Periodic Solutions and Their Connecting Orbits of Hamiltonian Systems

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This article deals with second order periodic Hamiltonian systems. We apply variational methods to obtain non-constant periodic solutions. For the time reversible Hamiltonian systems, there exist connecting orbits joining pairs of periodic solutions. Our methods can also be used to treat heteroclinic orbits connecting an equilibrium to a periodic solution. © 2001 Elsevier Science

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0. INTRODUCTION

This paper deals with periodic Hamiltonian system

$$\ddot{q} - V'(t, q) = 0,$$  \hspace{1cm} (HS)

where \(q: \mathbb{R} \to \mathbb{R}^n\), \(V \in C^2(\mathbb{R} \times \mathbb{R}^n, \mathbb{R})\) and \(V'(t, y) = D_y V(t, y)\). It is assumed that the potential function \(V\) satisfies the following conditions:

(V1) There is a set \(\mathcal{K} \subset \mathbb{R}^n\) such that if \(\eta \in \mathcal{K}\) then \(V(t, \eta) = \inf_{\eta' \in \mathbb{R}^n} V(t, \eta') = V_0\) for all \(t \in \mathbb{R}\).

(V2) There are positive numbers \(\mu_1, \mu_2\) and \(\rho_0\) such that if \(|y - \eta| \leq \rho_0\) for some \(\eta \in \mathcal{K}\), then \(\mu_2 |y - \eta|^2 \geq V(t, y) - V_0 \geq \mu_1 |y - \eta|^2\) for all \(t \in \mathbb{R}\). Moreover, if \(\eta_i, \eta_j \in \mathcal{K}\) and \(i \neq j\), then \(|\eta_i - \eta_j| > 8\rho_0\).

(V3) There is a \(\mu_0 > 0\) such that if \(V(t, y) \leq V_0 + \mu_0\) for some \(t \in \mathbb{R}\) then \(|y - \eta| \leq \rho_0\) for some \(\eta \in \mathcal{K}\).

(V4) \(V\) is \(T\)-periodic in \(t\).

(V5) \(V\) is \(T_i\)-periodic in \(y_i\), \(1 \leq i \leq n\).

(V6) \(V(t, y) = V(-t, y)\) for all \(t \in \mathbb{R}\) and \(y \in \mathbb{R}^n\).

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The problem outlined in (HS) and (V1)–(V6) is motivated through physical examples. The idea of unforced pendulum is governed by the equation

$$\ddot{q} + \sin q = 0. \quad (0.1)$$

If the pendulum is forced via a support which is moving vertically, then the motion is governed by

$$\ddot{q} + (1 + H(t)) \sin q = 0, \quad (0.2)$$

where $H(t)$ is the vertical displacement of the support at time $t$. For $n > 1$, (HS) can be viewed as a simple model for the $n$-pendulum problem with appropriate forcing.

In the case of the unforced pendulum, a phase plane analysis shows that (0.1) has a heteroclinic orbit connecting the adjacent minima of the potential. Physically, these orbits represent solutions for which the pendulum remains nearly vertical for a long period of time, makes one rotation, and then remains nearly vertical for a long period of time.

By (V1), any element of $\mathcal{K}$ is an equilibrium of (HS). In a recent work [St], Strobel showed that, for any $\eta_i \in \mathcal{K}$, there is a heteroclinic orbit $q$ of (HS) which satisfies

$$q(t) \to \eta_i \quad \text{as} \quad t \to -\infty$$

and

$$q(t) \to \mathcal{K} \setminus \{\eta_i\} \quad \text{as} \quad t \to \infty.$$ 

Moreover, for any pair of $\eta_i, \eta_j \in \mathcal{K}$, they can be joined by a chain of heteroclinics. If additional nondegeneracy conditions are satisfied, there exist multibump heteroclinic orbits originating at $\eta_i$ and terminating at $\eta_j$.

The goal of this paper is to investigate non-constant periodic solutions and their connecting orbits of (HS). The potential $V$ is only determined up to an additive constant, so we may assume that $V_0 = 0$. Let $E_1 = \{z \in W^{1,2}(\mathbb{R}, \mathbb{R}^n) \mid z(t+T) = z(t)\}$ and

$$\hat{I}_1(z) = \int_0^T \mathcal{L}(z) \, dt,$$

where $\mathcal{L}(q) = \frac{1}{2} |\dot{q}|^2 + V(t, q)$, the Lagrangian associated with (HS). It is known that the critical points of $\hat{I}_1$ in $E_1$ are the periodic solutions of (HS).

The case of $V(t, y) = F_e(t) W(y)$ is of particular interest in connection with the study of (0.2). We consider the case where the function $F_e$ oscillates slowly between its maximum and minimum. As a simple example, let
\( F_t(t) = F(\varepsilon t) \), where \( F \) is a positive non-constant periodic function and \( \varepsilon \) is a small positive number. We have the following existence result for the periodic solutions of (HS).

**Theorem 1.** For any given \( F \) and \( W \), if \( V(t, y) = F(\varepsilon t) W(y) \) and \( \varepsilon \) is sufficiently small, then (HS) possesses at least two non-constant periodic solutions.

A periodic solution obtained in Theorem 1 is a local minimizer of \( \hat{I}_1 \). Our strategy is to add penalization to \( \hat{I}_1 \) so that a local minimizer of \( \hat{I}_1 \) becomes a global minimizer to a penalized functional. Without further comment, a non-constant periodic solution will be simply called a periodic solution in what follows.

The proof of Theorem 1 will be carried out in a more general setting, in which some notation is now introduced. For \( g_1, g_2 \in K \), let

\[
L = \sup_{t \in \mathbb{R}} \left( \| V(t, y) \| + \frac{1}{2} \right) |t| + \frac{1}{2} \sum_{i=1}^{2} B_{r_0}(g_i)
\]

and \( h(r) = \min(\mu_1 r^2, \mu_0) \). For \( j_1 < j_2 \), set

\[
\hat{E}(j_1, j_2) = \{ z \in W^{1, 2}([j_1, j_2], \mathbb{R}^n) | z(j_1) = \eta_1 \text{ and } z(j_2) = \eta_2 \},
\]

\[
\check{E}(j_1, j_2) = \{ z \in W^{1, 2}([j_1, j_2], \mathbb{R}^n) | z(j_1) = \eta_2 \text{ and } z(j_2) = \eta_1 \},
\]

\[
\hat{\delta}(j_1, j_2) = \inf_{z \in \hat{E}(j_1, j_2)} I(z) \quad \text{and} \quad \check{\delta}(j_1, j_2) = \inf_{z \in \check{E}(j_1, j_2)} I(z),
\]

where

\[
I(z) = \int_{j_1}^{j_2} L(z) \, dt.
\]

**Theorem 2.** Assume that (V1)-(V5) are satisfied. Suppose there are \( k_0 < k_1 < k_2 < k_3 < k_4 = k_0 + T \) such that

\[
\hat{\delta}(k_1, k_2) < \min(\hat{\delta}(k_0, k_1), \check{\delta}(k_2, k_3)) \quad (0.3)
\]

\[
\check{\delta}(k_3, k_4) < \min(\check{\delta}(k_2, k_3), \hat{\delta}(k_0, k_1)) \quad (0.4)
\]

and

\[
\min(k_3 - k_2, k_1 - k_0) > 6 \rho_0 + 2(\hat{\delta}(k_1, k_2) + \check{\delta}(k_3, k_4)) + \rho_0 \sqrt{2\theta(\rho_0)}/\theta(r), \quad (0.5)
\]
where
\[
r = \min \left( 1, \frac{\rho_0}{2}, \sqrt{\frac{\rho_0^2}{8\mu_2}}, \frac{\rho_0}{2}, \frac{\sqrt{2\theta(\rho_0)}}{A}, \frac{\bar{\theta}}{4A} \right)
\]  
(0.6)

and \( \bar{\theta} = \min(\bar{\sigma}(k_0, k_1) - \bar{\sigma}(k_1, k_2), \bar{\sigma}(k_2, k_3) - \bar{\sigma}(k_1, k_2), \bar{\sigma}(k_2, k_1) - \bar{\sigma}(k_3, k_4), \bar{\sigma}(k_0, k_1) - \bar{\sigma}(k_2, k_4)) \). Then (HS) possesses a non-constant periodic solution.

The hypothesis of Theorem 2 may look complicated, but it is not hard to verify. Note that there is a monotonicity property for \( \bar{\sigma} \) and \( \bar{\sigma} \), depending on the choice of the corresponding boundary points \( j_1 \) and \( j_2 \). An example of such a verification will be given in the proof of Theorem 1.

