( A \), we have
\[
\int_A f_1 \, d\mu \leq \int_A (f_2 + a) \, d\mu \leq \int_A f_2 \, d\mu + a \wedge \mu(A) \leq \int_A f_2 \, d\mu + a
\]
by using Propositions 13 and 15. Analogously, from \( f_2 \leq f_1 + a \) on \( A \), we have
\[
\int_A f_2 \, d\mu \leq \int_A f_1 \, d\mu + a.
\]
Consequently,
\[
\left| \int_A f_1 \, d\mu - \int_A f_2 \, d\mu \right| \leq a.
\]

**Lemma 2.** Let \( h_\alpha, \alpha \in [0, \infty] \), be a nonnegative decreasing extended real-valued function of \( \alpha \), if \( \alpha_0 = \sup_{\alpha \in [0, \infty]} [\alpha \wedge h_\alpha] \), then \( h_{\alpha_0+0} \geq \alpha_0 \geq h_{\alpha_0-0} \), where \( h_{\alpha_0-0} = \lim_{\alpha \to \alpha_0^-} h_\alpha \), \( h_{\alpha_0+0} = \lim_{\alpha \to \alpha_0^+} h_\alpha \).

**Proof.** If \( \alpha_0 = 0 \), then \( h_\alpha = 0 \) for any \( \alpha > 0 \); If \( \alpha_0 = \infty \), then \( h_\alpha = \infty \) for
any $a \in [0, \infty)$. In these two cases, the conclusion of the lemma is obvious.

Now, suppose $a_0 \in (0, \infty)$. For every $a > a_0$, since $a \land h_a \leq a_0$, we have $h_a \leq a_0$, therefore $h_{a_0+0} \leq a_0$. On the other hand, we assume that $h_{a_0-0} < a_0$, then there exists $a' < a_0$, such that $h_{a'} < a_0$. Using the monotonicity of $h_a$, then $h_a \leq h_{a'} < a_0$ as $a \geq a'$ holds there. Consequently,

$$\sup_{a \in [0, a')} [a \land h_a] = \sup_{a \in (0, a')} [a \land h_a] \lor \sup_{a \in (a, \infty)} [a \land h_a]$$

$$\leq \sup_{a \in [0, a')} a \lor \sup_{a \in (a, \infty)} h_a$$

$$< a_0,$$

we obtain a contradiction. Therefore, $h_{a_0-0} \geq a_0$. The proof of the lemma is complete.

Now, we denote $\{x : f \geq a\}$ by $F_a$ and $\lim_{a \to a_0^+} F_a$, by $F_{a_0}^+$, then we have

**Theorem 10.**

1. $\int_A f \, d\mu = a \Leftrightarrow \mu(A \cap F_a) \geq a \geq \mu(A \cap F_{a_0}^+)$,

2. $\int_A f \, d\mu > a \Rightarrow \mu(A \cap F_a) > a$,

3. $\int_A f \, d\mu \geq a \Leftrightarrow \mu(A \cap F_a) \geq a$.

**Proof.**

1. From the definition of $\int_A f \, d\mu$ directly, it is obvious that $\mu(A \cap F_a) \geq a \geq \mu(A \cap F_{a_0}^+) \Rightarrow \int_A f \, d\mu = a$. For the converse proposition, we write $h_a = \mu(A \cap F_a)$, then $h_a$ is decreasing with respect to $a$, and $\int_A f \, d\mu = \sup_{a \in (0, a)} [a \land h_a]$. Looking to the fact that $h_{a_0-0} = h_a$, the desired conclusion is obtained by using Lemma 2.

2. Denote $\int_A f \, d\mu$ by $a_0$, then, from $a_0 > a$, we have

$$\mu(A \cap F_a) \geq \mu(A \cap F_{a_0}^+) \geq a_0 > a.$$

3. From the definition of $\int_A f \, d\mu$, it is obvious that

$$\mu(A \cap F_a) \geq a \Rightarrow \int_A f \, d\mu \geq a.$$

The converse proposition is true by using (1) and (2).

