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TOPOLOGY BECOMES ALGEBRAIC WITH VIETORIS AND NOETHER

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An inspiring article by Dieudonné recently published in this journal [1] describes how Emmy Noether persuaded Heinz Hopf that some of his arguments in combinatorial topology could be more perpicuous if the traditional use of Betti numbers and torsion coefficients were replaced by homology *groups*. Since these groups were finitely generated abelian groups, they are direct products of cyclic groups; in this decomposition; the number of infinite cyclic groups gives the corresponding Betti numbers, and the torsion coefficients are similarly determined (Dieudonné disguises the abelian groups by calling them Z-modules; the historical term is abelian group [7]). Heinz Hopf, with Paul Alexandroff, had visited Princeton University in 1927–1928. Then (and now) fixed-point theorems were a central interest. The famous Brouwer fixed-point theorem for the n-cell had been extended by Lefschetz [5] to a fixed-point formula for mappings of a manifold into itself. Hopf found a further extension from n-manifolds to n-complexes, and published an announcement [2], submitted January 9, 1928, of his result in the *Proceedings* of the National Academy of Sciences, U.S.A. (At that time, and during the whole 50 year period when the mathematician Edwin B. Wilson was its managing editor, the *Proceedings* encouraged the publication of research announcements in Mathematics). Hopf's detailed proof was then published [3] in the Mathematische Zeitschrift (submitted 10 April 1928). It carries a "Zusatz bei der Korrektur" . . . "Thanks to a suggestion of Miss E. Noether I am able to substantially simplify the proof of the central theorem I (§ 3) of this paper . . ." As Dieudonné's current article notes, the simplification was subsequently published (again with credit to Emmy Noether) in the Göttinger Nachrichten [4], moreover the seemingly innocuous modification (the use of homology groups) was to have for reaching consequences.

Dieudonné goes on to mention that W. Mayer made good use of this in a 1929 paper [6]. "He was at that time in Vienna and did not mention Emmy Noether at all in this paper. However, by that time the spirit of 'modern algebra'; had spread to many German universities . . ." This statement accords with a favorite view that

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Mathematical ideas originated in Göttingen and then spread to lesser places. It was not always that simple.

Combinatorial topology had long been active in Vienna, as witness Tietze's paper [7]. In 1926, Leopold Vietoris, then a privat-dozent in Vienna, became interested in the problem of describing homology not just for manifolds or for complexes, but more generally for compact metric spaces. He did devise such a theory, known today as Vietoris homology. To do so, he couldn't possibly use just Betti numbers and torsion coefficients, because such a space might have "infinitely" many holes. He had to use homology groups — and he did, in a paper [9] which was communicated to the Netherlands Academy by none other than L.E.J. Brouwer at the meeting of May 29, 1926. There, at least 19 months before Hopf, he speaks of homology groups (with integral coefficients) and connectivity groups (with coefficients modulo 2) for a space. He went on to publish a full account [11] in the Mathematische Annalen (submitted June 28, 1926), where he also explained how Poincaré's Betti numbers and torsion coefficients can be obtained (by methods used by Tietze for the fundamental group) from these homology groups. Also he presented his results at a meeting of the German Mathematical Society [10] on Sept. 24, 1926; the chairman of that session was Paul Alexandroff, who was then in close touch with both E. Noether and Heinz Hopf. Since Vietoris and Mayer were both in Vienna 1925–27, it seems likely that Mayer heard about Vietoris homology before he wrote his 1929 paper defining the homology of abstract chain complexes. Vienna was at that time a very lively intellectual center (H. Hahn, K. Menger, S. Freud and the logical positivists, as well as many economists). Probably ideas passed back and forth in both directions between Vienna and Göttingen (and Berlin, Moscow and Paris).

I have searched without success for earlier uses of homology groups. Hopf, Alexandroff and others all used the addition of chains without ever saying "group", and I have not found groups mentioned in earlier papers by Alexander, Veblen, or Lefschetz. At that time a standard source for combinatorial topology was Veblen's colloquium lectures [8]. Veblen, from his education at Chicago, surely knew about groups. In the lectures he of course speaks of the (Poincaré) fundamental group but I could find *no* homology group there. At the very end of his *second* edition (1921) homology groups do appear, but not at the beginning where they would have aided the understanding. Instead, there is a maze of then-standard instructions for calculations with incidence matrices, with little explanation of the purpose of the calculation. From such a book, without a teacher, it was exceedingly difficult to understand combinatorial topology; in 1931 I tried with Veblen's book and failed.

The decision to use homology groups was then important in at least two ways: It made it possible to define homology for general spaces (as with Vietoris) and it simplified and clarified proofs (as with Noether and Hopf). Dieudonné praises Noether for her leadership in liberating linear algebra "from the plague of matrices and determinants from which it had been suffering for a century". But there is a current plague of texts on linear algebra which are all matrix computation with no mention of the meaning of matrices for linear transformations. The liberation is not yet complete. In the meantime Vietoris (now 93 and alive and well in Innsbruck) deserves real credit for his important initiative in homology.

When Vietoris saw the above manuscript he responded promptly (on March 16, 1985), saying in part (my translation from the German):

"Without doubt H. Poincaré and his comtemporaries knew that the Betti numbers and the torsion coefficients were invariants of groups, whose elements were (classes of) cycles under the operation of addition . . . Then one worked with the numerical invariants rather than with the invariant groups. That was a matter of 'taste'."

Note added in proof

There is also a 1926 paper by Emmy Noether which discusses the use of homology groups; Jahresbericht der Deutschen Mathematiker Vereinigung, Vol. 34 (1926), p. 104.

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