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ON SYMMETRIC AND SKEW-SYMMETRIC DETERMINANTAL VARIETIES

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INTRODUCTION

LET E and F be differentiable vector bundles over a compact oriented manifold M and f: $E \to F$ a bundle map between them. The locus of points in M where the rank of f does not exceed a given integer r is called the *degeneracy locus* of rank r of f, denoted M_r. As such a locus is defined locally by the vanishing of all the $(r+1)$ by $(r+1)$ minors of the matrix of f, in the algebraic category it is called a *determinantal variety*. In [17] Thom observed that the cohomology class of the degeneracy locus of a general map f: $E \to F$ should be a polynomial in the Chern classes of E and F. This polynomial was found by Porteous in [16]:

$$[M_r] = (-1)^{(m-r)(n-r)}\Delta_{m-r,n-r}\left(\frac{c_t(E)}{c_t(F)}\right)$$

where if $a(t) = \sum_{-\infty}^{\infty} a_k t^k$ is a formal series, then by definition

$$\Delta_{p,q}(a) = \det\begin{pmatrix} a_p & a_{p+1} & \cdots & a_{p+q-1} \\ a_{p-1} & a_p & & \vdots \\ a_{p-q+1} & \cdots & \cdots & a_p \end{pmatrix}.$$

The purpose of this article is to derive the analogous formulas for symmetric and skew-symmetric maps; these are expressed in Theorems 1 and 8. By a "squaring principle", these results can be extended to a twisted bundle map f: $E \to E^* \otimes L$, where L is a line bundle (see Theorem 10). As applications of these formulas, we compute the degree of a determinantal variety, symmetric or not, and derive some numerical criteria for a variation of Hodge structure to be nontrivial.

We are indebted to the referee for detailed criticisms and to James Damon for pointing out an alternative method of deriving the degeneracy locus formulas (Theorems 1 and 8).

§1. THE DEGENERACY LOCUS FORMULAS

(a) *Symmetric and skew-symmetric maps*

Let V be a vector space and V^* its dual vector space. A linear map f: $V \to V^*$ is said to be *symmetric* if $(f(x), y) = (f(y), x)$ for all x, y in V, where (,) is the dual pairing between V^* and V. Equivalently, f is symmetric if $f = {}^t f$. Relative to a basis for V and its dual basis

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for V^* a symmetric map is represented by a symmetric matrix. The notion of a symmetric map generalizes in the obvious way to a bundle map $f\colon E \to E^*$ from a vector bundle to its dual—f is a symmetric bundle map if it is symmetric in each fiber. A skew-symmetric bundle map is defined analogously.

A linear map f from a rank n vector bundle $\pi\colon E \to M$ to its dual is a section of the bundle $\operatorname{Hom}(E, E^*) = E^* \otimes E^*$ if in addition f is symmetric, then it is a section of the subbundle $\operatorname{Sym}^2 E^*$ in $E^* \otimes E^*$. The fiber of $\operatorname{Sym}^2 E^*$ at a point x in M is on the one hand the symmetric linear maps from E_x to E_x^*, and on the other, the quadratic forms on E_x. So we may think of f as giving us a quadric in each fiber of E and the degeneracy locus M_r of f as the set of points x such that the quadric in the fiber E_x has rank at most r.

Now, in the vector space of all $n \times n$ symmetric complex matrices, the locus W_r of those of rank r or less is an analytic subvariety of codimension $\binom{n-r+1}{2}$. Thus, if we denote by $W_r(E)$ the sub-fiber bundle of $\operatorname{Sym}^2 E^*$ whose fiber at $x \in M$ consists of all quadrics of rank at most r in E_x, and f a section of $\operatorname{Sym}^2 E^*$, then the expected codimension of the locus $M_r = \pi(f(M) \cap W_r(E))$ is $\binom{n-r+1}{2}$. If this is the case, its class is given by the

THEOREM 1. *Let E be a smooth complex vector bundle of rank n over a compact oriented manifold M and $f\colon E \to E^*$ a general symmetric bundle map. Then the cohomology class of the degeneracy locus M_r is*

$$2^{n-r}\begin{vmatrix} c_{n-r} & c_{n-r+1} & c_{n-r+2} & \\ c_{n-r-2} & c_{n-r-1} & c_{n-r} & \\ \vdots & & c_{n-r-2} & \\ \vdots & & & \ddots \\ & \cdots & & c_1 \end{vmatrix}$$

where $c_i = c_i(E^*)$.

Note 2. Here, and throughout this paper, "general" may be given the following meaning: Let $\operatorname{Hom}(E, F)$ (resp. $\operatorname{Hom}^s(E, E^* \otimes L)$, $\operatorname{Hom}^{ss}(E, E^* \otimes L)$) be the bundle of maps from E to F (resp. symmetric maps from E to $E^* \otimes L$; skew-symmetric maps from E to $E^* \otimes L$); and let $W_r(E, F)$ (resp. $W_r^s(E, E^* \otimes L)$, $W_r^{ss}(E, E^* \otimes L)$) be the subvariety of maps of rank r or less. Then f is said to be general if it is transverse to the appropriate locus almost everywhere.