In the classical measure theory, for two measurable functions $f_1$ and $f_2$, if $f_1 = f_2$ a.e., then their integrals are equal. Is it still true for the fuzzy integral? The following theorem will answer this question:

**Theorem 11.** Whenever $f_1 = f_2$ a.e., $\int f_1 \, d\mu = \int f_2 \, d\mu$ holds, if and only if $\mu$ is null-additive.
Proof. If \( \mu \) is null-additive, then for any \( \alpha \geq 0 \), from \( \mu(\{f_1 \neq f_2\}) = 0 \), we have

\[
\mu(\{f_2 \geq \alpha\}) \leq \mu(\{f_1 \geq \alpha\} \cup \{f_1 \neq f_2\}) = \mu(\{f_1 \geq \alpha\})
\]

\[
\leq \mu(\{f_2 \geq \alpha\} \cup \{f_1 \neq f_2\}) = \mu(\{f_2 \geq \alpha\}).
\]

It follows that \( \int f_1 \, d\mu = \int f_2 \, d\mu \). Conversely, for any \( E \in \mathcal{F} \) and \( F \in \mathcal{F} \) with \( \mu(F) = 0 \), take

\[
f_1(x) = \infty, \quad x \in E.
\]

\[
= 0, \quad x \notin E.
\]

\[
f_2(x) = \infty, \quad x \in E \cup F.
\]

\[
= 0, \quad x \notin E \cup F.
\]

then \( f_1 = f_2 \) a.e. It follows, from \( \int f_1 \, d\mu = \int f_2 \, d\mu \), that

\[
\mu(E) = \mu(E \cup F).
\]

That is, \( \mu \) is null-additive.

Corollary 1. If \( \mu \) is null-additive, then whenever \( E \in \mathcal{F} \), \( F \in \mathcal{F} \), \( \mu(F) = 0 \), \( \int_{E \cup F} f \, d\mu = \int E f \, d\mu \) holds.

Wang [9] gave the definition of \( F \)-mean convergence of a sequence of functions and asserted that \( F \)-mean convergence implies convergence in measure (in a more particular context). Now, we show that these two concepts are equivalent.

Definition 8. \( \{f_n\} \) is said to \( F \)-mean converge to an a.e. finite measurable function \( f \), if

\[
\lim_{n \to \infty} \int |f_n - f| \, d\mu = 0.
\]

Theorem 12. \( F \)-mean convergence is equivalent to convergence in measure.

Proof. If \( \{f_n\} \) does not converge in measure to an a.e. finite measurable function \( f \), then there exist \( \epsilon > 0 \), \( \delta > 0 \), and a sequence \( \{n_i\} \), such that

\[
\mu(\{x: |f_{n_i} - f| \geq \epsilon\}) > \delta
\]
for every $i$. It follows that

$$
\int |f_{n_i} - f| \, d\mu \geq \varepsilon \wedge \delta > 0
$$

for every $i$. That is to say, $\{f_n\}$ does not $F$-mean converge to $f$. Conversely, if $\{f_n\}$ converges in measure to an a.e. finite measurable function $f$, then for $\varepsilon > 0$, there exists $n$, such that

$$
\mu(\{x: |f_n - f| \geq \varepsilon\}) < \varepsilon
$$

as $n \geq n_0$. It follows, by using Theorem 10, that

$$
\int |f_n - f| \, d\mu < \varepsilon.
$$

That is, $\{f_n\}$ $F$-mean converges to $f$.

5. The Convergence Theorems

In this section, we show some important convergence theorems of the sequence of fuzzy integrals under as weak as possible conditions. Throughout this section, $f, f_n, n = 1, 2, \ldots$ represent nonnegative extended real-valued measurable functions.

