A Characterization of the Closure of Inverse M-Matrices

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ABSTRACT

We give the explicit form of a matrix A to belong to $\overline{M^{-1}}$, the closure of inverse M-matrices. This completes a result from a previous paper of the authors and C. R. Johnson and answers a question which was raised there to characterize singular matrices in $\overline{M^{-1}}$.

1. DEFINITIONS AND NOTATION

We denote by M_n the class of all n-by-n M-matrices, i.e. matrices all of whose off-diagonal entries are nonpositive (thus belonging to Z_n) and whose inverse exists and is nonnegative. The class of all matrices whose inverse belongs to M_n, so-called inverse M-matrices, will be denoted by M_n^{-1}. Also,

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$\overline{M_n^{-1}}$ will be the closure of M_n^{-1}, i.e. the set of all matrices which are limits of convergent sequences of matrices in M_n^{-1}. Finally, we denote by $\mathcal{Q}_{rn}$ the set of all nonnegative r-by-n matrices, $r \leqslant n$, which contain in each column exactly one nonzero entry. Sometimes we write simply $\mathcal{Q}$ without subscripts if the dimensions are not specified.

We shall also mention connections with some other notions.

As is well known, the core decomposition [1] of a matrix A is expressing A as a sum $C + N$, where C is a matrix satisfying $\operatorname{rank}(C^2) = \operatorname{rank}(C)$, N is nilpotent, and $CN = NC$. This decomposition is unique, and is obtained, e.g., by finding a nonsingular matrix P such that $PAP^{-1} = J$ is in the Jordan normal form and decomposing J as $J_C + J_N$, J_N containing the blocks corresponding to the zero eigenvalue of A (and zero blocks elsewhere), and J_C all the remaining blocks (and zero blocks elsewhere). Then $C = P^{-1}J_C P$, $N = P^{-1}J_N P$.

Following the terminology in [3], we say that a matrix A is an MM-matrix if all powers of A are M-matrices.

2. RESULTS

The following theorems were proved in [2]:

THEOREM A. *A nonnegative n-by-n matrix A is in $\overline{M_n^{-1}}$ if and only if $(A + \alpha I)^{-1} \in M$ for all $\alpha > 0$, or, equivalently, if $(A + D)^{-1} \in M$ for all diagonal matrices D with positive diagonal entries.*

THEOREM B. *If $A \in M_r^{-1}$ and $Q \in \mathcal{Q}_{rn}$ then*

$$Q^T A Q \in \overline{M_n^{-1}}.$$

THEOREM C. *A nilpotent n-by-n matrix A is in $\overline{M_n^{-1}}$ if and only if it is nonnegative and $A^2 = 0$, i.e., if for some permutation matrix P,*

$$PAP^T = \begin{pmatrix} 0 & W \\ 0 & 0 \end{pmatrix},$$

where the diagonal blocks on the right-hand side are square and $W \geqslant 0$.

Theorem A implies the following

THEOREM D. *A nonnegative n-by-n matrix A is in* $\overline{M_n^{-1}}$ *if and only if*

$$(A+D)^{-1} \in M$$

for all nonnegative diagonal matrices D such that A + D is nonsingular.

Proof. If $A \in \overline{M^{-1}}$ and D is a diagonal matrix with nonnegative diagonal entries, then $A+D$ is the limit of the sequence $\{A_k\}$, $A_k = A + D + (1/k)I$, $k = 1,2,\dots,$ and $A_k \in M^{-1}$. Thus $A + D \in \overline{M^{-1}}$. If $A+D$ is, in addition, nonsingular, then $A + D \in M^{-1}$.

Conversely, let $A+D$ be in M^{-1} whenever D is a nonnegative diagonal matrix for which $A+D$ is nonsingular. Denote by A_k the matrix $A_k = A + (1/k)I$, $n = 1,2,\dots,$ and omit eventually those A_k for which A_k is singular. Then $\lim_{k\to\infty} A_k = A$, and since A_k is in M^{-1}, then A is in $\overline{M^{-1}}$. ∎

We shall complete these theorems by showing that a certain converse of Theorems B and C holds:

THEOREM 1. *An n-by-n matrix A belongs to* $\overline{M_n^{-1}}$ *if and only if there exist a permutation matrix P, a diagonal matrix D with positive diagonal entries, a matrix* $B \in M^{-1}$, *and a matrix* $Q \in \mathcal{Q}$ *without a zero row such that*

$$D^{-1}PAP^TD = \begin{pmatrix} 0 & UBQ & UBV+W \\ 0 & Q^TBQ & Q^TBV \\ 0 & 0 & 0 \end{pmatrix} \tag{0}$$

for some nonnegative matrices U, V, and W.