Equivalently, f is general if the $(r+1) \times (r+1)$ minors of a local matrix representative for f generate the ideal of the locus M_r in the local ring of germs of C^∞ functions around almost all points of M_r and M_r has the correct dimension.

We first observe that the problem of finding the class of M_r may be reduced to the case where M is the Grassmannian and E is the universal subbundle on M. For by choosing sufficiently many C^∞ sections of E^* we can embed M differentiably in the Grassmannian $G(n, m)$ where m is the dimension of the space of sections chosen. Under this embedding the dual of the universal subbundle S^* pulls back to E^*.

$$\begin{array}{ccc} E^* & & S^* \\ \downarrow & & \downarrow \\ M & \overset{}{\underset{i}{\hookrightarrow}} & G(n, m). \end{array}$$

Most general sections of $\operatorname{Sym}^2 S^*$ restrict to general sections of $\operatorname{Sym}^2 E^*$ on M. So if the polynomial $P_r(c_1(S^*), \ldots, c_k(S^*))$ gives the locus where a section of $\mathrm{Sym}^2 S^*$ has rank at most r, then its pullback

$$j^*P_r(c_1(S^*), \ldots, c_n(S^*)) = P_r(c_1(E^*), \ldots, c_n(E^*))$$

gives the degeneracy locus M_r.

One way of getting a section of $\mathrm{Sym}^2 S^*$ over $G(n, m)$ is to take a smooth quadric in $\mathbb{C}^m$ and restrict it to each n-plane in $\mathbb{C}^m$. The problem may now be stated in purely geometric terms:

Given a smooth quadric Q in $\mathbb{C}^m$, let M_r be the locus of n-planes Λ in $\mathbb{C}^m$ such that the restriction of Q to Λ has rank at most r. Find the cohomology class of M_r in $H^*(G(n, m))$.

It turns out to be simpler, in the matter of indices, to consider the locus $S_r = M_{n-r}$; we will show

$$S_r \sim 2^r \sigma_{r,r-1,\ldots,1} \tag{3}$$

in the homology of $G(n, m)$. We start by proving the

PROPOSITION (4): *If Q: $W \to W^*$ is a quadric and $V \subset W$ a linear subspace of codimension k in W, then*

$$\mathrm{rank}\, Q|_V \geq \mathrm{rank}\, Q - 2k .$$

Proof. Note that if i is the inclusion of V in W, then the quadric $Q|_V$ corresponds to the map

$$\tilde{Q}\colon V \xrightarrow{i} W \xrightarrow{Q} W^* \xrightarrow{{}^t i} V^*;$$

since composing with i or ${}^t i$ can drop the rank of a map by at most k, the proposition follows.

Now, to compute the class of S_r in general, we intersect it with the Schubert cycles of the complementary codimension, i.e. we compute

$$S_r \cdot \sigma_b, \quad \text{where} \quad \sum b_i = n(m-n) - \binom{r+1}{2}.$$

Now $S_r \cdot \sigma_b$ is nonempty if and only if there are n-planes Λ in S_r such that

$$\dim(\Lambda \cap V_{m-n+i-b_i}) \geq i \quad \text{for} \quad i = 1, \ldots, n,$$

where the V's form a generic flag of the indicated dimensions in $\mathbb{C}^m$. Let Q_i be the section $Q \cap V_{m-n+i-b_i}$. It is a smooth quadric of rank $m-n+i-b_i$; furthermore, because $\Lambda \cap Q$ has rank at most $n-r$ by the definition of Λ, $(\Lambda \cap V_{m-n+i-b_i}) \cap Q_I$ also has rank at most $n-r$. By Proposition (4),

$$n - r \geq (m-n+i-b_i) - 2(m-n-b_i).$$

Therefore,

$$b_i \leq m - r - i.$$

By the definition of the Schubert cycles, $b_i \leq m - n$. Hence,

$$(1.5) \qquad \begin{aligned} b_i &\leq m - n && \text{for} \quad 1 \leq i \leq n - r, \\ b_i &\leq m - r - i && \text{for} \quad n - r + 1 \leq i \leq n. \end{aligned}$$

Since by assumption

$$\sum_i b_i = n(m - n) - \binom{r+1}{2},$$

equality must hold in (1.5). In other words, $S_r \cdot \sigma_b \neq 0$ only if

$$b = m - n, \ldots, m - n, m - n - 1, \ldots, m - n - r.$$

This proves that S_r is a single-condition Schubert cycle:

$$S_r \sim a\sigma_{r,r-1,\ldots,1} \text{ for some constant } a.$$

To determine a we have

$$a = S_r \cdot \sigma_{\underbrace{m-n,\ldots,m-n}_{n-r},m-n-1,\ldots,m-n-r}.$$

So a is the number of Λ in S_r satisfying the Schubert conditions

$$\dim(\Lambda \cap V_{n-r+2i}) \geq n - r + i, \qquad i = 0, \ldots, r.$$

and this number is readily computed: letting $\Lambda_{n-r+i} = \Lambda \cap V_{n-r+2i}$, we see that $\Lambda_{n-r} = V_{n-r}$, while for $i = 1, \ldots, r$, Λ_{n-r+i} must be spanned by $\Lambda_{n-r+i-1}$ and either one of the two null-vectors of the quadratic form induced by Q in the 2-dimensional vector space

$$(V_{n-r+2i} \cap \Lambda^{\perp}_{n-r+i-1})/\Lambda_{n-r+i-1}.$$

Thus $a = 2^r$, and Formula (3) is established.