The monotone convergence theorem was presented by Sugeno [11] (in a more particular context), but his proof is wrong, since

$$
\mu(\{x: \lim_{n \to \infty} f_n \geq \alpha\}) = \lim_{n \to \infty} \mu(\{x: f_n \geq \alpha\})
$$

was asserted. In fact, this equality is not always true. For example, take a fuzzy measure space $(X, \mathcal{F}, \mu)$ with $\mu(X) = 1$, and $f_n = 1 - (1/n)$, $n = 1, 2, \ldots$. then $\lim_{n \to \infty} f_n = 1$, and for $\alpha = 1$, it holds

$$
\mu(\{x: \lim_{n \to \infty} f_n \geq 1\}) = \mu(X) = 1.
$$

but

$$
\mu(\{x: f_n \geq 1\}) = \mu(\phi) = 0
$$

for every $n$.

Using an equivalent definition of fuzzy integral, Ralescu and Adams [4] proved the monotone convergence theorem. Now, we give another proof of this theorem in a somewhat more extensive context.
THEOREM 13 (Monotone convergence theorem). Let \( \{f_n\} \) be increasing, and \( A \in \mathcal{F} \), then

\[
\lim_{n \to \infty} \int f_n \, d\mu = \int \lim_{n \to \infty} f_n \, d\mu.
\]

Proof: By Proposition 11, we can let \( A = X \) without losing generality. If \( f = \lim_{n \to \infty} f_n \), then \( f \) is a nonnegative extended real-valued measurable function. Write \( c = \int f \, d\mu \). If \( c = 0 \), then, by Proposition 13,

\[
0 \leq \int f_n \, d\mu \leq \int f \, d\mu = 0. \quad n = 1, 2, \ldots
\]

The conclusion of the theorem is obviously true.

Now, suppose that \( 0 < c < \infty \). For \( \varepsilon \in (0, c/2) \), there exists \( a_0 \), such that \( c \geq a_0 \wedge \mu(\{f \geq a_0\}) > c - \varepsilon \): of course, \( a_0 > \varepsilon \). We write \( E_n = \{f_n \geq a_0 - \varepsilon\} \), then \( \{E_n\} \) is an increasing sequence of sets, and \( E_n \supseteq E = \bigcup_{n=1}^{\infty} E_n \). It follows, by monotonicity and continuity of \( \mu \), that \( \mu(E_n) \to \mu(E) \).

Furthermore, since \( f_n / f \), we have \( E \supseteq \{f \geq a_0\} \) (for any \( x \in \{f \geq a_0\} \), there exists \( n \), such that \( f_n(x) \geq a_0 - \varepsilon \), i.e., \( x \in E_n \subseteq E \)), it follows that \( \mu(E) \supseteq \mu(\{f \geq a_0\}) \). Therefore, there exists \( n_0 \), such that \( \mu(E_n) \geq \mu(\{f \geq a_0\}) - \varepsilon \) as \( n \geq n_0 \). Thus, we have

\[
\int f_n \, d\mu = \sup_{\alpha \in [0, \infty]} (\alpha \wedge \mu(\{f_n \geq \alpha\})) \geq (\alpha_0 - \varepsilon) \wedge \mu(\{f_n \geq \alpha_0 - \varepsilon\})
\]

\[
\geq (\alpha_0 - \varepsilon) \wedge [\mu(\{f \geq a_0\}) - \varepsilon] - \varepsilon
\]

\[
\geq c - 2\varepsilon
\]

for all \( n \geq n_0 \). On the other hand, by using Proposition 13, we have

\[
\int f_n \, d\mu \leq c.
\]

Consequently,

\[
\lim_{n \to \infty} \int f_n \, d\mu = c.
\]

Finally, if \( c = \infty \), then \( \mu(\{f \geq \alpha\}) = \infty \) for all \( \alpha \in [0, \infty) \). For arbitrarily given \( N > 0 \), write \( F_N = \{f_n \geq N\} \), then \( F_N \supseteq F_N^\infty = \bigcup_{n=1}^{\infty} F_n^\infty \), and therefore \( \mu(F_N^\infty) \supseteq \mu(F_N^\infty) \). On the other hand, since \( f_n / f \), we have \( F_n^\infty \supseteq \{f \geq N + 1\} \), and therefore \( \mu(F_N^\infty) \supseteq \mu(\{f \geq N + 1\}) = \infty \). Thus, there exists \( n_0 \), such that \( \mu(F_N^\infty) \geq N \) as \( n \geq n_0 \). By using Theorem 10, we have \( \int f_n \, d\mu \geq N \) as \( n \geq n_0 \). The proof of the theorem is complete. \( \square \)
For a decreasing sequence, we have