After this paper was submitted to the journal, we learned from the referee an interesting work [BM] on the multibump orbits for the Lagrangian systems. For the slowly perturbed pendulum equation, if we identify \( \eta = (2n+1)\pi, n \in \mathbb{Z} \), as one point, then a corresponding result to Theorem 1.3 of [BM] is that there exists a trajectory \( q(t) \) homoclinic to \( \eta \) and “near” the bump of \( q(t) \) there is a periodic solution. If \( F_e \) has only one maximum and one minimum per period, a periodic solution obtained in Theorem 1 is a subharmonic solution with twice the minimal period. We could call it a two bump periodic solution (see e.g. [CR3]) and (0.5)–(0.6) give certain lower bound estimates for the distance between two bumps.

In [R1], Rabinowitz studied a class of Hamiltonian systems where a family \( \mathcal{K}_p \) of periodic solutions can be obtained as the global minimizers of a variational problem. Assuming \( \mathcal{K}_p \) consists of isolated points and (HS) is time reversible, he showed that, for any periodic solution \( p_1 \in \mathcal{K}_p \), there is a heteroclinic orbit connecting \( p_1 \) to an element of \( \mathcal{K}_p \setminus \{p_1\} \).

We intend to investigate whether the periodic solutions obtained in Theorem 2 can be joined by a connecting orbit. To the best of our knowledge, little seems to be known by using variational methods to find heteroclinic solutions joining pairs of local minimizers, particularly in the setting of connecting non-constant periodic solutions. With the aid of penalization, we show that such connecting orbits exist. Indeed, we are able to single out infinitely many connecting orbits by using different penalty functions. Also, our methods could be used to treat heteroclinic orbits joining a pair of periodic solutions at different critical levels of \( \tilde{I}_1 \).

In Section 4, we study multibump connecting orbits for the periodic solutions. The existence of multibump solutions of differential equations has been the object of continued investigation over the past decade [BS, CR1-3, dF2, KV, KKV, M1, M2, R2, S, Sp, St]. The first work on the variational approach to multibump solutions is due to Séré [S]. He found
multibump homoclinic solutions for first order Hamiltonian systems for which the existence of single bump solutions had been obtained in an earlier work [CES]. Subsequently, there have been further applications to second order and fourth order Hamiltonian systems to obtain homoclinic as well as heteroclinic solutions. Roughly speaking, a multibump solution comprise a number of one bump solutions nicely concatenated. A key requirement for the construction of multibump solutions is that the one bump solutions satisfy certain nondegeneracy conditions. This hypothesis plays the role in variational settings of the classical transversality conditions used in the study of analogous questions for dynamical systems. Namely the standard condition there is that the stable and unstable manifolds through an equilibrium point for the Poincaré map associated with a dynamical system intersect transversally at a homoclinic point. For a given potential $V$, it is no easy matter to verify if such a nondegeneracy condition or the classical transversality hypothesis holds. Instead of dealing with nondegeneracy hypotheses like above, we could follow penalization arguments to obtain multibump connecting orbits joining periodic solutions.

An additional periodic solution of (HS) can be obtained by the Mountain Pass Lemma. The detailed analysis is given in Section 2.

We are also interested in finding heteroclinic orbits joining an equilibrium to a non-constant periodic solution. As will be seen in Section 5, penalization method provides a way to obtain this kind of heteroclinic orbits of Hamiltonian systems.

All the results mentioned above are applicable to (0.2) if $1 + \dot{H}(t)$ is a positive slowly oscillating periodic function. It will be detailed in Section 6. The verification of hypotheses (0.3)–(0.6) can be done more efficiently by numerical computation. The penalization methods could also help compute solutions numerically.

1. EXISTENCE RESULTS FOR PERIODIC SOLUTIONS

In this section, we prove an existence result for the periodic solutions of (HS). We start with two technical lemmas.

**Lemma 1.** Suppose $z(t_1) \in \partial B_\epsilon(\eta_i)$, $z(t_2) \in \partial B_\epsilon(\eta_j)$ and $z(t) \in \mathbb{R}^n \setminus (\bigcup_{i \neq j} B_\epsilon(\eta_i)$ $B_\epsilon(\eta_j))$ for $t \in (t_1, t_2)$. If $i \neq j$ and $\rho \in (0, \rho_0]$, then

$$\int_{t_1}^{t_2} \mathcal{L}(z) \, dt \geq \frac{1}{2(t_2 - t_1)} (|\eta_i - \eta_j| - 2\rho)^2 + \theta(\rho)(t_2 - t_1).$$

(1.1)
\textbf{Proof.} Since
\[
|\eta_i - \eta_j| - 2p \leq |z(t_2) - z(t_1)| = \left| \int_{t_1}^{t_2} \dot{z}(t) \, dt \right| \leq \sqrt{t_2 - t_1 \left( \int_{t_1}^{t_2} |\dot{z}(t)|^2 \, dt \right)^{1/2}},
\]
this together with (V2) and (V3) yields (1.1).

\textbf{Lemma 2.} Suppose \( z(t) = \frac{t - t^*}{\sigma(t - T)} z(t_2) + \frac{t - t^*}{\delta(t - T)} z(t_1) \) for \( t \in (t_1, t_2) \). If \( z(t_1) \in \mathcal{X}_c \) and \( |z(t_2) - z(t_1)| = t_2 - t_1 \leq \rho_0 \), then
\[
\int_{t_1}^{t_2} \mathcal{L}(z) \, dt \leq A |z(t_2) - z(t_1)|.
\]

\textbf{Proof.} It directly follows from the mean value theorem.

\textbf{Proof of Theorem 2.} As noted earlier, we seek periodic solutions of (HS) as the critical points of \( \tilde{I}_1 \). To find local minimizers of \( \tilde{I}_1 \), we use a penalization method described as follows. Let \( \psi_i \in C^\infty(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}) \) be such that \( 0 \leq \psi_i \leq M_1, \psi_i(t + T, y) = \psi_i(t, y) \) and

\[
\psi_i(t, y) = \begin{cases} 
0 & \text{if } t \in [\tilde{t}_0, \tilde{t}_2 - \rho_0] \cup [\tilde{t}_3 + \rho_0, \tilde{t}_4 - \rho_0] \\
M_1 & \text{if } y \notin B_{3\rho_0}(\eta_2) \text{ and } t \in [\tilde{t}_2, \tilde{t}_3] \\
M_1 & \text{if } y \in B_{3\rho_0}(\eta_2) \text{ and } t \in (\tilde{t}_2 - \rho_0, \tilde{t}_3 + \rho_0) \\
0 & \text{if } y \notin B_{2\rho_0}(\eta_1) \text{ and } t \in (\tilde{t}_0, \tilde{t}_2) \\
0 & \text{if } y \in B_{2\rho_0}(\eta_1) \text{ and } t \in (\tilde{t}_2 - \rho_0, \tilde{t}_3 + \rho_0),
\end{cases}
\]

where \( \tilde{\rho} = \frac{\rho_0}{2} \), \( \tilde{\rho}_0 = k_0 + 3\rho_0 + (\hat{\alpha}(k_3, k_4) + \rho_0 \sqrt{2\theta(\rho_0)}) / \theta(r), \hat{t}_1 = k_1 - 3\rho_0 - (\hat{\alpha}(k_3, k_2) + \rho_0 \sqrt{2\theta(\rho_0)}) / \theta(r), \hat{t}_2 = k_2 + 3\rho_0 + (\hat{\alpha}(k_3, k_2) + \rho_0 \sqrt{2\theta(\rho_0)}) / \theta(r), \hat{t}_3 = k_3 - 3\rho_0 - (\hat{\alpha}(k_3, k_2) + \rho_0 \sqrt{2\theta(\rho_0)}) / \theta(r), \hat{t}_4 = k_4 + \rho_0 / \theta(r), \hat{t}_5 = \hat{t}_0 + T, \hat{t}^* = \min(\hat{t}_4 - \hat{t}_2, \hat{t}_1 - \hat{t}_0) \) and \( M_1 = \theta(r) + (\hat{\alpha}(k_3, k_2) + \hat{\alpha}(k_3, k_4)) / \hat{t}^* \). Set
\[
I_1(z) = \int_0^T \mathcal{L}(z) \quad \text{and} \quad \alpha_i = \inf_{z \in \mathcal{E}_0} I_1(z), \text{ where } \mathcal{L}(z) = \mathcal{L}(z) + \psi_i(t, z). \text{ There is a } p_i \in E_i \text{ such that}
\]
\[
I_1(p_i) = \alpha_i < \hat{\alpha}(k_1, k_2) + \hat{\alpha}(k_3, k_4).
\]