To express this Schubert cycle in terms of the Chern classes of the dual of the universal subbundle S^* we apply Giambelli's formula (Griffiths and Harris[11], p. 205). Under the natural isomorphism

$$\begin{array}{ccc} S^* & & Q \\ \downarrow & & \downarrow \\ G(n, V) & \longrightarrow & G(m - n, V^*), \end{array} \qquad m = \dim V,$$

the Schubert cycle $\sigma_{r,r-1,\ldots,1}$ in $G(n, V)$ corresponds to $\sigma_{r,r-1,\ldots,1}$ in $G(m - n, V^*)$. Giambelli's formula gives

$$\sigma_{r,r-1,\ldots,1} = \det \begin{pmatrix} \sigma_r & \sigma_{r+1} & \cdots \\ \sigma_{r-2} & \sigma_{r-1} & \cdots \\ & & \\ & & \sigma_1 \end{pmatrix} \text{ in } G(m - n, V)$$

which corresponds to

$$\sigma_{r,r-1,\dots,1} = \det \begin{pmatrix} c_r & c_{r+1} & \cdots & \\ c_{r-2} & c_{r-1} & & \\ & & \cdots & \\ & & & c_1 \end{pmatrix}, \qquad \text{where } c_i = c_i(S^*) \text{ in } G(n, V).$$

This completes the proof of Theorem 1.

Remark 6. Two cases of this formula, rank at most $k-1$ and rank 0, may be proved directly. Since M_{k-1} is the locus where $\det f = 0$ and

$$\operatorname{def} f\colon \det E \longrightarrow \det E^*$$

is a section of the line bundle $(\det E^*) \otimes (\det E^*)$,

$$[M_{k-1}] = c_1((\det E^*)^{\otimes 2}) = 2c_1(E^*).$$

For M_0 we note that it is the zero locus of a section of $\mathrm{Sym}^2 E^*$, and is represented by the top Chern class $c_{\binom{k+1}{2}}(\mathrm{Sym}^2 E^*)$. A direct, though somewhat tricky, algebraic computation yields

$$c_{\binom{k+1}{2}}(\mathrm{Sym}^2 E^*) = 2^k \begin{vmatrix} c_k & c_{k+1} & \cdots & \\ c_{k-2} & c_{k-1} & & \\ & & \ddots & \\ & & & c_1 \end{vmatrix}, \qquad \text{where } c_i = c_i(E^*).$$

Remark 7. The method of reduction to the Grassmannian in the proof of Theorem 2 is not practical for a Porteous locus, since if two bundles E and F are involved, it would be necessary to embed M into a product of Grassmannians $G(m, N) \times G(n, N')$ and it is not easy to carry out explicitly the Schubert calculus on such a product.

Remark. A different approach to the proof of Theorem 1 is to mimic the standard construction used in deriving Porteous' formula: that is, we pass to the Grassmannian bundle $G = G(n-r, E)$. If Q is the universal quotient bundle on G, then we have a natural inclusion

$$\mathrm{Sym}^2 Q^* \hookrightarrow \mathrm{Sym}^2 \pi^* E^*;$$

let Σ denote the quotient bundle and $\tilde{f}$ the image in Σ of the section $\pi^* f$ of $\mathrm{Sym}^2 \pi^* E^*$. Then M_r—the locus of $x \in M$ such that f_x is the pullback to E_x of a quadric on some r-dimensional quotient of E_x—is just the image in M of the zero locus of $\tilde{f}$. The class of M_r is then just the Gysin image $\pi_* e(\Sigma)$ of the Euler class of Σ. James Damon has shown us, via the referee, how this approach can be successfully carried out.

SKEW-SYMMETRIC MAPS

Again let E be a smooth complex vector bundle of rank n over a manifold M. A bundle map $f\colon E \to E^*$ is *skew-symmetric* if $f = -{}^t f$. Such a map may be viewed as a global section of the exterior bundle $\Lambda^2 E^*$ whose fiber at each point x in M consists of all the skew forms on the fiber E_x. Locally f is represented by a skew-symmetric matrix of functions. We now let M_r be the locus of points x in M where f_x has rank at most r and ask for the cohomology class of M_r. The result is

THEOREM 8. *Let E be a smooth complex vector bundle of rank n over a compact oriented manifold M and f: $E \to E^*$ a general skew-symmetric bundle map. Then for any even r, the cohomology class of the degeneracy locus M_r is*

$$\begin{vmatrix} c_{n-r-1} & c_{n-r} & \cdots & \\ c_{n-r-3} & c_{n-r-2} & & \\ & & \ddots & \\ & & & c_1 \end{vmatrix},$$

where $c_i = c_i(E^*)$.