**Theorem 14.** If \( \{f_n\} \) is decreasing and converges to \( f \) on \( A \in \mathcal{F} \), and if there exist \( n_0 \) and a constant \( c' \leq \int_A f \, d\mu \), such that

\[
\mu(\{f_{n_0} > c'\} \cap A) < \infty,
\]

then

\[
\lim_{n \to \infty} \int_A f_n \, d\mu = \int_A f \, d\mu.
\]

**Proof.** We can let \( A = X \) without losing generality. Write \( c = \int f \, d\mu \), we have \( \lim_{n \to \infty} \int f_n \, d\mu \geq c \). If \( c = \infty \), then the conclusion of the theorem is obviously true. If \( c < \infty \), we assume that \( \lim_{n \to \infty} \int f_n \, d\mu > c \), then there exists \( \delta > 0 \), such that

\[
\lim_{n \to \infty} \int f_n \, d\mu \geq c + \delta.
\]

Since \( \{f_n\} \) is decreasing, we have \( \int f_n \, d\mu \geq c + \delta \), and therefore, by using Theorem 10,

\[
\mu(\{f_n \geq c + \delta\}) \geq c + \delta
\]

for every \( n \). From the hypothesis that \( \mu(\{f_{n_0} > c'\}) < \infty \), we have

\[
\mu(\{f_{n_0} \geq c + \delta\}) \leq \mu(\{f_{n_0} > c'\}) < \infty,
\]

and since \( f_n \searrow f \), \( \{f_n \geq c + \delta\} \) is decreasing with respect to \( n \), we have

\[
\left| f_n \geq c + \delta \right| \searrow \bigcap_{n=1}^{\infty} \left| f_n \geq c + \delta \right| \neq \emptyset
\]

it follows, by using continuity of \( \mu \), that

\[
\mu(\{f \geq c + \delta\}) = \lim_{n \to \infty} \mu(\{f_n \geq c + \delta\}) \geq c + \delta.
\]

Consequently, from Theorem 10, \( \int f \, d\mu \geq c + \delta \). It is a contradiction. The proof of the theorem is complete. \( \blacksquare \)

**Corollary 2.** If \( f_n \searrow f \) and \( \mu \) is finite, then \( \int f_n \, d\mu \searrow \int f \, d\mu \).

In Theorem 14, the hypothesis of finiteness cannot be dropped. The following simple example will show it.
**EXAMPLE 4.** Let $X = (0, \infty)$, $\mathcal{F}$ be a Borel field on $X$ and $\mu$ be the Lebesgue measure. We take $f_n(x) = x/n$, $n = 1, 2, \ldots$. Then $f_n \to f \equiv 0$. Obviously, it does not satisfy the hypothesis of Theorem 14. Consequently, the fuzzy integrals $\int f_n \, d\mu = \infty$, $n = 1, 2, \ldots$ and $\int f \, d\mu = 0$, that is $\lim_{n \to \infty} \int f_n \, d\mu = \int f \, d\mu$.

**COROLLARY 3.** Let $\mu$ be null-additive, then

1. $f_n \to f$ a.e. $\Rightarrow \int f_n \, d\mu \to \int f \, d\mu$;
2. $f_n \not\to f$ a.e. and there exist $n_0$ and $c' \leq \int f \, d\mu$, such that $\mu(\{f_n > c'\}) < \infty \Rightarrow \int f_n \, d\mu \not\to \int f \, d\mu$.

Now, we can show a general convergence theorem.