In (0), *any one or two of the three block rows (and their corresponding block columns) can be void.*

COROLLARY. *Singular matrices in* $\overline{M_n^{-1}}$ *are exactly matrices A for which at least one of the following occurs:* (0) *holds and either (a)* $n_1 > 0$, *or (b)* $n_3 > 0$, *or (c) Q is p-by-*n_2 *where* $1 \leqslant p < n_2$; *here,* n_1, n_2, n_3 *denote the sizes of the three block rows on the right-hand side of* (0).

Proof of Theorem 1. First, we shall state two observations and prove some lemmas.

OBSERVATION 1. *If* $A \in \overline{M^{-1}}$ *and if* A *is nonsingular, then* $A \in M^{-1}$.

OBSERVATION 2. *If* $A \in \overline{M^{-1}}$, *then* $A \geqslant 0$ *(entrywise).*

LEMMA 1. *If* $A \in \overline{M^{-1}}$, *then each principal submatrix of* A *belongs to* $\overline{M^{-1}}$ *as well. If*

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$$

and A_{11} *is nonsingular, then the Schur complement* [4]

$$A/A_{11} = A_{22} - A_{21}A_{11}^{-1}A_{12}$$

also belongs to $\overline{M^{-1}}$.

Proof. This is well known if A is nonsingular. If A is singular, it follows from the fact that A is a limit of a convergent sequence of matrices in M^{-1}. ∎

LEMMA 2. *If* $A \in \overline{M^{-1}}$, *then* adj $A \in Z$, *i.e., all off-diagonal entries of* adj A *are nonpositive.*

Proof. Since in the nonsingular case adj $A = (\det A)A^{-1}$ and $\det A > 0$, $A^{-1} \in Z$, it suffices again to prove the singular case via limit considerations. ∎

LEMMA 3. *Let* $A = (a_{ij}) \in \overline{M_n^{-1}}$. *If* $a_{kk} = 0$, *then*

$$a_{ik}a_{kj} = 0 \qquad \text{for all} \quad i, j = 1, \dots, n. \tag{1}$$

Proof. Suppose $a_{kk} = 0$, but $a_{ik}a_{kj} \neq 0$ for some i, j. Then i, j, k are distinct and

$$\hat{A} = \begin{pmatrix} a_{ii} & a_{ij} & a_{ik} \\ a_{ji} & a_{jj} & a_{jk} \\ a_{ki} & a_{kj} & a_{kk} \end{pmatrix} \in \overline{M^{-1}},$$

by Lemma 1. However,

$$\left(\operatorname{adj}\hat{A}\right)_{ij}=a_{ik}a_{kj}>0,$$

a contradiction to Lemma 2. ∎

LEMMA 4. *If $A\in\overline{M^{-1}}$ has all diagonal entries equal to zero, then there exists a permutation matrix P such that*

$$PAP^{T}=\begin{pmatrix}0&0&W\\0&0&0\\0&0&0\end{pmatrix},\tag{2}$$

where the diagonal blocks are square and W is a nonnegative matrix which has in each row as well as in each column at least one positive entry. On the right-hand side of (2), *some of the block rows (and their corresponding block columns) can be void. The first block row (and block column) is void if and only if the third block row (and block column) is void.*

Proof. By Lemma 3, each index k belongs to exactly one of the following three categories:

(a) There exists an i such that $a_{ik}\neq 0$; then $a_{kj}=0$ for all j.
(b) For all $i,j=1,\dots,n$, $a_{ik}=a_{kj}=0$.
(c) There exists a j such that $a_{kj}\neq 0$; then $a_{ik}=0$ for all i.