By the construction of \( \psi_i \), we see that \( \alpha_i > 0 \) and hence \( p_i \notin \mathcal{X}_c \). We are going to show that \( p_i \) is a solution of (HS). Note that
\[
\text{there exist } t_1 \in (\hat{t}_0, \hat{t}_1) \text{ and } t_2 \in (\hat{t}_2, \hat{t}_3) \text{ such that } p_i(t_1) \in B_i(\eta_1) \text{ and } p_i(t_2) \in B_i(\eta_2);
\]
for otherwise,
\[ I_1(p_1) \geq \theta(r) \left( \frac{\tilde{a}(k_1, k_2) + \tilde{a}(k_3, k_4)}{\theta(r)} \right) > \tilde{c} \]
which violates (1.3). Let \( \tau_1 = \tau_1(p_1) = \inf \{ t \mid t \in (t_1, t_2) \text{ and } p_1(t) \in \overline{B_1(\eta_1)} \} \), \( \tau_2 = \tau_2(p_1) = \sup \{ t \mid t \in [t_1, \tau_1) \text{ and } p_1(t) \in \overline{B_1(\eta_1)} \} \), \( \tau_3 = \tau_3(p_1) = \inf \{ t \mid t \in [t_3, t_4 + T) \text{ and } p_1(t) \in \overline{B_1(\eta_1)} \} \), and \( \tau_4 = \tau_4(p_1) = \sup \{ t \mid t \in [t_3, t_4 + T) \text{ and } p_1(t) \in \overline{B_1(\eta_1)} \} \). We claim
\[ \tau_1 < \hat{t}_2 - 2\rho_0. \] (1.4)

Suppose (1.4) is false. It is clear from (1.3) that
\[ \int_{t_1}^{\tau_1} \mathcal{L}_1(p_1) \, dt = \inf_{z \in A_0} \int_{t_1}^{\tau_1} \mathcal{L}_1(z) \, dt, \] (1.5)
where \( A_0 = \{ z \in W^{1,2}_{\text{loc}}(\mathbb{R}, \mathbb{R}^n) \mid z(t_1) = p_1(t_1) \text{ and } z(\tau_3) = p_1(\tau_3) \} \). Define
\[
Z_0(t) = \begin{cases} 
\frac{t_1 + r - t}{r} p_1(t_1) + \frac{t - t_1}{r} \eta_1 & \text{if } t \in [t_1, t_1 + r) \\
\eta_1 & \text{if } t \in [t_1 + r, t_1 + r + \eta_1) \\
\hat{z}(t) & \text{if } t \in [t_1 + r + \eta_1, \tau_2) \\
\eta_2 & \text{if } t \in [\tau_2, \tau_3 - r) \\
\frac{\tau_3 - t}{r} \eta_2 + \frac{t - \tau_3 + r}{r} p_1(\tau_3) & \text{if } t \in (\tau_3 - r, \tau_3], 
\end{cases}
\]
where \( \hat{z} \in \hat{E}(k_1, k_2) \) and \( I(\hat{z}) = \hat{a}(k_1, k_2) \). Invoking Lemma 2 yields \( \int_{t_1}^{t_1 + r} \mathcal{L}(Z_0) \, dt \leq Ar \) and \( \int_{t_1 - r}^{t_1} \mathcal{L}(Z_0) \, dt \leq Ar \). It follows that
\[ \int_{t_1}^{\tau_3} \mathcal{L}_1(Z_0) \, dt \leq \hat{a}(k_1, k_2) + 2Ar. \]
This together with (1.5) shows that
\[ \int_{t_1}^{\tau_3} \mathcal{L}_1(p_1) \, dt < \hat{a}(k_1, k_2) + 2Ar. \] (1.6)
Combining (1.6) with (0.6) gives
\[ \int_{t_1}^{\tau_3} \mathcal{L}_1(p_1) \, dt < \hat{a}(k_1, k_2) + \rho_0 \sqrt{2\theta(\rho_0)}. \]
It follows from Lemma 1 that
\[ \tau_2 \geq \tau_3 - \frac{1}{\theta(r)}(\hat{\alpha}(k_1, k_2) + \rho_0 \sqrt{2\theta(\rho_0)}) \geq k_2 + \rho_0. \]  
(1.7)

Setting
\[
Z_1(t) = \begin{cases}
\eta_2 & \text{if } t \geq \tau_3 + r \\
\frac{\tau_3 + r - t}{r} p_1(\tau_3) + \frac{t - \tau_3}{r} \eta_2 & \text{if } t \in (\tau_3, \tau_3 + r) \\
p_1(t) & \text{if } t \in [\tau_2, \tau_3] \\
\frac{t - \tau_2 + r}{r} p_1(\tau_2) + \frac{\tau_2 - t}{r} \eta_1 & \text{if } t \in (\tau_2 - r, \tau_2) \\
\eta_1 & \text{if } t \leq \tau_2 - r,
\end{cases}
\]

we see that \( Z_1 \in \hat{E}(k_2, k_3) \) and
\[
\int_{\tau_3}^{\tau_2} \mathcal{L}_1(p_1) \, dt \geq \int_{\tau_3}^{\tau_2} \mathcal{L}_1(Z_1) \, dt - 2Ar > \hat{\alpha}(k_2, k_3) - 2Ar \geq \hat{\alpha}(k_1, k_2) + 2Ar.
\]

This is incompatible with (1.6), so (1.4) must be true. The same argument also shows that \( \tau_4 > \tilde{\tau}_3 + 2\rho_0, \tau_2 > \tilde{\tau}_1 + 2\rho_0 \) and \( \tau_4 < \tilde{\tau}_4 - 2\rho_0 \).

It remains to show that
\[ p_1(t) \in B_{2\rho_0}(\eta_3) \quad \text{if } t \in [\tilde{\tau}_3, \tilde{\tau}_4] \]  
(1.8)
and
\[ p_1(t) \in B_{2\rho_0}(\eta_1) \quad \text{if } t \in [\tilde{\tau}_3, \tilde{\tau}_2 + T]. \]
(1.9)

We only prove (1.8), since the other is analogue. The proof of (1.8) follows from a variational study of the flow near the equilibrium \( \eta_2 \) as stated in the following lemma.

**Lemma 3.** Let \( \hat{A} = \{ z \in W^{1,2}([\tilde{\tau}_3, \tilde{\tau}_4], \mathbb{R}^n) | z(\tilde{\tau}_3), z(\tilde{\tau}_4) \in \partial B_{\rho_0}(\eta_2) \} \). If \( p_1 \in \hat{A} \) and \([\int_{\tilde{\tau}_3}^{\tilde{\tau}_4} \mathcal{L}_1(p) \, dt = \min_{z \in \hat{A}} \int_{\tilde{\tau}_3}^{\tilde{\tau}_4} \mathcal{L}_1(z) \, dt \), then \( p(t) \in B_{2\rho_0}(\eta_3) \) for all \( t \in [\tilde{\tau}_3, \tilde{\tau}_4] \).

**Proof.** Suppose there is a \( \tau \in (\tilde{\tau}_3, \tilde{\tau}_4) \) such that \( p_1(\tau) \notin B_{2\rho_0}(\eta_3) \). Then we can find points \( \tau_5 < \tau_6 < \tilde{\tau}_3 < \tau_7 < \tau_8 \) in the interval \([\tilde{\tau}_3, \tilde{\tau}_4] \) such that
\( p_1(t_i) \in \partial B_{\rho_0}(\eta_2) \) if \( i = 3, 8 \), \( p_1(t_i) \in \partial B_{\rho_0}(\eta_2) \) if \( i = 4, 7 \), \( p_1(t_i) \in \partial B_{2\rho_0}(\eta_2) \) if \( i = 5, 6 \) and \( \rho_0 < |p_1(t_i) - \eta_2| < 2\rho_0 \) if \( t \in (t_4, t_5) \cup (t_6, t_7) \). If \( t_6 - t_5 < 2\rho \), letting

\[
Z_2(t) = \begin{cases} 
  p_1(t) & \text{if } t \neq (t_3, t_8) \\
  \frac{t-t_5}{t_8-t_5} p_1(t_5) + \frac{t-t_5}{t_8-t_5} p_1(t_8) & \text{if } t \in (t_3, t_8)
\end{cases}
\]

yields

\[
\int_{t_3}^{t_8} \mathcal{L}(Z_2) - \mathcal{L}(p_1) \, dt = \int_{t_3}^{t_8} \left[ \frac{1}{2(t_8-t_5)} |p_1(t_5) - p_1(t_8)|^2 + V(t, Z_2) \right] dt \\
- \int_{t_3}^{t_5} \mathcal{L}(p_1) \, dt.
\]

Now

\[
\rho_0 \leq |p_1(t_5) - p_1(t_4)| = \left| \int_{t_4}^{t_5} \dot{p}_1(t) \, dt \right| \leq \sqrt{t_5 - t_4} \left( \int_{t_4}^{t_5} |\dot{p}_1(t)|^2 \, dt \right)^{1/2}
\]

which implies that

\[
\int_{t_3}^{t_8} \mathcal{L}(p_1) \, dt \geq \frac{\rho_0^2}{2(t_8 - t_5)} > \frac{\rho_0^2}{2(t_8 - t_5)}.
\]