**THEOREM 15.** If $\{f_n\}$ converges to $f$ on $A \in \mathcal{F}$, and there exist $n_0$ and a constant $c' \leq \int f \, d\mu$, such that

$$\mu(\{\sup_{n > n_0} f_n > c'\} \cap A) < \infty,$$

then $\lim_{n \to \infty} \int_A f_n \, d\mu$ exists, and $\lim_{n \to \infty} \int_A f_n \, d\mu = \int_A f \, d\mu$.

**Proof.** Without losing generality, we suppose that $A = X$. Write $\tilde{f}_n = \sup_{i > n} f_i$ and $f_n = \inf_{i < n} f_i$, then $\tilde{f}_n$, $f_n$, $n = 1, 2, \ldots$, are measurable, and $\tilde{f}_n \to f$, $f_n \to f$. Since $f_n \leq f_n \leq \tilde{f}_n$, it follows that

$$\int f_n \, d\mu \leq \int f_n \, d\mu \leq \int \tilde{f}_n \, d\mu,$$

and therefore

$$\lim_{n \to \infty} \int f_n \, d\mu \leq \lim_{n \to \infty} \int f_n \, d\mu \leq \lim_{n \to \infty} \int \tilde{f}_n \, d\mu.$$ 

By using Theorems 13 and 14, we have

$$\lim_{n \to \infty} \int f_n \, d\mu = \lim_{n \to \infty} \int \tilde{f}_n \, d\mu = \int f \, d\mu.$$ 

Consequently, $\lim_{n \to \infty} \int f_n \, d\mu = \lim_{n \to \infty} \int f_n \, d\mu = \int f \, d\mu$. The theorem is proved.

**COROLLARY 4.** Whenever $f_n \to f$ a.e., and there exist $n_0$ and a constant $c' \leq \int f \, d\mu$ such that $\mu(\{\sup_{n > n_0} f_n > c'\}) < \infty$, then $\int f_n \, d\mu \to \int f \, d\mu$, if and only if $\mu$ is null-additive.
Proof: Using Theorem 11, it is easy to obtain the conclusion.

We give the following statement, which was shown by Ralescu and Adams [4], as a particular case of the preceding result.

**Corollary 5.** If \( \mu \) is finite and subadditive, then

\[
f_n \to f \text{ a.e. } \Rightarrow \int f_n \, d\mu \to \int f \, d\mu.
\]

**Proof.** From Propositions 8 and 3, subadditivity implies null-additivity. Using finiteness of \( \mu \), the desired result is clear.

Given that \( f_n \to f \) in measure, under what condition does there hold \( \int f_n \, d\mu \to \int f \, d\mu \)? Ralescu and Adams [4] required that \( \mu \) is subadditive, and presented an example to show that the subadditivity assumption cannot be dropped. However, this example is wrong. As a matter of fact, \( X = \{1, 2, \ldots\} \), \( \mathcal{F} = \mathcal{F}(X) \) and \( \mu \) was defined by \( \mu(\emptyset) = 0, \mu(\{n\}) = 1/n, n = 1, 2, \ldots, \mu(A) = 100 \) if \( A \) has not less than two elements. It is easy to see that \( \mu \) is not continuous from above at 0. In fact, take \( A_n = \{n, n + 1, \ldots\} \), then \( \mu(A_n) = 100, n = 1, 2, \ldots, \) and \( A_n \setminus \emptyset, \) but \( \lim_{n \to \infty} \mu(A_n) = 100 \neq \mu(\emptyset) \).

Therefore, \( \mu \) is not a fuzzy measure. Indeed, the subadditivity condition, that [4] required, is too strong for the desired result. We can show the following theorem, in which a necessary and sufficient condition will be given.

**Theorem 16.** Whenever \( \{f_n\} \) converges in measure to an a.e. finite measurable function \( f \) on \( A \in \mathcal{F} \), then \( \int f_n \, d\mu \to \int f \, d\mu \), if and only if \( \mu \) is autocontinuous.

**Proof.** Without any loss of generality, let \( A = X \) and \( f \) be finite everywhere.

**Sufficiency.** Write \( F_\alpha = \{f \geq \alpha\}, F_{\alpha + \delta} = \{f > \alpha\}, F_\alpha^n = \{f_n \geq \alpha\}, \) for \( \alpha \in [0, \infty] \), and \( c = \int f \, d\mu \).