If we choose the permutation matrix P in such a way that the diagonal entries of PAP^T begin with indices of category (a), followed by indices of category (b) and then those of category (c), we obtain the form (2). ∎

LEMMA 5. *Let $A\in\overline{M_n^{-1}}$ have all diagonal entries different from zero. Let r be the rank of A. Then*

(a) *there exists an r-by-r principal submatrix of A which is nonsingular;*
(b) *there exist a nonsingular nonnegative diagonal matrix D, a matrix $B\in M_r^{-1}$, and a matrix $Q\in\mathcal{Q}_{rn}$ with rank r such that*

$$DAD^{-1}=Q^{T}BQ.\tag{3}$$

Conversely, if A satisfies (3) *with $B\in M_r^{-1}$, $Q\in\mathcal{Q}_{rn}$ with rank $Q=r$, and D nonsingular nonnegative diagonal, then $A\in\overline{M^{-1}}$, has rank r, and has all diagonal entries different from zero.*

Proof. Let s be the order of the largest nonsingular principal submatrix of A. By our assumption, $s \geqslant 1$. If $s = n$, the assertion is trivially true. Without loss of generality, we can assume that $s < n$ and

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix},$$

where A_{11} is s-by-s and nonsingular. By Lemma 1, the Schur complement $A/A_{11} = A_{22} - A_{21}A_{11}^{-1}A_{12}$ is in $\overline{M_{n-s}^{-1}}$. We intend to show that A/A_{11} is the zero matrix, which will prove (a).

First, all its diagonal entries are equal to zero, since otherwise there would exist a nonsingular principal submatrix of A of order $s+1$. By Lemma 4, there exists an $(n-s)$-by-$(n-s)$ permutation matrix $\hat{P}$ such that $\hat{P}(A/A_{11})\hat{P}^T$ has the form (2).

It follows easily that for the original matrix A and an appropriate permutation matrix P,

$$PAP^T = \begin{pmatrix} A_{11} & A_{11}U_1 & A_{11}U_2 & A_{11}U_3 \\ V_1^TA_{11} & V_1^TA_{11}U_1 & V_1^TA_{11}U_2 & V_1^TA_{11}U_3 + W \\ V_2^TA_{11} & V_2^TA_{11}U_1 & V_2^TA_{11}U_2 & V_2^TA_{11}U_3 \\ V_3^TA_{11} & V_3^TA_{11}U_1 & V_3^TA_{11}U_2 & V_3^TA_{11}U_3 \end{pmatrix}, \tag{4}$$

where U_i, V_j are some matrices of the appropriate size. On the right-hand side of (4), some of the last three block rows (and corresponding block columns) can be void. In fact, we shall show that the second (and by Lemma 4, also the fourth) block row and block column are void.

Suppose this is not so, and denote by $\tilde{A}(\varepsilon_1, \varepsilon_2, \varepsilon_3)$ the matrix obtained from PAP^T by adding $\varepsilon_i I_i$ to the blocks $V_i^TA_{11}U_i^T$, $i = 1,2,3$. By Theorem D, the matrix $\tilde{A}(\varepsilon_1, \varepsilon_2, \varepsilon_3)$ is in M_n^{-1} for any $\varepsilon_1, \varepsilon_2, \varepsilon_3$ positive, since it is in $\overline{M_n^{-1}}$ by Theorem A, and is nonsingular, since it can be written as

$$\begin{pmatrix} I_0 & 0 & 0 & 0 \\ V_1^T & I_1 & 0 & 0 \\ V_2^T & 0 & I_2 & 0 \\ V_3^T & 0 & 0 & I_3 \end{pmatrix} \begin{pmatrix} A_{11} & A_{11}U_1 & A_{11}U_2 & A_{11}U_3 \\ 0 & \varepsilon_1 I_1 & 0 & W \\ 0 & 0 & \varepsilon_2 I_2 & 0 \\ 0 & 0 & 0 & \varepsilon_3 I_3 \end{pmatrix}.$$

Consequently, $[\tilde{A}(\varepsilon_1, \varepsilon_2, \varepsilon_3)]^{-1} \in M$ for every triple $\varepsilon_1, \varepsilon_2, \varepsilon_3$ of positive numbers.