Proof. The proof here is exactly analogous to the symmetric case. First, we may reduce the problem to the following:

Given a nondegenerate skew form Q on $\mathbb{C}^m$, let S_r be the locus of n-planes Λ in $\mathbb{C}^m$ such that $Q|_\Lambda$ has rank at most $n - r$. Find the cohomology class of S_r in $H^*(g(n, m))$.

Secondly, since Proposition 4 applies just as well to skew forms as to quadrics, we see that

$$S_r \sim a \cdot \sigma_{r-1, r-2, \ldots, 1}$$

$$\begin{aligned} a &= S_r \cdot \sigma_{m-n, \ldots, m-n, m-n-1, \ldots, m-n-r+1} \\ &= \#\{\Lambda \in S_r\colon \dim \Lambda \cap V_{n-r+2i-1} \geq n - r + i, \qquad i = 1, \ldots, r\}. \end{aligned}$$

Finally, the number a is easy to compute: if Λ satisfies these conditions and $\Lambda_i = \Lambda \cap V_{n-r+2i-1}$, we must have

$$\begin{aligned} \Lambda_1 &= V_{n-r+1} \\ \Lambda_2 &= \text{span of } \Lambda_1 \cap V_{n-r+3} \text{ and } \Lambda_1 \\ \Lambda_3 &= \text{span of } \Lambda_2 \cap V_{n-r+5} \text{ and } \Lambda_2 \\ &\vdots \end{aligned}$$

and finally

$$\Lambda = \Lambda_r = \text{span of } \Lambda_{r-1} \cap V_{n+r-1} \text{ and } \Lambda_{r-1}.$$

Thus, $a = 1$,

$$S_r \sim \sigma_{r-1, r-2, \ldots, 1}$$

and Theorem 8 follows from Giambelli's formula as before.

(b) ***Twisted symmetric and skew-symmetric maps (the squaring principle)***

Let E be a complex vector bundle and L a complex line bundle over M. A bundle map f: $E \to E^* \otimes L$ is locally given by a square matrix of functions; f is said to be *symmetric* or *skew-symmetric* according as the local matrix is symmetric or skew-symmetric around each point. Equivalently, if $1_L: L \to L$ is the identity map and $f^*: E \otimes L^* \to E^*$ the dual of f, then f is *symmetric* if $f = f^* \otimes 1_L$, and *skew-symmetric* if $f = -f^* \otimes 1_L$. Here we identify $E \otimes L^* \otimes L$ with E by the natural pairing on L^* and L:

$$\langle\ ,\ \rangle: E \otimes L^* \otimes L \xrightarrow{\sim} E.$$

Given such a twisted map f, one may again ask for the cohomology class of its degeneracy locus.

In case the line bundle L is a square, say $L = K^2$, the answer can be easily computed from Theorem 1 for

$$(\mathrm{Sym}^2 E^*) \otimes L = \mathrm{Sym}^2 (E^* \otimes K).$$

We claim that even when L is not the square of a line bundle, the same formula holds. To justify this, we will mimic the idea of the splitting principle and establish the following "squaring principle".

PROPOSITION 9. (The squaring principle). *Given a line bundle L over a manifold M, there exist a manifold X, a map $\sigma: X \to M$ and a line bundle K on X such that*

(a) $\sigma^* L \cong K \otimes K$
(b) *the induced map in cohomology $\sigma^*: H^*(M, \mathbb{Z}) \to H^*(X, \mathbb{Z})$ is injective.*

This proposition says that we may compute the class of the degeneracy locus of a twisted symmetric bundle map $f: K \to E^* \otimes L$ by positing the existence of a line bundle $K = \sqrt{L}$ with $K^{\otimes 2} \cong L$ and applying Theorem 2 to $E \otimes K$; since the formula involves only even multiples of $c_1(K)$ and we have $2c_1(K) = c_1(L)$, the answer will in the end involve only the Chern classes of E and L. The skew-symmetric case being completely analogous, we have the

THEOREM 10. (The degeneracy locus formula for twisted symmetric maps). *Let E be a rank k complex vector bundle and L a complex line bundle over M.*

(a) *If $f: E \to E^* \otimes L$ is a general symmetric map, then the cohomology class of M_r is the same as in Theorem 1, but with $c_i = c_i(E^* \otimes \sqrt{L})$;*

(b) *If $f: E \to E^* \otimes L$ is a general skew-symmetric map, then the cohomology class of M_r is the same as in Theorem 8, but with $c_i = c_i(E^* \otimes \sqrt{L})$.*