Likewise,

\[
\int_{t_3}^{t_6} \mathcal{L}(p_1) \, dt > \frac{\rho_0^2}{2(t_8 - t_5)}.
\]

Moreover, it follows from (V2) that

\[
\int_{t_3}^{t_8} \left[ \frac{1}{2(t_8-t_5)} |p_1(t_5) - p_1(t_8)|^2 + V(t, Z_2) \right] dt < \frac{2r^2}{t_8-t_5} + 2\mu_2 r^3.
\]

Invoking (0.6) yields

\[
\int_{t_3}^{t_8} \mathcal{L}(Z_2) - \mathcal{L}(p_1) \, dt < \frac{2r^2}{t_8-t_5} + 2\mu_2 r^3 - \frac{\rho_0^2}{t_8-t_5} \leq 0,
\]

which is absurd since \( \int_{t_3}^{t_8} \mathcal{L}(p) \, dt = \min_{z \in \mathcal{A}} \int_{t_3}^{t_8} \mathcal{L}(z) \, dt \).
We next consider the case that \( t_8 - t_3 \geq 2r \). Let

\[
Z_3(t) = \begin{cases} 
    p_1(t) & \text{if } t \notin (t_3, t_8) \\
    \eta_2 & \text{if } t \in [t_3 + r, t_8 - r] \\
    \frac{t_3 + r - t}{r} p_1(t_3) + \frac{t - t_3}{r} \eta_2 & \text{if } t \in (t_3, t_3 + r) \\
    \frac{t - t_8 + r}{r} p_1(t_8) + \frac{t_8 - t}{r} \eta_2 & \text{if } t \in (t_8 - r, t_8).
\end{cases}
\]

Applying Lemma 2 gives \( \int_{t_3}^{t_8} \mathcal{L}_1(Z_3) \, dt \leq 2Ar \). On the other hand, by (V2)

\[
\int_{t_4}^{t_5} \mathcal{L}_1(p_1) \, dt \geq \frac{\rho_0^2}{2(t_5 - t_4)} + \theta(\rho_0)(t_5 - t_4) \geq \rho_0 \sqrt{2\theta(\rho_0)}.
\]

Likewise,

\[
\int_{t_6}^{t_7} \mathcal{L}_1(p_1) \, dt \geq \rho_0 \sqrt{2\theta(\rho_0)}.
\]

Hence using (0.6) yields

\[
\int_{t_3}^{t_8} \mathcal{L}_1(Z_3) - \mathcal{L}_1(p_1) \, dt = \int_{t_3}^{t_8} [\mathcal{L}_1(Z_3) - \mathcal{L}_1(p_1)] \, dt \leq 2Ar - 2\rho_0 \sqrt{2\theta(\rho_0)} < 0,
\]

which leads to the same contradiction as above. The proof is complete.

**Remark 1.** (a) We refer to [BS, BM, CR1-3, dF2, KV, KKV] for some variational results analogous to Lemma 3. They have been used to study multibump solutions of various equations.

(b) The existence of periodic solutions with minimal (i.e. primitive) period \( mT, m \in \mathbb{N} \setminus \{1\} \), will be investigated in a forthcoming paper.

## 2. Multiplicity Results for Periodic Solutions

Our aim in this section is to use Mountain Pass Lemma to obtain additional periodic solutions of (HS). Let \( p_1 \) be a periodic solution obtained in Theorem 2. For \( z \in E_1 \), define

\[
G_1(z) = \{(t, z(t)) \mid t \in [0, T]\}.
\]
Let $A_1 = \{(t, y) | \psi_1(t, y) = 0\}$,
\[ \Gamma = \{v \in C([0, 1], E_1) | v(0) = p_1 \text{ and } v(1) = \eta_1 \} \] (2.1)
and
\[ \beta = \inf_{v \in \Gamma} \max_{a \in [0, 1]} \hat{I}_1(v(a)). \] (2.2)

**Theorem 3.** If the hypotheses of Theorem 2 are satisfied, then $\beta > \alpha_1$ and (HS) possesses at least two periodic solutions.

**Proof.** We have already obtained $p_1$ in Theorem 2, the second periodic solution immediately follows from the Mountain Pass Lemma if $\beta > \alpha_1$. We argue indirectly. Suppose $\beta = \alpha_1$. Then there is a sequence $\{v_m\} \subset \Gamma$ such that
\[ \max_{a \in [0, 1]} \hat{I}_1(v_m(a)) \to \alpha \quad \text{as} \quad m \to \infty. \]

For fixed $m$, we set
\[ a_m = \sup \{\bar{a} | \bar{a} \in (0, 1) \text{ and } G_1(v_m(a)) \subset \bar{A}_1 \text{ if } a < \bar{a}\} \] (2.3)
and $u_m = v_m(a_m)$. It is clear that $G_1(u_m) \cap \partial A_1 \neq \emptyset$. Since
\[ \lim_{m \to \infty} I_1(u_m) = \lim_{m \to \infty} \hat{I}_1(u_m) = \alpha_1, \] (2.4)
there is a $p \in E_1$ such that along a subsequence $u_m \to p$ in $E_1$. Consequently there is an $s \in [0, T]$ such that $p(s) \in \partial A_1$. On the other hand, repeating the proof of Theorem 1 yields $G_1(p) \subset \bar{A}_1$. We thus get a contradiction which completes the proof.

**Remark 2.** (a) It is clear that $p \notin X$, since $\hat{I}_1(p) = \beta > \alpha_1 > 0$.
(b) In case $V(t, y) = F_1(t) W(y)$ and $\varepsilon$ is sufficiently small, it will be seen that $p$ is a non-constant periodic solution. A detailed proof will be given in Section 6.

Theorem 2 and Theorem 3 still hold if (V1)-(V4) and the following condition are satisfied:

(V7) For any $r_0 > 0$ there is a $M > 0$ such that
\[ \sup_{t \in \mathbb{R}} \|D_2^2V(t, y)\| \leq M \quad \text{if} \quad |y| \leq r_0. \]
3. CONNECTING ORBITS JOINING PERIODIC SOLUTIONS

In this section, a connecting orbit for the periodic solutions of (HS) will be established. Throughout Sections 3–5, it is assumed that (HS) is time reversible; i.e., condition (V6) is satisfied. Let $E_1^0 = W^{1,2}([0, T], \mathbb{R}^n)$ and

$$\mathcal{X}_1^0 = \{ p \in E_1^0 \mid I_1(p) = \inf_{z \in E_1} I_1(z) \}. $$

The additional hypothesis (V6) ensures that the periodic solutions obtained in Theorem 2 are local minimizers among a larger family of functions than the periodic ones.

**Proposition 1.** Assume that the hypotheses of Theorem 2 are satisfied. Suppose $k_4 - k_3 = k_2 - k_1$ and $V((k_2 + k_3)/2 + t, y) = V((k_2 + k_3)/2 - t, y)$ for all $t \in \mathbb{R}$ and $y \in \mathbb{R}^n$. Then $p(t) = p(-t)$ for any $p \in \mathcal{X}_1^0$ and

$$\inf_{z \in K_1} I_1(z) = \inf_{z \in E_1} I_1(z) \quad (3.1)$$

**Remark 3.** (a) In view of Theorem 2, the choices of $k_2$ and $k_3$ are not unique.

(b) We may assume without loss of generality that $(k_2 + k_3)/2 = 0$.

**Proof.** From the hypotheses, it is not difficult to check that

$$\bar{a}(k_1, k_2) = \bar{a}(k_3, k_4).$$

Also, in the proof of Theorem 2, the penalty function $\psi_1$ can be chosen to satisfy $\psi_1(t, y) = \psi_1(-t, y)$ for all $t \in \mathbb{R}$ and $y \in \mathbb{R}^n$.

For $w \in W^{1,2}([0, \frac{T}{2}], \mathbb{R}^n)$, define

$$\hat{w}(t) = \begin{cases} w(t), & t \in \left[0, \frac{T}{2}\right] \\
w(T-t), & t \in \left[\frac{T}{2}, T\right] \end{cases}$$

and extend $\hat{w}$ to $\mathbb{R}$ as a $T$-periodic function. Then $\hat{w} \in E_1$ and

$$\int_0^T \mathcal{L}_1(\hat{w}) \, dt = \int_{\frac{T}{2}}^T \mathcal{L}_1(\hat{w}) \, dt.$$ 

For $p \in \mathcal{X}_1^0$, let $e_1 = \int_0^T \mathcal{L}_1(p) \, dt$, $e_2 = \int_{\frac{T}{2}}^T \mathcal{L}_1(p) \, dt$ and choose the portion of $p$ which gives the smaller $e_i$ (or either if they are equal). Reflecting the
portion of \( p \) chose above about \( t = \frac{T}{2} \) as an even function yields a \( \hat{p} \in E_1 \) and

\[
I_1(\hat{p}) \leq I_1(p).
\]

Thus \( \hat{p} \in \mathcal{E}' \). Since \( \hat{p} \) coincides with \( p \) on a subinterval of \([0, T]\), \( \hat{p} = p \) and \( p \) must be an even periodic function.