Since

$$\tilde{A}^{-1}(\varepsilon_1,\varepsilon_2,\varepsilon_3)=\begin{pmatrix} A_{11}^{-1} & -\dfrac{1}{\varepsilon_1}U_1 & -\dfrac{1}{\varepsilon_2}U_2 & -\dfrac{1}{\varepsilon_3}U_3+\dfrac{1}{\varepsilon_1\varepsilon_3}U_1W \\ 0 & \dfrac{1}{\varepsilon_1}I_1 & 0 & -W \\ 0 & 0 & \dfrac{1}{\varepsilon_2}I_2 & 0 \\ 0 & 0 & 0 & \dfrac{1}{\varepsilon_3}I_3 \end{pmatrix} \times\begin{pmatrix} I_0 & 0 & 0 & 0 \\ -V_1^T & I_1 & 0 & 0 \\ -V_2^T & 0 & I_2 & 0 \\ -V_3^T & 0 & 0 & I_3 \end{pmatrix}, \tag{5}$$

we obtain that its (1,2) block is $-\varepsilon_1^{-1}U_1$. Hence

$$U_1 \geqslant 0.$$

The (1,4) block is $-\varepsilon_3^{-1}U_3+(\varepsilon_1\varepsilon_3)^{-1}U_1W$. It follows that

$$U_1W \leqslant 0,$$

since otherwise this bock would not, for sufficiently small positive ε_1, be nonpositive. Since W is nonnegative and has in each row a positive entry, it follows that

$$U_1 = 0.$$

Returning to (4), it follows that our assumption that the second block row is nonvoid implies $V_1^TA_{11}U_1=0$, so that A has a zero diagonal entry, a contradiction.

Thus the second and fourth block row and column are void, and we can just consider $\tilde{A}(\varepsilon_2)$ instead of $\tilde{A}(\varepsilon_1,\varepsilon_2,\varepsilon_3)$. By (5), we have

$$\tilde{A}^{-1}(\varepsilon_2)=\begin{pmatrix} A_{11}^{-1}+\dfrac{1}{\varepsilon_2}U_2V_2^T & -\dfrac{1}{\varepsilon_2}U_2 \\ -\dfrac{1}{\varepsilon_2}V_2^T & \dfrac{1}{\varepsilon_2}I_2 \end{pmatrix}.$$

Since $\tilde{A}^{-1}(\varepsilon_2) \in M$ for all $\varepsilon_2 > 0$, we obtain

$$U_2 \geqslant 0, \qquad V_2^T \geqslant 0.$$

In addition,

$$U_2V_2^T \in Z$$

since otherwise, for sufficiently small positive ε_2, the $(1,1)$ block would not be in Z.

It follows that $U_2V_2^T$ is diagonal. No column of U_2 can be zero, since the corresponding diagonal entry of A would be zero. Similarly, no row of V_2^T can be zero. Thus both U_2 and V_2 have exactly one nonzero entry in each column, and these entries are in the same position. Consequently, there is a diagonal matrix D_1 with positive diagonal entries, such that

$$V_2 = U_2D_1. \tag{6}$$

By (4),

$$PAP^T = \begin{pmatrix} I_0 \\ V_2^T \end{pmatrix} A_{11}(I_0, U_2).$$

This can also be written, using (6), as

$$A = P^T\begin{pmatrix} I_0 & 0 \\ 0 & D_1^{1/2} \end{pmatrix}\begin{pmatrix} I_0 \\ D_1^{1/2}U_2^T \end{pmatrix} A_{11}\big(I_0, U_2D_1^{1/2}\big)\begin{pmatrix} I_0 & 0 \\ 0 & D_1^{-1/2} \end{pmatrix}P. \tag{7}$$

There exists a diagonal nonsingular matrix D with positive diagonal entries, such that

$$DP^T\begin{pmatrix} I_0 & 0 \\ 0 & D_1^{1/2} \end{pmatrix} = P^T. \tag{8}$$

Denoting then by Q the matrix

$$\big(I_0, U_2D_1^{1/2}\big)P,$$

then $Q \in \mathcal{Q}_{rn}$ has rank r and, by (7) and (8),

$$DAD^{-1} = Q^TA_{11}Q_1,$$

i.e. (3) with $B = A_{11}$.

The converse follows from Theorem B and the observation that, B belonging to M^{-1}, all of its diagonal entries are positive and this property is preserved by multiplication by Q^T and Q. ∎

Let us return to the proof of Theorem 1.