Proof of Proposition 9. Let $f_L: M \to \mathbb{C}P^\infty$ be the classifying map of the line bundle L, and $g: \mathbb{C}P^\infty \to \mathbb{C}P^\infty$ the classifying map of $S^{\otimes 2}$, where S is the universal subbundle over $\mathbb{C}P^\infty$. Define X to be the fiber product of f_L and g. Then we have the following communative diagram

$$\begin{tikzcd}
 & & & S^{\otimes 2} \arrow[dl] \\
 & X \arrow[r, "h"] \arrow[dd, "\sigma"] & \mathbb{C}P^\infty \arrow[dd, "g"] & \\
L \arrow[dr] & & & S \arrow[dl] \\
 & M \arrow[r, "f_L"'] & \mathbb{C}P^\infty &
\end{tikzcd}$$

and

$$\sigma^* L = \sigma^* f_L^* S = h^* g^* S = h^*(S^{\otimes 2}) = (h^* S)^{\otimes 2}.$$

Hence X satisfies Property (a) of the proposition.

To show the injectivity of σ^*: $H^*(M)\to H^*(X)$, we first note that g^*: $H^*(\mathbb{C}P^\infty)\to H^*(\mathbb{C}P^\infty)$ is injective, because $H^*(\mathbb{C}P^\infty)=\mathbb{Z}[a]$ where $a=c_1(S)$, and

$$g^*(a)=g^*(c_1(S))=c_1(g^*(S))=c_1(S^{\otimes 2})=2a.$$

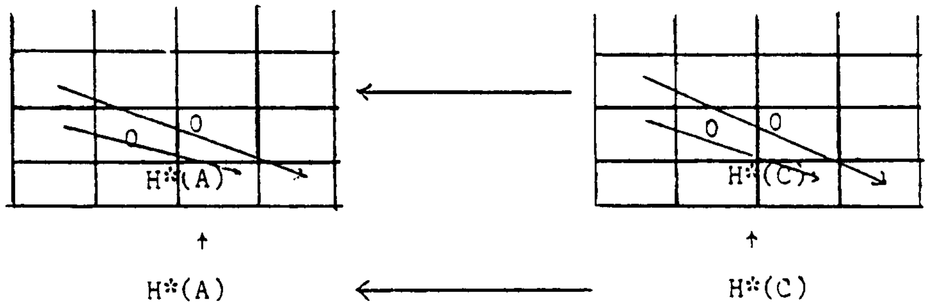
The injectivity of σ now follows from the following topology lemma.

LEMMA 11. *Let X be the fiber product of $A\to C$ and $B\to C$,*

$$\begin{array}{ccc} X & \longrightarrow & B \\ f\downarrow & & \downarrow g \\ A & \longrightarrow & C \end{array}$$

Assume C to be simply connected. If g^*: $H^*(C)\to H^*(B)$ is injective, then so is f^*: $H^*(A)\to H^*(X)$.

Proof. Recall from homotopy theory that every map is a fibering up to homotopy equivalence. So we may view g: $B\to C$ as a fibering, say with fiber F, and f: $X\to A$ has the same fiber. Since C is simply connected, the E^2 term of the spectral sequence of the fibering g is the group $H^*(C, H^*(F))$. Because g^* is injective, the differentials which hit



the bottom row must be all zero. By the commutativity of the two spectral sequence diagrams, the same is true of the differentials of f: $X\to A$. Hence f^*: $H^*(A)\to H^*(X)$ is injective.

Q.E.D.

(c) *Application: The degrees of determinantal varieties*

As an application of the degeneracy locus formulas we will now compute the degrees of determinantal varieties in a projective space.

PROPOSITION 12. (a) *Let V be the space of all $m\times n$ matrices and V_k those of rank at most k. Then in the projective space $\mathbb{P}(V)$,*

$$\deg \mathbb{P}(V_k)=\prod_{\alpha=0}^{m-k-1}\frac{\binom{n+\alpha}{m-1-\alpha}}{\binom{n-k+\alpha}{m-k+-\alpha}}.$$

(b) *Let W be the space of all $n\times n$ symmetric matrices and S_r those of corank at least r. Then in $\mathbb{P}(W)$*

$$\deg \mathbb{P}(S_r)=\prod_{\alpha=0}^{r-1}\frac{\binom{n+\alpha}{r-\alpha}}{\binom{2\alpha+1}{\alpha}}$$

(c) *If in* (b) *the symmetric matrices are replaced by the skew symmetric matrices and* $n-r$ *is even, then*

$$\deg P(S_r) = \frac{1}{2^{r-1}} \prod_{\alpha=0}^{r-2} \frac{\binom{n+\alpha}{r-1-\alpha}}{\binom{2\alpha+1}{\alpha}}.$$

Remark. In (b), $\mathbb{P}(W)$, $= \mathbb{P}^{\binom{n+1}{2}-1}$ is the space of all quadrics in $\mathbb{P}^{n-1}$ and $\mathbb{P}(S_r)$ is the subvariety of quadrics of corank of at least n.