The proof of (3.1) is similar.

In the remaining of the paper, it is assumed that

(P) \( \mathcal{E}' \) consists of isolated points.

For \( z \in E_1' \) there is a constant \( c \) such that

\[
\|z\|_{L^\infty} \leq \frac{c}{2} \|z\|.
\]  

(3.2)

Let \( \mathcal{B}_r(z) \) denote an open ball about \( z \) of radius \( r \), and \( \mathcal{B}_r(\mathcal{E}') = \bigcup_{z \in \mathcal{E}'} \mathcal{B}_r(z) \). Let

\[
\overline{\gamma} = \inf \{ \|p - p'\|_{L^\infty} \mid p, p' \in \mathcal{E}' \text{ and } p \neq p' \}.
\]  

(3.3)

**Proposition 2.** Assumed that (P) and the hypotheses of Proposition 1 are satisfied. Then \( \mathcal{E}' \) contains only a finite number of minima of \( I_1 \) and \( \overline{\gamma} > 0 \). Moreover, there is a continuous function \( d_1 : (0, \overline{\gamma}/8c) \to (0, \infty) \) such that \( \lim_{r \to 0^+} d_1(p) = 0 \) and

\[
I_1(z) \geq I_1(p_1) + d_1(p)
\]  

(3.4)

for \( z \in E_1' \setminus \mathcal{B}_r(\mathcal{E}') \), where \( p_1 \) is an element of \( \mathcal{E}' \).

**Proof.** If the assertion of the proposition is false, there is a sequence \( \{z_m\} \subset E_1' \setminus \mathcal{B}_r(\mathcal{E}') \) such that

\[
\lim_{m \to \infty} I_1(z_m) = I_1(p_1).
\]  

(3.5)

Hence along a subsequence, \( z_m \) converges to \( p \in E_1' \) as \( m \to \infty \). Set \( v_m = z_m - p \). It is clear that \( \|v_m\| \geq \rho \). Since

\[
I_1(z_m) = \int_0^T \left[ \frac{1}{2} |\dot{p}|^2 + \dot{p} \cdot \dot{v}_m + \frac{1}{2} (|\dot{v}_m|^2 + |v_m|^2) - \frac{1}{2} |v_m|^2 + V(t, z_m) + \psi_1(t, z_m) \right] dt,
\]
it follows that
\[ I_1(z_m) \geq \frac{1}{2} \rho^2 + I_1(p) + \int_0^T \left[ \dot{p} \cdot \dot{v}_{z_m} - \frac{1}{2} |v_m|^2 + V(t, z_m) - V(t, p) + \psi_1(t, z_m) - \psi_1(t, p) \right] dt. \]

This implies \( \lim_{m \to \infty} I_1(z_m) \geq \frac{1}{2} \rho^2 + I_1(p) \), contrary to (3.5). The proof is complete.

Next, a variational problem for connecting orbits of (HS) will be formulated. Let \( p_1', p_1 \in \mathcal{K} \) and

\[ \Gamma(p_1', p_1) = \{ z \in W_{loc}^{1,2}(\mathbb{R}, \mathbb{R}^n) \mid z(t) \to p_1'(t) \text{ uniformly as } t \to -\infty \text{ and } z(t) \to p_1(t) \text{ uniformly as } t \to \infty \}. \quad (3.6) \]

To find connecting orbits of (HS), we add to \( V \) a penalty function described as follows. Let \( \bar{p}, \bar{t}_0, \bar{t}_1, \bar{t}_2, \bar{t}_3, \bar{t}_4, M_1 \) and \( \psi_1 \) be defined as in the proof of Theorem 2. Furthermore, in view of (V6), \( \psi_1 \) can be chosen as an even function of \( t \); i.e., \( \psi_1(t, y) = \psi_1(-t, y) \). For fixed \( N \in \mathbb{N} \), let \( \Psi \in C^0(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}) \) be such that \( 0 \leq \Psi \leq M_1 \) and

\[ \Psi(t, y) = \begin{cases} 
\psi_1(t, y) & \text{if } t \in (-\infty, \bar{t}_1] \cup [\bar{t}_0 + NT, \infty) \\
M_1 & \text{if } y \notin B_{\psi_1}(\eta_1) \text{ and } t \in (\bar{t}_1, \bar{t}_0 + NT) \\
0 & \text{if } y \in B_{\psi_1}(\eta_1) \text{ and } t \in (\bar{t}_1, \bar{t}_0 + NT).
\end{cases} \]

Let \( A = \{(t, y) \mid \Psi(t, y) = 0\} \). From the proof of Theorem 2, there exist \( t_1, t_1' \in (\bar{t}_0, \bar{t}_1) \) such that \( p_1(t_1), p_1'(t_1') \in B(\eta_1) \). For \( z \in W_{loc}^{1,2}(\mathbb{R}, \mathbb{R}^n) \), define \( G(z) = \{(t, z(t)) \mid t \in \mathbb{R}\} \),

\[ a_{\ell}(z) = \int_{t_1 + (\ell + 1)T}^{t_1 + \ell T} \left[ \mathcal{L}(z) + \Psi(t, z) - \mathcal{L}(p_1) \right] dt \quad \text{if } \ell \in \mathbb{N}, \]

\[ a_\ell(z) = \int_{t_1 + (\ell + 1)T}^{t_1 + \ell T} \left[ \mathcal{L}(z) + \Psi(t, z) - \mathcal{L}(p_1) \right] dt \quad \text{if } \ell \geq N \]

and

\[ a_0(z) = \int_{t_1}^{t_1 + NT} \left[ \mathcal{L}(z) + \Psi(t, z) \right] dt. \]

It is clear that \( a_\ell(z) \geq 0 \) for all \( \ell \). Set

\[ J(z) = \sum_{\ell = -\infty}^{0} a_\ell(z) + \sum_{\ell = N}^{\infty} a_\ell(z) \quad (3.7) \]
and
\[
Y(t) = \begin{cases} 
    p'_1(t) & \text{if } t \in (-\infty, t'_1] \\
    \frac{t'_1 + r - t}{r} p'_1(t'_1) + \frac{t - t'_1}{r} \eta_1 & \text{if } t \in (t'_1, t'_1 + r) \\
    \eta_1 & \text{if } t \in [t'_1 + r, t_1 + NT - r] \\
    \frac{t_1 + NT - t}{r} \eta_1 + \frac{t - (t_1 + NT - r)}{r} p_1(t_1 + NT) & \text{if } t \in (t_1 + NT - r, t_1 + NT) \\
    p_1(t) & \text{if } t \in [t_1 + NT, \infty). 
\end{cases}
\]

Direct calculation shows that \( J(Y) \leq 2Ar. \) This implies that \( \beta(p'_1, p_1) \leq 2Ar, \) where for \( p, p' \in \mathcal{X} \) we define
\[
\beta(p, p') = \inf_{z \in \Gamma(p, p')} J(z). \tag{3.8}
\]

**Theorem 4.** Assumed that (P) and the hypotheses of Proposition 1 are satisfied. If \( p_1, p'_1 \in \mathcal{X} \) and
\[
\beta(p'_1, p_1) = \inf_{p, p' \in \mathcal{X}} \beta(p', p),
\]
then there is a connecting orbit \( q \) of (HS) which satisfies
\[
q(t) \to p_1(t) \text{ uniformly as } t \to \infty \tag{3.9}
\]
and
\[
q(t) \to p'_1(t) \text{ uniformly as } t \to -\infty. \tag{3.10}
\]

**Proof.** Let \( \{z_m\} \subset \Gamma(p'_1, p_1) \) be a minimizing sequence for \( J. \) It is not difficult to show that \( \{z_m\} \) is bounded in \( W^{1,2}_{\text{loc}}(\mathbb{R}, \mathbb{R}^n). \) Hence there is a \( q \in W^{1,2}_{\text{loc}}(\mathbb{R}, \mathbb{R}^n) \) such that along a subsequence, \( z_m \to q \) weakly in \( W^{1,2}_{\text{loc}}(\mathbb{R}, \mathbb{R}^n) \) and strongly in \( L^\infty_{\text{loc}}(\mathbb{R}, \mathbb{R}^n). \) By the weak lower semicontinuity of \( a_t, \)
\[
\sum_{t = -\tilde{N}}^{0} a_t(q) + \sum_{t = \tilde{N}}^{\hat{N}} a_t(q) \leq \lim_{m \to \infty} J(z_m)
\]
for all $\tilde{N} > N$. It follows that
\[ J(q) \leq \beta(p'_1, p_1) \leq 2 Ar. \tag{3.11} \]

Then the argument used in the proof of Theorem 2 shows that $G(q) \subset \hat{A}$.