Let $A \in \overline{M^{-1}}$. Then there exists a permutation matrix P such that PAP^T has the form

$$PAP^T = \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix},$$

A_{22} containing all nonzero diagonal entries of A, A_{11} those zero diagonal entries a_{kk} for which $a_{ik} = 0$ for all i, and A_{33} the remaining diagonal entries (i.e. zero diagonal entries a_{kk} for which $a_{ik} \neq 0$ for some i). By Lemma 3 we have $A_{11} = 0$, $A_{21} = 0$, $A_{31} = 0$, $A_{32} = 0$, $A_{33} = 0$, so that

$$PAP^T = \begin{pmatrix} 0 & A_{12} & A_{13} \\ 0 & A_{22} & A_{23} \\ 0 & 0 & 0 \end{pmatrix}.$$

By Lemma 5, there exists a diagonal matrix D_2 with positive diagonal entries and a matrix $Q \in \mathcal{Q}_{rn_2}$, where r is the rank of A_{22}, such that

$$D_2 A_{22} D_2^{-1} = Q^T B Q,$$

where $B \in M_r^{-1}$. Without loss of generality, we can assume that $Q = (I, Z)$. Choosing D as block diagonal with diagonal blocks I_1, D_2, and I_3, we have then

$$D^{-1}PAP^T D = \begin{pmatrix} 0 & \tilde{A}_{12} & \tilde{A}_{13} & \tilde{A}_{14} \\ 0 & B & BZ & \tilde{A}_{24} \\ 0 & Z^T B & Z^T BZ & \tilde{A}_{34} \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{9}$$

Denote by $\tilde{A}(\varepsilon_1, \varepsilon_3, \varepsilon_4)$ the matrix

$$\tilde{A}(\varepsilon_1, \varepsilon_3, \varepsilon_4) = \begin{pmatrix} \varepsilon_1 I_1 & \tilde{A}_{12} & \tilde{A}_{13} & \tilde{A}_{14} \\ 0 & B & BZ & \tilde{A}_{24} \\ 0 & Z^T B & Z^T BZ + \varepsilon_3 I_3 & \tilde{A}_{34} \\ 0 & 0 & 0 & \varepsilon_4 I_4 \end{pmatrix}. \tag{10}$$

Since this matrix is nonsingular for every triple $\varepsilon_1, \varepsilon_3, \varepsilon_4$ of positive numbers, it belongs to M_n^{-1} by Theorem D for every such triple. Therefore, for some nonnegative matrices $B_{12}, B_{13}, B_{14}, B_{24}, B_{34}$,

$$\tilde{A}^{-1}(\varepsilon_1, \varepsilon_3, \varepsilon_4) = \begin{pmatrix} \frac{1}{\varepsilon_1}I_1 & -B_{12} & -B_{13} & -B_{14} \\ 0 & B^{-1} + \frac{1}{\varepsilon_3}ZZ^T & -\frac{1}{\varepsilon_3}Z & -B_{24} \\ 0 & -\frac{1}{\varepsilon_3}Z^T & \frac{1}{\varepsilon_3}I_3 & -B_{34} \\ 0 & 0 & 0 & \frac{1}{\varepsilon_4}I_4 \end{pmatrix}, \tag{11}$$

where

$$\begin{pmatrix} B_{12} & B_{13} \end{pmatrix} = \frac{1}{\varepsilon_1}\begin{pmatrix} \tilde{A}_{12} & \tilde{A}_{13} \end{pmatrix}\begin{pmatrix} B^{-1} + \frac{1}{\varepsilon_3}ZZ^T & -\frac{1}{\varepsilon_3}Z \\ -\frac{1}{\varepsilon_3}Z^T & \frac{1}{\varepsilon_3}I \end{pmatrix}, \tag{12}$$

$$\begin{pmatrix} B_{24} \\ B_{34} \end{pmatrix} = \frac{1}{\varepsilon_4}\begin{pmatrix} B^{-1} + \frac{1}{\varepsilon_3}ZZ^T & -\frac{1}{\varepsilon_3}Z \\ -\frac{1}{\varepsilon_3}Z^T & \frac{1}{\varepsilon_3}I \end{pmatrix}\begin{pmatrix} \tilde{A}_{24} \\ \tilde{A}_{34} \end{pmatrix}, \tag{13}$$