(i) If \( c = \infty \), then \( \mu(F_\alpha) = \infty \) for every \( \alpha \in [0, \infty) \). Arbitrarily giving \( \alpha \in (0, \infty) \), we have

\[
F_\alpha^n \supseteq F_{\alpha + 1} - \{|f - f_n| \geq 1\}.
\]

Since \( \mu \) is autocontinuous from below, there exists \( \delta = \delta(\alpha) > 0 \), such that whenever \( \mu(B) < \delta, B \in \mathcal{F} \), it holds \( \mu(F_{\alpha + 1} - B) \geq \alpha \). And since \( f_n \to f \) in measure, there exists \( n_0 = n_0(\alpha) \), such that \( \mu(|f - f_n| \geq 1|) < \delta \) as \( n \geq n_0 \). Consequently,

\[
\mu(F_\alpha^n) \geq \mu(F_{\alpha + 1} - \{|f - f_n| \geq 1\}) \geq \alpha
\]
as $n \geq n_0$. By using Theorem 10, it follows that $\int f_n \, d\mu \geq \alpha$ as $n \geq n_0$, that is, $\int f_n \, d\mu \to \infty$.

(ii) If $c < \infty$, from Theorem 10, we have, for every $\alpha > c$,

$$\mu(F_\alpha) \leq \mu(F_{c+0}) \leq c < \infty,$$

therefore $\{F_\alpha\}_{\alpha > c}$ is a $\mu$-bounded chain.

For arbitrarily given $\varepsilon > 0$, since

$$F_{a+\varepsilon}^n \subset F_a \cup \{|f - f_n| \geq \varepsilon\},$$

and since autocontinuity of $\mu$ implies local-uniform autocontinuity from above, by using the fact that $f_n \to f$ in measure, there exists $n_0$, such that

$$\mu(F_{a+\varepsilon}^n) \leq \mu(F_a \cup \{|f - f_n| \geq \varepsilon\}) \leq \mu(F_a) + \varepsilon$$

as $n \geq n_0$ for every $\alpha > c$. On the other hand, for every $\alpha \in [0, c]$, we have, from Theorem 10, $\mu(F_\alpha) \geq \alpha$, and therefore

$$\alpha \wedge \mu(F_{a+\varepsilon}^n) \leq \alpha \wedge \mu(F_a)$$

holds for any $n$. Consequently, for all $\alpha \geq 0$,

$$\alpha \wedge \mu(F_{a+\varepsilon}^n) \leq \alpha \wedge \mu(F_a) + \varepsilon.$$

and therefore,

$$(\alpha + \varepsilon) \wedge \mu(F_{a+\varepsilon}^n) \leq [(\alpha \wedge \mu(F_{a+\varepsilon}^n))] + \varepsilon \leq [\alpha \wedge \mu(F_a)] + 2\varepsilon.$$  

Obviously, the above inequality holds also for $\alpha \in [-\varepsilon, 0)$, we have

$$(\alpha + \varepsilon) \wedge \mu(F_{a+\varepsilon}^n) \leq [\alpha \wedge \mu(F_a)] + 2\varepsilon \leq \int f \, d\mu + 2\varepsilon$$

for all $\alpha \in [-\varepsilon, \infty)$. It follows that

$$\int f_n \, d\mu \leq \int f \, d\mu + 2\varepsilon$$

as $n \geq n_0$.