To show $q$ is a connecting orbit of $(HS)$, we first prove (3.9). For $z \in W^s_{loc}(\mathbb{R}, \mathbb{R}^n)$, let $\sigma_i(z)$ denote the restriction of $z$ on $[\ell T, (\ell + 1) T)$. By (3.11) and Proposition 2, for any $\rho > 0$, there is an $\hat{\ell} = \hat{\ell}(\rho)$ such that if $\ell \geq \hat{\ell}$ then
\[ \sigma_i(q) \in B_{\rho}(\bar{p}) \text{ for some } \bar{p} \in \mathcal{K}_1. \tag{3.12} \]

Furthermore, there is a $p \in \mathcal{K}_1$ such that $\sup \{ \ell | \sigma_i(q) \in B_{\rho}(p) \} = \infty$. We claim

there is an $\hat{\ell} \in \mathbb{N}$ such that $\sigma_i(q) \in B_{\rho}(p)$ for all $\ell \geq \hat{\ell}$. \tag{3.13}

Suppose (3.13) is false. Given $\rho_1 > 0$, we can find $\ell_3 > \ell_2 > \ell_1$ such that
\[ \sigma_{i}(q) \in B_{\rho}(p) \text{ for } i = 1, 3 \text{ and } \sigma_3(q) \in B_{\rho}(p') \text{ for some } p' \in \mathcal{K}_1 \setminus \{ p \}, \tag{3.14} \]

where $c$ is the same constant as in (3.2). Since $z_m \to q$ strongly in $L^\infty_{loc}(\mathbb{R}, \mathbb{R}^n)$, there is an $\tilde{m} = \tilde{m}(\rho_1) \in \mathbb{N}$ such that
\[ \| \sigma_i(z_m - q) \|_{L^\infty} < \frac{\rho_1}{2} \tag{3.15} \]

and
\[ J(z_m) < \beta(p'_1, p_1) + \rho_1 \tag{3.16} \]

if $m \geq \tilde{m}$. Consequently $\| \sigma_i(z_m - p') \|_{L^\infty} < \rho_1$ and $\| \sigma_i(z_m - p) \|_{L^\infty} < \rho_1$ for $i = 1, 3$. Set

\[
\begin{align*}
\tau_m(t) &= \begin{cases} 
\frac{z_m(t)}{\rho_1} & \text{if } t \in (-\infty, (\ell_1 + 1) T - \rho_1) \cup [(\ell_1 + 1) T + \rho_1, \infty) \\
\frac{(\ell_1 + 1) T - t}{\rho_1} z_m'(t) + \frac{t - (\ell_1 + 1) T + \rho_1}{\rho_1} p((\ell_1 + 1) T) & \text{if } t \in [(\ell_1 + 1) T - \rho_1, (\ell_1 + 1) T) \\
p(t) & \text{if } t \in [(\ell_1 + 1) T, \ell_3 T) \\
\frac{\ell_3 T + p - t}{\rho_1} z_m(\ell_3 T + p) + \frac{t - \ell_3 T}{\rho_1} z_m(\ell_3 T + p) & \text{if } t \in (\ell_3 T, \ell_3 T + p),
\end{cases}
\end{align*}
\]
where $z'_m = z_m((\ell + 1)T - \rho_1)$. Direct calculation yields

$$
\int_{sT}^{sT+\rho_1} \left( \mathcal{L}(w_m) + \Psi(t, w_m) - \mathcal{L}(p_1) \right) dt
\leq \int_{sT}^{sT+\rho_1} \frac{1}{\rho_1^2} (|z_m(\ell_3T + \rho_1) - p(\ell_3T + \rho_1)|^2 + |p(\ell_3T + \rho_1) - p(\ell_3T)|^2) dt
+ \int_{sT}^{sT+\rho_1} [V(t, w_m) + \Psi(t, w_m) - \mathcal{L}(p_1)] dt
\leq b_1 \rho_1,
$$

where $b_1$ is a constant independent of $m$ and $\rho_1$. Likewise,

$$
\int_{(s+1)T-\rho_1}^{(s+1)T} \left( \mathcal{L}(w_m) + \Psi(t, w_m) - \mathcal{L}(p_1) \right) dt \leq b_2 \rho_1
$$

with $b_2$ independent of $m$ and $\rho_1$.

We now state a technical lemma whose proof will be given at the end of the section.

**Lemma 4.** Let $p, p' \in \mathcal{X}$ and $p \neq p'$. There exist positive numbers $\tilde{\rho}$ and $C_1$ such that, for any $z \in W^{1,2}_{loc}(\mathbb{R}, \mathbb{R}^n)$, if

$$
\|\sigma_1(z - p')\|_{L^\infty} < \tilde{\rho}
$$

and

$$
\|\sigma_1(z - p)\|_{L^\infty} < \tilde{\rho}
$$

for some $\tilde{\ell} > \ell$, then

$$
\int_{sT}^{(s+1)T} \left( \mathcal{L}(z) + \psi_1(t, z) - \mathcal{L}(p) \right) dt \geq C_1,
$$

where $C_1 = C_1(p, p')$, a constant independent of $\ell$ and $\tilde{\ell}$.

To proceed with the proof of Theorem 4, we apply Lemma 4 to obtain

$$
\int_{(s+1)T}^{((s+1)T)} \left( \mathcal{L}(z_m) + \Psi(t, z_m) - \mathcal{L}(p_1) \right) dt \geq C_1.
$$
if \( r_1 < \min(\beta, C_1 / (b_1 + b_2 + 1)) \). Then for large \( m \)

\[
J(w_m) \leq J(z_m) - C_1 + b_1 \rho_1 + b_2 r_1
\]

\[
< \beta(p_1', p_1) + r_1 - C_1 + b_1 \rho_1 + b_2 r_1 < \beta(p_1', p_1),
\]

contary to (3.8). Therefore (3.13) must hold.

It remains to show \( p = p_1 \) to complete the proof of (3.9). Suppose \( p \neq p_1 \), then for any \( r_2 > 0 \), there is an \( \ell_4 \in \mathbb{N} \) such that \( \sigma_{\ell_4}(q) \in \mathcal{B}_{2}(p) \). If \( m \) is large enough then \( J(z_m) < \beta(p_1', p_1) + r_2 \) and \( \|\sigma_{\ell_4}(z_m - p)\|_{L^\infty} < r_2 \). Moreover, \( \|\sigma_{\ell_4}(z_m - p)\|_{L^\infty} < r_2 \) if \( \ell_4 \) is sufficiently large. Define

\[
v_m(t) = \begin{cases} 
  z_m(t) & \text{if } t \in (-\infty, (\ell_4 + 1)T - \rho_2] \\
  (\ell_4+1)T - t_z_m^* + t - (\ell_4+1)T + \rho_2 & \text{if } t \in ((\ell_4+1)T - \rho_2, (\ell_4+1)T) \\
  p(t) & \text{if } t \in [(\ell_4+1)T, \infty),
\end{cases}
\]

where \( z_m^* = z_m((\ell_4 + 1)T - \rho_2) \). Then

\[
\int_{(\ell_4+1)T - \rho_2}^{(\ell_4+1)T} (\mathcal{L}(v_m) + \Psi(t, v_m) - \mathcal{L}(p_1)) \, dt \leq b_3 r_2,
\]

where \( b_3 \) is a constant independent of \( m \) and \( r_2 \). By Lemma 4

\[
\int_{(\ell_4+1)T - \rho_2}^{(\ell_4+1)T} (\mathcal{L}(z_m) + \Psi(t, z_m) - \mathcal{L}(p_1)) \, dt \geq C_1
\]

if \( r_2 < \min(\beta, C_1 / (b_1 + 1)) \). It follows that

\[
J(v_m) \leq J(z_m) - C_1 + b_3 r_2 < \beta(p_1', p_1) + r_2 - C_1 + b_3 r_2 < \beta(p_1', p_1).
\]

Now, we get

\[
\beta(p_1', p) < \beta(p_1', p_1),
\]

a contradiction which completes the proof of (3.9). The proof of (3.10) is the same.

To show \( q \) satisfies (HS), we note that \( q + \delta \varphi \in \Gamma(p_1', p_1) \) if \( \varphi \in C_0^\infty(\mathbb{R}, \mathbb{R}^n) \) and \( \delta \in \mathbb{R} \). Let \( f(\delta) = J(q + \delta \varphi) \). Then \( f \) is a \( C^1 \) function of \( \delta \) and \( f(\delta) \geq f(0) \). Suppose the support of \( \varphi \) lies in \([-\ell, \ell]\), then

\[
f'(0) = \int_{-\ell}^{\ell} \left[ \dot{q} \cdot \dot{\varphi} + (V'(t, q) + \Psi'(t, q)) \cdot \varphi \right] \, dt \geq 0. \tag{3.20}
\]
Due to the freedom of choice of $\varphi$ and $G(q) \in \hat{A}$, (3.20) shows
\[
\int_{-\infty}^{\infty} \left[ \dot{q} \dot{\varphi} + V'(t, q) \cdot \varphi \right] dt = 0
\]
for all $\varphi \in C_0^\infty(\mathbb{R}, \mathbb{R}^n)$. So $q$ is a weak solution of (HS). Standard arguments then yield that $q$ is a classical solution of (HS).