$$B_{14} = \frac{1}{\varepsilon_1\varepsilon_4}\tilde{A}_{14} - \frac{1}{\varepsilon_1\varepsilon_4}\begin{pmatrix} \tilde{A}_{12} & \tilde{A}_{13} \end{pmatrix}\begin{pmatrix} B^{-1} + \frac{1}{\varepsilon_3}ZZ^T & -\frac{1}{\varepsilon_3}Z \\ -\frac{1}{\varepsilon_3}Z^T & \frac{1}{\varepsilon_3}I \end{pmatrix}\begin{pmatrix} \tilde{A}_{24} \\ \tilde{A}_{34} \end{pmatrix}. \tag{14}$$

By (12),

$$\begin{aligned} B_{12} &= \frac{1}{\varepsilon_1}\tilde{A}_{12}B^{-1} + \frac{1}{\varepsilon_1\varepsilon_3}(\tilde{A}_{12}Z - \tilde{A}_{13})Z^T, \\ B_{13} &= -\frac{1}{\varepsilon_1\varepsilon_3}(\tilde{A}_{12}Z - \tilde{A}_{13}), \end{aligned} \tag{15}$$

which implies

$$\tilde{A}_{12}Z - \tilde{A}_{13} \leqslant 0.$$

Also, $(\tilde{A}_{12}Z - \tilde{A}_{13})Z^T \geqslant 0$, since otherwise B_{12} would not be nonnegative for sufficiently small positive ε_3. Since $Z^T \geqslant 0$ and has in each row a positive entry, we obtain

$$\tilde{A}_{12}Z - \tilde{A}_{13} = 0.$$

By (15),

$$\tilde{A}_{12}B^{-1} \geqslant 0.$$

Denoting the left-hand side by U, we have

$$\tilde{A}_{12} = UB, \qquad \tilde{A}_{13} = UBZ.$$

Thus

$$\begin{pmatrix} A_{12} & \tilde{A}_{13} \end{pmatrix} = UBQ, \qquad U \geqslant 0. \tag{16}$$

Similarly, it follows from (13) that

$$\begin{pmatrix} \tilde{A}_{24} \\ \tilde{A}_{34} \end{pmatrix} = Q^T BV, \qquad V \geqslant 0. \tag{17}$$

Finally, (14) yields

$$B_{14} = \frac{1}{\varepsilon_1 \varepsilon_4} \tilde{A}_{14} - \frac{1}{\varepsilon_1 \varepsilon_4} \begin{pmatrix} \tilde{A}_{12} & \tilde{A}_{13} \end{pmatrix} \begin{pmatrix} B^{-1} & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \tilde{A}_{24} \\ \tilde{A}_{34} \end{pmatrix},$$

or

$$B_{14} = \frac{1}{\varepsilon_1 \varepsilon_4} \left(\tilde{A}_{14} - UBV \right),$$

i.e.,

$$\tilde{A}_{14} - UBV \geqslant 0.$$

Denoting the left-hand side by W, we have

$$\tilde{A}_{14} = UBV + W, \qquad W \geqslant 0. \tag{18}$$

However, (15), (16), (17), and (18) prove (0).

Conversely, let (0) hold with the conditions given above. Since Q does not have a zero row, we can assume without loss of generality that Q has the form

$$Q = (I, Z).$$

Then (0) can be written as

$$D^{-1}PAP^TD = \begin{pmatrix} 0 & UB & UBZ & UBV+W \\ 0 & B & BZ & BV \\ 0 & Z^TB & Z^TBZ & Z^TBV \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Define the matrix $\tilde{A}(\varepsilon_1, \varepsilon_3, \varepsilon_4)$ as in (10), with $\tilde{A}_{12} = UB$, $\tilde{A}_{13} = UBZ$, $\tilde{A}_{14} = UBV + W$, $\tilde{A}_{24} = BV$, $\tilde{A}_{34} = Z^TBV$. The matrix $\tilde{A}(\varepsilon_1, \varepsilon_3, \varepsilon_4)$ is then nonsingular for every triple $\varepsilon_1, \varepsilon_3, \varepsilon_4$ of positive numbers, and its inverse is given by (11), (12), (13), and (14), i.e.