Because the computation of all three parts are very similar, we will carry out only the symmetric case.

Let $(X_{ij})_{1\le i,j\le n}$, $X_{ij}=X_{ji}$, be the homogeneous coordinates on $\mathbb{P}(W)$. Then $\mathbb{P}(S_r)$ is the locus where rank $(X_{ij}) \le n-r$. Since (X_{ij}) may be regarded as a section of $\mathrm{Hom}^s(\mathcal{O}^{\oplus n}, \mathcal{O}^{\oplus n}\otimes\mathcal{O}(1))$ over $\mathbb{P}(W)$, $\mathbb{P}(S_r)$ is the degeneracy locus of corank r of this bundle.

If we write formally

$$\mathrm{Sym\,Hom}\,(\mathcal{O}^{\oplus n}, \mathcal{O}^{\oplus n}\otimes\mathcal{O}(1)) = \mathrm{Sym}^2\left(\mathcal{O}^{\oplus n}\otimes\mathcal{O}\left(\frac{1}{2}\right)\right) = \mathrm{Sym}^2\left(\mathcal{O}\left(\frac{1}{2}\right)^{\oplus n}\right),$$

then by Theorem 1.10, the cohomology class of $\mathbb{P}(S_r)$ is given by

$$2^r \begin{vmatrix} c_r & c_{r+1} & & \\ c_{r-2} & c_{r-1} & & \\ & & \ddots & \\ & & & c_1 \end{vmatrix}$$

where

$$c_i = c_i\left(\mathcal{O}\left(\frac{1}{2}\right)^{\oplus n}\right).$$

Since

$$c\left(\mathcal{O}\left(\frac{1}{2}\right)^{\oplus n}\right) = \left(1+\frac{1}{2}\omega\right)^n = \sum_{i=0}^{n}\binom{n}{i}\frac{\omega^i}{2^i},$$

the degree of $\mathbb{P}(S_r)$ is

$$\text{(13)} \qquad \frac{1}{2^{\binom{r}{2}}} \begin{vmatrix} \binom{n}{r} & \binom{n}{r+1} & & \\ \binom{n}{r-2} & \binom{n}{r-1} & & \\ \vdots & & \ddots & \\ & & & \binom{n}{1} \end{vmatrix}$$

To complete the proof of the proposition, then, we have to evaluate this determinant; we do this in slightly greater generality in the following lemma, whose proof may be found in [1]:

LEMMA 14. *For any n, r and $a_1, \ldots, a_r$ such that $n \geq a_i + r - 1$ for all i,*

$$\det \begin{pmatrix} \binom{n}{a_1} & \binom{n}{a_1 - 1} & \cdots & \binom{n}{a_1 - r + 1} \\ \vdots & & & \vdots \\ \binom{n}{a_r} & \binom{n}{a_r - 1} & \cdots & \binom{n}{a_r - r + 1} \end{pmatrix} = \prod_{\alpha=1}^{r} (n + r - 1 - a_\alpha)!/(a_\alpha)! \prod_{1 \leq i < j \leq r} (a_i - a_j)$$

Applying this to the matrix in (13) with rows and columns reversed, we take

$$a_i = 2i - 1$$

and compute

$$\prod_{1 \leq i < j \leq r} (a_j - a_i) = \prod_{1 \leq i < j \leq r} (2j - 2i) = 2^{\binom{r}{2}} \prod_{i=1}^{r-1} i!;$$

we conclude that

$$\deg \mathbb{P}(S_r) = \frac{1}{2^{\binom{r}{2}}} \det \begin{pmatrix} \binom{n}{r} & & \cdots & \binom{n}{2r-1} \\ \binom{n}{r-2} & & & \\ & & \ddots & \\ & \cdot & & \binom{n}{1} \end{pmatrix}$$

$$= \prod_{\alpha=1}^{r} \frac{(\alpha - 1)!(n + \alpha - 1)!}{(2\alpha - 1)!(n + r - 2\alpha)!}$$

$$= \prod_{\alpha=0}^{r-1} \frac{\alpha!(n + \alpha)!}{(2\alpha + 1)!(n - r + 2\alpha)!}$$

$$= \prod_{\alpha=0}^{r-1} \frac{\binom{n + \alpha}{r - \alpha}}{\binom{2\alpha + 1}{\alpha}}.$$