**Remark 4.** (a) Actually we have obtained infinitely many connecting orbits by taking different $N$.

(b) If $p_1^* = p_1$, the solution obtained in Theorem 4 is a homoclinic orbit.

**Proof of Lemma 4.** Suppose the assertion of the lemma is false. Then there exist $\{a_i\}, \{\bar{a}_i\}$ and $\{z_i\}$ such that
\[
\lim_{i \to \infty} \|a_i(z_i - p')\|_{L^\infty} = 0,
\]
\[
\lim_{i \to \infty} \|\bar{a}_i(z_i - p)\|_{L^\infty} = 0
\]
and
\[
\lim_{i \to \infty} \int_{\ell_i T}^{(\ell_i + 1) T} (\mathcal{L}(z_i) + \psi_1(t, z_i) - \mathcal{L}(p)) dt = 0.
\]

Let's first treat the case where $\ell_i = \ell$ and $\tilde{\ell}_i = \tilde{\ell}$ for all $i$. Then along a subsequence,

\[ z_i \to \bar{q} \quad \text{in} \quad W^{1,2}([\ell T, (\ell + 1) T], \mathbb{R}^n). \]

It follows that
\[
\int_{\ell T}^{(\ell + 1) T} (\mathcal{L}(\bar{q}) + \psi_1(t, \bar{q}) - \mathcal{L}(p)) dt = 0
\]
and consequently $\bar{q}$ is a solution of (HS). This violates the basic uniqueness theorem for the initial value problems, since $\bar{q}(t) = p'(t)$ if $t \in (\ell T, (\ell + 1) T)$ and $\bar{q}(t) = p(t)$ if $t \in (\tilde{\ell} T, (\tilde{\ell} + 1) T)$. Thus for any given $\tilde{\ell} > \ell$, there exist $\bar{\varphi} = \bar{\varphi}(\tilde{\ell} - \ell)$ and $C_1 = C_1(p, p', \tilde{\ell} - \ell)$ such that if (3.17) and (3.18) are satisfied, then (3.19) holds.

Suppose $C_1(p, p', \tilde{\ell} - \ell) \to 0$ along a sequence as $\tilde{\ell} - \ell \to \infty$, here and throughout to the end of the proof we suppress the subscript $i$ from the notation. Let $C_2 = \min \{ C_1(p_k, p_j, 1) | p_k, p_j \in \mathcal{X}, p_k \neq p_j \}$. Then if $\ell \to \infty$ is sufficiently large, where $\rho^* < \min(d, (\rho_1(1/c), C_2)$ and $d_i$ is a function defined in Proposition 2. It follows from Proposition 2 that
\(\|s_j(z) - x_i\|_{L^\infty} < \hat{p}(1)\) for every \(j \in [\ell, \tilde{\ell}]\). Since \(\|s_j(z) - p\|_{L^\infty} < \hat{p}(1)\) and \(\|s_{j+1}(z) - p\|_{L^\infty} < \hat{p}(1)\), there exists an \(\ell_0 \in [\ell, \tilde{\ell}]\) such that

\[\|s_{\ell_0}(z) - p\|_{L^\infty} < \hat{p}(1)\text{ and }\|s_{\ell_0+1}(z) - p\|_{L^\infty} < \hat{p}(1)\]

for some \(p^* \in \mathcal{X}_i \setminus \{p\}\). Now

\[
\int_{\ell_0T}^{(\ell_0+2)T} (\mathcal{L}(z) + \psi_1(t, z) - \mathcal{L}(p)) \, dt < C_1(p, p^*, \tilde{\ell} - \ell) < p^* \prec C_2,
\]

we get a contradiction which completes the proof.

### 4. MULTIBUMP CONNECTING ORBITS

In this section, we apply penalization methods to study multibump connecting orbits of (HS). Let \(N, N_1, N_2\) be integers and \(\bar{p}, \tilde{\ell}\) be as in the proof of Theorem 2. Let \(\Psi \in C^\infty(\mathbb{R} \times \mathbb{R}^n, \mathbb{R})\) be such that \(0 \leq \Psi \leq M_1\) and

\[
\Psi(t, y) = \begin{cases} 
\psi_1(t, y) & \text{if } t \in D_1 \\
M_1 & \text{if } y \notin B_{\rho_0}(\eta_1) \text{ and } t \in D_2 \\
0 & \text{if } y \notin B_{\rho}(\eta) \text{ and } t \in D_2,
\end{cases}
\]

where \(-N < N_1 < N_2 < N\), \(D_1 = (-\infty, \tilde{\ell}_1 - NT) \cup (\tilde{\ell}_0 + N_1T, \tilde{\ell}_1 + N_2T) \cup (\tilde{\ell}_0 + NT, \infty)\) and \(D_2 = (\tilde{\ell}_1 - NT, \tilde{\ell}_0 + N_1T) \cup (\tilde{\ell}_1 + N_2T, \tilde{\ell}_0 + NT)\). From the proof of Theorem 2, for each \(p_i \in \mathcal{X}_i\) there exists an \(s_i \in (\tilde{\ell}_0, \tilde{\ell}_1)\) such that \(p_i(s_i) \in B_{\rho}(\eta_i)\). Choose \(p_1, p_2, p_3 \in \mathcal{X}_i\). For \(z \in W^{1,2}_{\text{loc}}(\mathbb{R}, \mathbb{R}^n)\), define

\[
\hat{a}_\ell(z) = \int_{\tilde{\ell}_0T}^{\tilde{\ell}_1 + (\ell+1)T} [\mathcal{L}(z) + \Psi(t, z) - \mathcal{L}(p)] \, dt \quad \text{if } \ell \leq -N - 1,
\]

\[
\hat{a}_{-N}(z) = \int_{\tilde{\ell}_0T}^{\tilde{\ell}_1 - (N-1)T} [\mathcal{L}(z) + \Psi(t, z)] \, dt,
\]

\[
\hat{a}_{-N+1}(z) = \int_{\tilde{\ell}_0T}^{\tilde{\ell}_1 + (\ell+1)T} [\mathcal{L}(z) + \Psi(t, z) - \mathcal{L}(p_1)] \, dt \quad \text{if } -N < \ell \leq -N_1 - 1,
\]

\[
\hat{a}_{-N+2}(z) = \int_{\tilde{\ell}_0T}^{\tilde{\ell}_1 + (\ell+1)T} [\mathcal{L}(z) + \Psi(t, z) - \mathcal{L}(p_1)] \, dt \quad \text{if } N_1 \leq \ell < N_2,
\]

\[
\hat{a}_{-N+3}(z) = \int_{\tilde{\ell}_0T}^{\tilde{\ell}_1 + (\ell+1)T} [\mathcal{L}(z) + \Psi(t, z)] \, dt \quad \text{if } N_2 \leq \ell < N - 1,
\]

\[
\hat{a}_{N-1}(z) = \int_{\tilde{\ell}_0T}^{\tilde{\ell}_1 + NT} [\mathcal{L}(z) + \Psi(t, z)] \, dt,
\]
and
\[ \hat{a}_\ell(z) = \int_{\tau_\ell}^{\tau_{\ell+1}} \left[ \mathcal{L}(z) + \dot{\Psi}(t, z) - \mathcal{L}(p_1) \right] dt \quad \text{if} \quad N \leq \ell. \]

Then \( \hat{a}_\ell(z) \geq 0 \) for all \( \ell \). Set
\[ \hat{J}(z) = \sum_{\ell=-\infty}^{\infty} \hat{a}_\ell(z). \]

For \( p, p' \in \mathcal{X}_1 \), define
\[ \hat{\beta}(p, p') = \inf_{z \in \Gamma(p, p')} \hat{J}(z). \quad (4.2) \]

**Theorem 5.** Suppose that \( (P) \) and the hypotheses of Proposition 1 are satisfied. Assume that
\[ \min(k_3 - k_2, k_1 - k_0) > 6\rho_0 + \frac{2}{\theta(r)} \left( \tilde{a}(k_1, k_2) + \tilde{a}(k_3, k_4) + 2\rho_0 \sqrt{2\theta(\rho_0)} \right). \]

If \( \hat{\beta}(p_3, p_1) = \inf_{p_3, p_1 \in \mathcal{X}_1} \hat{\beta}(p_3, p_1) \), then there is a connecting orbit \( q \) of (HS) which satisfies
\[ q(t) \to p_3(t) \quad \text{uniformly as} \quad t \to -\infty \quad (4.3) \]

and
\[ q(t) \to p_1(t) \quad \text{uniformly as} \quad t \to \infty. \quad (4.4) \]

Moreover, for any sufficiently small positive number \( \rho_1 \), if \( N_2 - N_1 \) is chosen large enough, then there exist \( N_3, N_4 \in (N_1, N_2) \cap \mathbb{Z} \) and \( p_4 \in \mathcal{X}_1 \) such that
\[ \sigma_\ell(q) \in \mathcal{B}_{\rho_1}(p_4) \quad \text{for} \quad \ell \in [N_3, N_4]. \]

**Proof.** We only prove the last assertion of the theorem, since the other parts follow from an argument similar to the proof of Theorem 4.