Analogously, for $\varepsilon > 0$ and all $\alpha > c$, since

$$F_{a-\varepsilon}^n \supset F_a - \{|f - f_n| \geq \varepsilon\},$$
using local-uniform autocontinuity from below of \( \mu \) and the fact that \( f_n \to f \) in measure, there exists \( n_1 \), such that

\[
\mu(F_{\alpha-\varepsilon}^n) \geq \mu(F_{\alpha}) - \varepsilon
\]
as \( n \geq n_1 \). Furthermore, since

\[
F_{c-\varepsilon}^n \Rightarrow F_c - \{|f - f_n| \geq \varepsilon\},
\]
by using autocontinuity from below of \( \mu \), there exists \( n_2 \), such that

\[
\mu(F_{c-\varepsilon}^n) \geq \mu(F_{c}) - \varepsilon
\]
as \( n \geq n_2 \). Thus, for \( \alpha \in [0, c] \), we have

\[
\mu(F_{\alpha-\varepsilon}^n) \geq \mu(F_{c-\varepsilon}^n) \geq \mu(F_{\alpha}) - \varepsilon \geq c - \varepsilon \geq \alpha - \varepsilon
\]
as \( n \geq n_2 \). Consequently, as \( n \geq n'_0 = n_1 \lor n_2 \),

\[
(\alpha - \varepsilon) \land \mu(F_{\alpha-\varepsilon}^n) \geq (\alpha - \varepsilon) \land \mu(F_{\alpha}) - \varepsilon \geq \alpha \land \mu(F_{\alpha}) - 2\varepsilon
\]
for all \( \alpha \geq 0 \). Since

\[
\int f_n \, d\mu \geq (\alpha - \varepsilon) \land \mu(F_{\alpha-\varepsilon}^n)
\]
for all \( \alpha \geq \varepsilon \), and for \( \alpha \in [0, \varepsilon) \), obviously holds

\[
\int f_n \, d\mu \geq 0 \geq \alpha - \varepsilon \geq (\alpha - \varepsilon) \land \mu(F_{\alpha-\varepsilon}^n),
\]
we have

\[
\int f_n \, d\mu \geq \alpha \land \mu(F_{\alpha}) - 2\varepsilon
\]
for all \( \alpha \geq 0 \), therefore,

\[
\int f_n \, d\mu \geq \int f \, d\mu - 2\varepsilon
\]
as \( n \geq n'_0 \).

Now, it is clear that

\[
\left| \int f_n \, d\mu - \int f \, d\mu \right| \leq 2\varepsilon
\]
as $n \geq n_0 \lor n_0'$, namely, $\int f_n \, d\mu \to \int f \, d\mu$. The proof of sufficiency is complete.

**Necessity.** Let $A \in \mathcal{F}$ and a sequence of sets $\{B_n\} \subset \mathcal{F}$ with

$$\lim_{n \to \gamma} \mu(B_n) = 0.$$

If $\mu(A) < \infty$, we take $\alpha > \mu(A)$ and define

$$f(x) = \begin{cases} a, & x \in A, \\ 0, & x \notin A. \end{cases}$$

$$f_n(x) = \begin{cases} a, & x \in A \triangle B_n, \\ -0, & x \notin A \triangle B_n, \quad n = 1, 2, \ldots. \end{cases}$$

Obviously, for arbitrarily given $\varepsilon > 0$,

$$\mu(\{|f - f_n| \geq \varepsilon\}) \leq \mu(B_n) \to 0,$$

namely, $f_n \to f$ in measure. Therefore, by hypothesis of the theorem.

$$\int f_n \, d\mu \to \int f \, d\mu.$$ 

Consequently, from the fact that $\int f_n \, d\mu = \alpha \land (A \triangle B_n)$ and $\int f \, d\mu = \alpha \land \mu(A) = \mu(A)$, we have

$$\lim_{n \to \infty} \mu(A \triangle B_n) = \mu(A).$$

If $\mu(A) = \infty$, it is sufficient to prove that $\mu$ is autocontinuous from below at $A$. For arbitrarily given $N > 0$, we define

$$f(x) = N + 1, \quad x \in A.$$

$$= 0, \quad x \notin A.$$

$$f_n(x) = N + 1, \quad x \in A - B_n.$$

$$= 0, \quad x \notin A - B_n, \quad n = 1, 2, \ldots.$$

It is clear that $f_n \to f$ in measure, and therefore $\int f_n \, d\mu \to \int f \, d\mu$. Since $\int f \, d\mu = N + 1$ and $\int f_n \, d\mu = (N + 1) \land \mu(A - B_n)$, $n = 1, 2, \ldots$, it follows that there exists $n_0$, such that $\mu(A - B_n) \geq N$ as $n \geq n_0$. That is,

$$\lim_{n \to \infty} \mu(A - B_n) = \infty.$$ 

The proof of necessity is complete.