Note that if $r = n - 1$, then

$$\deg \mathbb{P}(S_r) = \frac{0!1! \ldots (n-2)!n!(n+1)! \ldots (2n-2)!}{1!3! \ldots (2n-3)!1!3! \ldots (2n-3)!}$$

$$= \frac{1}{(n-1)!} \frac{1!2!3! \ldots (2n-2)!}{(1!3! \ldots (2n-3)!)^2}$$

$$= \frac{1}{(n-1)!} \frac{2!4!6! \ldots (2n-2)!}{1!3!5! \ldots (2n-3)!}$$

$$= \frac{1}{(n-1)!} \cdot 2 \cdot 4 \cdot 6 \cdot \ldots \cdot (2n-2)$$

$$= 2^{n-1},$$

as expected (the variety $\mathbb{P}(S_{n-1})$ of double hyperplanes is just the quadric Veronese embedding of $(\mathbb{P}^{n-1})^*$ in $\mathbb{P}^{\binom{n+1}{2}-1}$); while if $r = 1$ we have

$$\deg \mathbb{P}(S_1) = n$$

as expected. If $r = 2$,

$$\deg \mathbb{P}(S_2) = \frac{(n+1)!}{3!(n-2)!} = \binom{n+1}{3},$$

and if $r = 3$,

$$\deg \mathbb{P}(S_3) = \frac{n}{3}\binom{n+2}{5}.$$

Remark. The computation of Proposition 12 in fact proves something slightly more general; namely, if L is a complex line bundle over a compact oriented manifold M and (s_{ij}) a transversal $n \times n$ symmetric matrix of global sections of L, then the cohomology class of the locus where this matrix has corank at least r is

$$\left(\prod_{\alpha=0}^{r-1} \binom{n+\alpha}{r-\alpha} \Big/ \binom{2\alpha+1}{\alpha}\right) c_1(L)^{\binom{r+1}{2}}.$$

Analogous formulas of course hold for a general matrix or a skew-symmetric matrix of sections of a line bundle.

APPLICATIONS TO VARIATION OF HODGE STRUCTURE

Let $\pi: X \to B$ be a family of curves of genus g over a compact base space, and $H^{1,0}$ and $H^{0,1}$ the associated Hodge bundles. In this section we will deduce, from the symmetric degeneracy locus formula, inequalities among the Chern numbers of the base space and the Hodge bundles when the variation of Hodge structure is nontrivial. For the definitions of these terms, see for instance [9] or [18].

Let S be the universal subbundle and Q the universal quotient bundle on the Grassmannian $G(k, n)$, and let $\phi: B \to G(g, 2g)/\Gamma$ be the period map of the family of curves above. In this context the pullback to B of the universal subbundle over $G(g, 2g)$ is the Hodge bundle $H^{1,0}$ and the pullback to B of the universal quotient bundle is $\cup H^1(X_t, \mathbb{C})/H^{1,0}(X_t) = H^{0,1}$. So the tangent bundle to $G(g, 2g)/\Gamma$ pulls back to $\mathrm{Hom}(H^{1,0}, H^{0,1})$ and the differential of the period map gives a bundle map

$$\phi_*: TB \to \mathrm{Hom}(H^{1,0}, H^{0,1}).$$

It is well known that the period map may be given locally by $g \times 2g$ matrices of the form $(I, Z(t))$, $t \in B$, where the matrix $Z(t)$ is *holomorphic* as a function of t and is *symmetric* by the Hodge–Riemann bilinear relations (Griffiths[10], pp. 613 and 616). So if v is a tangent vector to B at t, then $\phi_*(v)$ is a symmetric map from $H_t^{1,0}$ to $H_t^{0,1}$, i.e.

$$\phi_*: TB \to \mathrm{Sym}^2 (H^{1,0})^*.$$

Write E for the bundle $H^{1,0}$ as well as its pullback to TB. Then the differential of the period map may be viewed as a section of $\mathrm{Sym}^2 E^*$ over TB. Since $\phi_*(\lambda v) = \lambda\phi_*(v)$ for λ a complex number and v a tangent vector to B, if we projectivize the tangent bundle TB, ϕ_* gives rise to a section of $\mathrm{Sym}^2 E^* \otimes \mathcal{O}(1)$ over $P = \mathbb{P}(TB)$, where $\mathcal{O}(1)$ is the universal line bundle over P which restricts to the hyperplane bundle on each fiber.

We will study several cases separately.

(a) ***The base B is a curve and there are no singular fibers***

In this case the line bundle $\det(\mathrm{Sym}^2 E^* \otimes \mathcal{O}(1))$ has a holomorphic section $\det(\phi_*)$ over P. Assume that the variation of Hodge structure is not constant. Then it follows from the local Torelli theorem that the differential of the period map ϕ_* has maximal rank at some point, i.e. $\det(\phi_*)$ is not identically zero. Therefore,

$$c_1(\det(\mathrm{Sym}^2 E^* \otimes \mathcal{O}(1))) \geq 0.$$

Noting that $c_1(\mathrm{Sym}^2 E^*) = (g+1)c_1(E^*)$, this simplifies to

$$g(g(B)-1) \geq c_1(E). \tag{15}$$

By a result of Griffiths [9, Corollary 7.10, p. 147] when the base is a complete curve and there are no singular fibers, $c_1(E) \geq 0$, with strict inequality if the variation of Hodge structure is nontrivial. Combined with (15), this gives

PROPOSITION 15.1. *A family of smooth curves over $\mathbb{P}^1$ or an elliptic curve necessarily has trivial variation of Hodge structure.*