As in the proof of Theorem 2, a simple calculation shows that
\[ \hat{\beta}(p_3, p_1) \leq 4A \rho. \]
Taking a minimizing sequence in \( \Gamma(p_3, p_1) \) for \( \hat{J} \), we obtain a connecting orbit \( q \) of (HS) which satisfies (4.3), (4.4) and
\[ \hat{J}(q) = \hat{\beta}(p_3, p_1). \quad (4.5) \]
For sufficiently small $r_3$, if $N_2 - N_1$ is large enough, there exist $N_3, N_4 \in (N_1, N_2) \cap \mathbb{Z}$ and a $p_4 \in \mathcal{X}_i$ such that
\[ \sigma_i(q) \in \mathcal{B}_{p_4}(\mathcal{X}_i) \quad \text{for} \quad \ell \in (N_1, N_2) \cap \mathbb{Z} \tag{4.6} \]
and
\[ \sigma_i(q) \in \mathcal{B}_{p_4}(p_4) \quad \text{for} \quad \ell = N_3, N_4. \tag{4.7} \]

If there were an $\ell \in (N_3, N_4) \cap \mathbb{Z}$ such that $\sigma_i(q) \notin \mathcal{B}_{p_4}(p_4)$, then an argument used in the proof of Theorem 4 would yield a contradiction to (4.2). Thus the proof is complete.

5. CONNECTION BETWEEN EQUILIBRIA AND PERIODIC SOLUTIONS

We are now going to find a heteroclinic orbit connecting an equilibrium to a periodic solution. We use the same $\bar{\rho}, \hat{t}_0$ as before and a penalty function as described follows. Let $\bar{\psi} \in C^\infty(\mathbb{R} \times \mathbb{R}^n, \mathbb{R})$ be such that $0 \leq \bar{\psi} \leq M_1$ and
\[ \bar{\psi}(t, y) = \begin{cases} \bar{\psi}_1(t, y) & \text{if} \quad t \in [\hat{t}_0, \infty) \\ M_1 & \text{if} \quad y \notin B_{\rho_0}(\eta_1) \text{ and } t \in (-\infty, \hat{t}_0) \\ 0 & \text{if} \quad y \in B_{\rho}(\eta_1) \text{ and } t \in (-\infty, \hat{t}_0). \end{cases} \]

For $z \in W^{1,2}_{\text{loc}}(\mathbb{R}, \mathbb{R}^n)$, define
\[ \tilde{a}_\ell(z) = \int_{\hat{t}_0 + \ell \bar{\tau}}^{t_1 + (\ell + 1) \bar{\tau}} \left[ \mathcal{L}(z) + \bar{\psi}(t, z) \right] dt \quad \text{if} \quad \ell \in \mathbb{N} \cup \{0\} \]
\[ \tilde{a}_\ell(z) = \int_{\hat{t}_0 + \ell \bar{\tau}}^{t_1 + (\ell + 1) \bar{\tau}} \left[ \mathcal{L}(z) + \bar{\psi}(t, z) - \mathcal{L}(p_1) \right] dt \quad \text{if} \quad \ell \in \mathbb{N}. \]

It is clear that $\tilde{a}_\ell(z) \geq 0$ for all $z$. Set
\[ \tilde{J}(z) = \sum_{\ell \in \mathbb{Z}} \tilde{a}_\ell(z). \]

For $\eta \in \mathcal{X}_i$ and $p \in \mathcal{X}_i$, let
\[ \tilde{\mathcal{F}}(\eta, p) = \{ z \in W^{1,2}_{\text{loc}}(\mathbb{R}, \mathbb{R}^n) \mid z(t) \to \eta \text{ as } t \to -\infty \text{ and } z(t) \to p(t) \text{ uniformly as } t \to \infty \}. \]
and
\[ \hat{J}(\eta, p) = \inf_{z \in F(\eta, p)} J(z). \] (5.1)

**Theorem 6.** Assumed that (P) and the hypotheses of Proposition 1 are satisfied. If \( \eta_1 \in \mathcal{X}, \ p_1 \in \mathcal{X}_1 \) and
\[ \hat{J}(\eta_1, p_1) = \inf_{p \in \mathcal{X}_1} J(\eta_1, p), \]
then there is a heteroclinic orbit \( q \) of (HS) which satisfies
\[ q(t) \to \eta_1 \text{ as } t \to -\infty \] (5.2)
and
\[ q(t) \to p_1(t) \text{ uniformly as } t \to \infty. \] (5.3)

Except for using different penalty functions, the proof is essentially the same as that of Theorem 3. We omit it.

6. FORCED PENDULUM PROBLEMS

We turn to the case of \( V(t, y) = F(t) W(y) \). It is in particular led to (0.2) if \( W(y) = \cos y + 1 \). Let \( f_1 = \max_{t \in \mathbb{R}} F(t), \ f_2 = \min_{t \in \mathbb{R}} F(t), \ F_M = f_1 - (f_1 - f_2)/10, \) and \( F_m = f_2 + (f_1 - f_2)/10 \). For such a \( V \), it is clear that
\[ \hat{\alpha}(j_1, j_2) = \alpha(j_1, j_2). \] Define
\[ \alpha_M(j_1, j_2) = \inf_{z \in \hat{B}(j_1, j_2)} \int_{j_1}^{j_2} \frac{1}{2} |z|^2 + F_M W(z) \, dt \]
and
\[ \alpha_m(j_1, j_2) = \inf_{z \in \hat{B}(j_1, j_2)} \int_{j_1}^{j_2} \frac{1}{2} |z|^2 + F_m W(z) \, dt. \]

Pick \( k_0 < k_1 < k_2 < k_3 < k_4 = k_0 + T \) such that \( F_i(t) \geq F_M \) if \( t \in [k_0, k_1] \cup [k_2, k_3] \). It follows that \( \hat{\alpha}(k_0, k_1) \geq \alpha_M(k_0, k_1) \) and \( \hat{\alpha}(k_2, k_3) \geq \alpha_M(k_2, k_3) \).

On the other hand, there are \( \tilde{k}_1, \tilde{k}_2 \in (k_1, k_2) \) and \( \tilde{k}_3, \tilde{k}_4 \in (k_3, k_4) \) such that \( F_i(t) \leq F_m \) if \( t \in [\tilde{k}_1, \tilde{k}_2] \cup [\tilde{k}_3, \tilde{k}_4] \). Hence \( \hat{\alpha}(k_1, k_2) < \alpha_m(\tilde{k}_1, \tilde{k}_2) \) and \( \hat{\alpha}(k_3, k_4) < \alpha_m(\tilde{k}_3, \tilde{k}_4) \). If \( \epsilon \) is small enough, \( k_1 - k_0, k_3 - k_2, \tilde{k}_2 - \tilde{k}_1, \) and \( \tilde{k}_4 - \tilde{k}_3 \) can be chosen to satisfy (0.3)–(0.6). Thus we see that a non-constant periodic solution has been obtained by Theorem 2.
Next, we apply Theorem 3 to get another periodic solution. It is not difficult to check that the minimax value $\beta = \beta(\varepsilon)$, defined by (2.2), is uniformly bounded for $\varepsilon \in (0, 1)$. Set $W_\varepsilon = \{ y \mid y \in \mathbb{R}^n \setminus K_\varepsilon \text{ and } W'(y) = 0 \}$ and $W_m = \inf_{y \in W_\varepsilon} W(y)$. Clearly, (V2) and (V5) imply that $W_m > 0$. Let $T_0$ be the minimal period of $F$. If $\varepsilon$ is sufficiently small, then $\hat{I}_1(y) \geq W_m T_0^{-1} > \beta(\varepsilon)$, for all $y \in W_\varepsilon$. Thus the periodic solution obtained by Theorem 3 is not a constant function. This completes the proof of Theorem 1.

**Remark 5.** As noted in the introduction, we are working with twice the minimal period if $F_\varepsilon$ has only one maximum and one minimum per period.

Theorems 4–6 can also be carried over to the case of $V(t, y) = F_\varepsilon(t) W(y)$ in the same manner. We omit its detail.

REFERENCES


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