By using Theorem 12, we can give the following statement:
Corollary 6. Whenever \( \{f_n\} \) F-mean converges to an a.e. finite measurable function \( f \) on \( A \in \mathcal{F} \), then \( \int_A f_n \, d\mu \to \int_A f \, d\mu \), if and only if \( \mu \) is autocontinuous.

Now, we use Example 2 to construct an example in which \( f_n \to f \) in measure, but \( \int f_n \, d\mu \not\to \int f \, d\mu \).

Example 5. Let \( X = \{1, 2, \ldots\} \), \( \mathcal{F} = 2^X \), \( \mu(E) = k \cdot \sum_{i \in E} (1/2^i) \) for \( E \in \mathcal{F} \), where \( k \) is the number of points in \( E \). As we already know, \( \mu \) is autocontinuous from below, but not from above. Take \( f(x) = \chi_{\{x\}}(x) \), \( f_n(x) = \chi_{\{x\},n}(x) \), then, for arbitrary \( \varepsilon \in (0, 1) \),

\[
\mu(\{|f-f_n| > \varepsilon\}) = \mu(\{|n|\}) = (1/2^n) \to 0.
\]

namely, \( f_n \to f \) in measure. But, \( \int f \, d\mu = \frac{1}{2} \) and \( \int f_n \, d\mu = 1 \), \( n = 1, 2, \ldots \), namely, \( \int f_n \, d\mu \not\to \int f \, d\mu \).

Theorem 17 will relate to the concept of \( L^1(\mu) \) given in [4].

Theorem 17. If \( \mu \) is uniformly autocontinuous, \( \{f_n\} \) converges in measure to an a.e. finite measurable function \( f \) on \( A \in \mathcal{F} \), then

1. \( \int_A f \, d\mu = \infty \iff \) there exists \( n_0 \), such that \( \int_A f_n \, d\mu = \infty \) as \( n \geq n_0 \).
2. \( f \in L^1(\mu) \iff \) there exists \( n_0 \), such that \( f_n \in L^1(\mu) \) as \( n \geq n_0 \).

Proof. It is sufficient to prove that, if \( \int f \, d\mu = \infty \), then there exists \( n_0 \), such that \( \int f_n \, d\mu = \infty \) as \( n \geq n_0 \). We use the notations given in the proof of Theorem 16. Let \( \int f \, d\mu = \infty \), then \( \mu(F_\alpha) = \infty \) for every \( \alpha \in [0, \infty) \). Since

\[
F_\alpha = F_{\alpha+1} - \{|f-f_n| \geq 1\},
\]

it follows, by using uniform autocontinuity of \( \mu \) and the fact that \( f_n \to f \) in measure, that there exists \( n_0 \), such that

\[
\mu(F_\alpha) \geq \mu(F_{\alpha+1} - \{|f-f_n| \geq 1\}) = \infty
\]

for every \( \alpha \in [0, \infty) \) and \( n \geq n_0 \). Consequently, \( \int f_n \, d\mu = \infty \) as \( n \geq n_0 \).

Theorem 18. If \( \{f_n\} \) uniformly converges to a finite \( f \) on \( A \in \mathcal{F} \), then

\[
\int_A f_n \, d\mu \to \int_A f \, d\mu.
\]

Proof. For arbitrarily given \( \varepsilon > 0 \), since \( f_n \to f \) uniformly, there exists \( n_0 \).
such that $|f(x) - f_n(x)| \leq \varepsilon$ for all $x \in A$ as $n \geq n_0$. It follows, by using Theorem 9, that

$$\left| \int_A f \, d\mu - \int_A f_n \, d\mu \right| \leq \varepsilon$$

as $n \geq n_0$. □

REFERENCES