Remark. 15.2. The twisted degeneracy locus formula (Theorem 10) gives exactly the same inequality as (15), as follows. If the variation of Hodge structure over B is not trivial, then by the local Torelli theorem the period map ϕ has maximal rank at some point of B. Hence the degeneracy locus M_{g-1} is not all of B. By Theorem 10,

$$[M_{g-1}] = -2c_1\left(E\otimes\mathcal{O}\left(-\frac{1}{2}\right)\right) = -2(c_1(E) - g(g(B)-1)).$$

Because this is an effective divisor,

$$g(g(B)-1) - c_1(E) \geq 0.$$

Note that here because the tangent bundle TB is a line bundle, $P = \mathbb{P}(TB) = B$ and $\mathcal{O}(-1)$ is the tangent bundle TB.

Remark 15.3. Proposition 15.1 also follows from the fact that the classifying space $G(g, 2g)/\Gamma$ is a bounded domain in some $\mathbb{C}^n$. If the base is $\mathbb{P}^1$, the period map gives a holomorphic map from $\mathbb{P}^1$ into $\mathbb{C}^n$ and so must be a constant. If the base is a torus, the period map can be lifted to a holomorphic map $\tilde{\phi}$ on the universal covering $\mathbb{C}$ of the torus

$$\begin{tikzcd} \mathbb{C} \arrow[d] \arrow[dr, dashed, "\tilde{\phi}"] & \\ B \arrow[r, "\phi"] & G(g, 2g)/\Gamma \subset \mathbb{C}^n. \end{tikzcd}$$

Since there are no bounded entire holomorphic functions on C except the constants, $\tilde{\phi}$ and therefore ϕ also are constant.

Remark 15.4. To underscore the fact that the inequality (15) holds only if the variation of Hodge structure is nontrivial, consider a product family of smooth curves over $\mathbb{P}^1$. Inequality (15) gives $-g \geq 0$, which is clearly false. The argument breaks down because in this case $\det(\phi_*)$ is identically zero, so that the degeneracy locus M_{g-1} does not have the expected dimension.

(b) ***The base B is a curve and there are singular curves***

Let D be the divisor on B parametrizing the singular fibers. The period map ϕ is defined only over $B - D$. Now one version of the theorem on regular singular point says that the differential of the period map has at most simple poles over D. Therefore ϕ_* is a section of $\operatorname{Sym}^2 E^* \otimes \mathcal{O}(D) \otimes \mathcal{O}(1)$ over $P = \mathbb{P}(TB)$, where we write $\mathcal{O}(D)$ for the pullback of the line bundle to P. Applying the twisted degeneracy locus formula (Theorem 10), we get

$$g(g(B)-1) + \frac{\deg D}{2} \geq c_1(E). \tag{16}$$

If the nontriviality of the variation of Hodge structure is measured by the degree of the Hodge bundle $H^{1,0}$, then (16) says that *the more nontrivial the variation of Hodge structure, either the larger the genus or the base curve or the greater the number of singular fibers.*

(c) ***The base B is a surface and there are no singular fibers***

The differential of the period map ϕ_* is a section of $\operatorname{Sym}^2 E^* \otimes \mathcal{O}(1)$ over $P = \mathbb{P}(TB)$, which is a compact threefold. If it is a general section, then the degeneracy locus formula gives an effective cycle of codimension 3 on P,

$$M_{g-2} = -4(c_1c_2 - c_3) \geq 0 \tag{17}$$

where

$$c_i = c_i\left(E \otimes \mathcal{O}\left(\frac{1}{2}\right)\right).$$

Recall that if E is a rank g vector bundle and L is a line bundle, then the Chern classes of the tensor product $E \otimes L$ are given by

$$c_r(E \otimes L) = \sum_{i=0}^{r} \binom{g-i}{r-i} c_i(E) c_1^{r-i}(L).$$

Recall also that if $x = c_1(\mathcal{O}(1))$ is the hyperplane class on P and $\pi: P \to B$ is the projection map, then the Gysin map π_* is given by

$$\pi_* x = 1,$$
$$\pi_* x^2 = c_1(-B) = -c_1(B),$$
$$\pi_* x^3 = c_2(-B) = c_1^2(B) - c_2(B),$$

and so on. Using these facts, (1.20) may be simplified to

$$2((g-1)c_1^2(E) + 2c_2(E)) + (g+1)(g-1)c_1(E)c_1(B) + \binom{g+1}{3}(c_1^2(B) - c_2(B)) \geq 0.$$

This formula holds whenever the degeneracy locus of the differential of the period map, considered as a section of $\operatorname{Sym}^2 E^*$ over TB, has the expected codimension 3.

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Note. It is mentioned in Barth[2] that the formulas for the cohomology class of a symmetric determinantal variety has also appeared in the following paper:

19. T. JOSEFIAK, A. LASCOUX and P. PRAGACZ: Classes of determinantal varieties associated with symmetric and antisymmetric matrices. *Izvestiia* **45** (1981), 662–673.

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