$$B_{12} = \frac{1}{\varepsilon_1}U, \qquad B_{13} = 0,$$

$$B_{24} = \frac{1}{\varepsilon_4}V, \qquad B_{34} = 0,$$

$$B_{14} = \frac{1}{\varepsilon_1\varepsilon_4}W.$$

Since ZZ^T is diagonal, the matrix $\tilde{A}^{-1}(\varepsilon_1, \varepsilon_3, \varepsilon_4)$ is in Z and $\tilde{A}(\varepsilon_1, \varepsilon_3, \varepsilon_4)$ is thus in M_n^{-1} for every triple $\varepsilon_1, \varepsilon_3, \varepsilon_4$ of positive numbers. Thus $A \in \overline{M_n^{-1}}$ as the limit of the convergent sequence $A^{(k)} = P^TD\tilde{A}(1/k, 1/k, 1/k)D^{-1}P$ of matrices in M_n^{-1}. ■

We add two observations which may be of interest.

THEOREM 2. *Let $A \in \overline{M_n^{-1}}$, and let $A = C + N$ be the core decomposition of A. Then both C and N belong to $\overline{M_n^{-1}}$.*

Proof. If A is nonsingular, $N=0$ and the result is true. Let now A be singular. By (0), A is similar to the matrix

$$\tilde{A}=\begin{pmatrix} 0 & UBQ & UBV+W \\ 0 & Q^TBQ & Q^TBV \\ 0 & 0 & 0 \end{pmatrix},$$

where $B\in M^{-1}$, $Q\in\mathcal{Q}$, and U,V,W are nonnegative matrices.

The core $\tilde{C}$ of $\tilde{A}$ is the matrix

$$\tilde{C}=\begin{pmatrix} 0 & UBQ & UBV \\ 0 & Q^TBQ & Q^TBV \\ 0 & 0 & 0 \end{pmatrix}.$$

Indeed,

$$\tilde{C}^2=\begin{pmatrix} 0 & U\hat{B}Q & U\hat{B}V \\ 0 & Q^T\hat{B}Q & Q^T\hat{B}V \\ 0 & 0 & 0 \end{pmatrix},$$

where $\hat{B}=BQQ^TB$.

Since $r(\tilde{C}^2)\leqslant r(\tilde{C})$, to show that equality is obtained, it suffices to prove the implication

$$\tilde{C}^2x=0 \quad\to\quad \tilde{C}x=0.$$

Thus let

$$\tilde{C}^2\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}=0,$$

i.e.,

$$U\hat{B}Qx_2+U\hat{B}Vx_3=0, \tag{19}$$

$$Q^T\hat{B}Qx_2+Q^T\hat{B}Vx_3=0. \tag{20}$$

Since both B and QQ^T are nonsingular and since Q^T has full column rank, (20) implies that

$$Qx_2+Vx_3=0.$$

But then both equations obtained analogously to (19) and (20) from

$$\tilde{C}x = 0$$

are fulfilled.

The matrix

$$\tilde{N} = \begin{pmatrix} 0 & 0 & W \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

is clearly nilpotent and satisfies $\tilde{N}\tilde{C} = \tilde{C}\tilde{N} = 0$. Thus

$$\tilde{A} = \tilde{C} + \tilde{N}$$

is the unique core decomposition of $\tilde{A}$, and both matrices $\tilde{C}$ and $\tilde{N}$ are in $\overline{M_n^{-1}}$ by Theorem 1. The same is, of course, true for the corresponding decomposition of the original matrix A. ∎

REMARK. A similar result for the closure $\overline{M}$ of M-matrices does not hold. For

$$A = \begin{pmatrix} 0 & -1 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix},$$

the core decomposition is easily seen to be $A = C + N$ with

$$C = \begin{pmatrix} 0 & -1 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad N = \begin{pmatrix} 0 & 0 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

It is obvious that C is not in $\overline{M}$.

THEOREM 3. *Let B in (0) satisfy the additional condition that B^{-1} is an $\underline{MM}$-matrix and $QQ^T = I_r$. Then A is an $\overline{MM^{-1}}$-matrix, i.e., all its powers are in $\overline{M^{-1}}$.*

Proof. This follows from the fact that for each positive integer s, the sth power of A has the same structure as A with B^s instead of B. Since B^s is in M^{-1}, the result follows. ∎

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Received May 1987; final manuscript accepted 23 September 1987