On minimal $\pi$-character of points in extremally disconnected compact spaces

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Abstract

We consider the relationship between $\pi$-character, refinement number (=weak density) and $\pi$-weight in complete Boolean algebras. As an application we shall show that every extremally disconnected compact space contains a point which is not an accumulation point of any countable discrete subset, provided that minimal $\pi$-character and $\pi$-weight coincide.

Keywords: Ultrafilter, $\pi$-character, $\pi$-weight, refinement number, discretely untouchable point.

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Introduction

The starting point of our investigation was the famous Frolik theorem, stating that no infinite extremally disconnected compact space is homogeneous. In [8], Dow asked whether every extremally disconnected compact space contains a point, which is not an accumulation point of any countable discrete set. It turns out that this problem leads to a careful examination of mutual relationship between $\pi$-weight and $\pi$-character of ultrafilters in complete Boolean algebras. Our results in Section 1 partly overlap with the recent paper of Bozeman [4]. We shall show that a minimal $\pi$-character of an ultrafilter equals the refinement number as defined in [15] in every Boolean algebra, which is either homogeneous or complete.
For ccc complete Boolean algebras we show that the lower bound for $\pi$-character of ultrafilters is given by cardinal characteristics of functions and subsets of natural numbers.

The notation used throughout the paper is the standard one. The Boolean operations will be denoted by $\lor$ (join) and $\land$ (meet). Ult($\mathcal{B}$) denotes the Stone space of a Boolean algebra $\mathcal{B}$ and the completion of $\mathcal{B}$ is denoted by $\text{Compl}(\mathcal{B})$.

If $\mathcal{B}$ is a Boolean algebra and $X \subseteq \mathcal{B}$, then $X^+$ will stand for $X \setminus \{0\}$. For $u \in \mathcal{B}$, $X \upharpoonright u = \{x \land u : x \in X\}$. We shall say that a set $Y \subseteq \mathcal{B}$ is dense below $X$, if $(\forall x \in X^+) (\exists y \in Y^+) (y \preceq x)$. A set dense below $\mathcal{B}$ is called a $\pi$-base of $\mathcal{B}$; the smallest size of a $\pi$-base is called a $\pi$-weight of $\mathcal{B}$ and is denoted by $\pi w(\mathcal{B})$. If $\mathcal{U}$ is a ultrafilter on $\mathcal{B}$ and $X$ is dense below $\mathcal{U}$, then $X$ is called a local $\pi$-base of $\mathcal{U}$. The $\pi$-character of an ultrafilter $\mathcal{U}$ is
\[
\pi X(\mathcal{U}) = \min\{|X| : X \text{ is dense below } \mathcal{U}\}.
\]
If $\phi$ is a cardinal invariant of Boolean algebras, we shall say that a Boolean algebra $\mathcal{B}$ is homogeneous in $\phi$, if $\phi(\mathcal{B} \upharpoonright u) = \phi(\mathcal{B})$ for every $u \in \mathcal{B}^+$.

For the unexplained notation, see e.g. [14].

1. $\pi$-characters

Let us remind several notions and cardinal invariants of Boolean algebras, which we shall deal with.

**Definition 1.1.** (i) Let $\mathcal{B}$ be a Boolean algebra, $X \subseteq \mathcal{B}^+$. Call an element $u \in \mathcal{B}$ independent with respect to the set $X$, if both $u$ and $-u$ meet each $x \in X$, i.e. $(\forall x \in X) (x \land u \neq 0 \neq x \land -u)$.

(ii) The refinement number of $\mathcal{B}$ is
\[
\text{r}(\mathcal{B}) = \min\{|X| : \text{there is no } u \in \mathcal{B} \text{ independent w.r.t. } X\}.
\]

The name "refinement number" was first used in [15] for the Boolean algebra $\mathcal{P}(\mathbb{N})/\text{fin}$. The same notion is called "reaping number" in [11] and "weak density" in [4, 5].

**Notation 1.2.** The set of all partitions of the unity of a Boolean algebra $\mathcal{B}$ will be denoted by Part($\mathcal{B}$). If $k$ is a positive integer, then
\[
\text{Part}_k(\mathcal{B}) = \{P \in \text{Part}(\mathcal{B}) : |P| \leq k\},
\]
\[
\text{Part}_{\text{fin}}(\mathcal{B}) = \bigcup \{\text{Part}_k(\mathcal{B}) : k \in \mathbb{N}\}.
\]

It is immediately clear that for an arbitrary Boolean algebra $\mathcal{B}$ we have
\[
\text{r}(\mathcal{B}) = \min\{|X| : X \subseteq \mathcal{B}^+ \text{ and } (\forall P \in \text{Part}_2(\mathcal{B})) (\exists p \in P) (\exists x \in X) (x \preceq p)\}.
\]
We may, of course, consider not only partitions into two pieces, but more general $k$-partitions, all finite partitions or all partitions of the unity of $B$ in the right-hand side of formula (*). This way we get formally more general refinement numbers of $B$, namely $r_1(B)$, $r_{\infty}(B)$ and $r_{\infty}(B)$. Thus $r(B)$ is the same as $r_2(B)$. Obviously,

$$r_2(B) \leq r_3(B) \leq \cdots \leq r_{\infty}(B) \leq r_{\infty}(B).$$

Given some system of elements of a Boolean algebra, we ask, whether the system is a local $\pi$-base of some ultrafilter.

**Lemma 1.3.** Let $B$ be a Boolean algebra, $X \subseteq B$. For every $P \in \text{Part}_{\text{fin}}(B)$ let us define $w_P = \bigvee \{ p \in P : (\exists x \in X^+) \ x \leq p \}$.

(i) The set $X$ is a local $\pi$-base of some ultrafilter on $B$ if and only if $w_P \neq \emptyset$ for every $P \in \text{Part}_{\text{fin}}(B)$.

(ii) $X$ is a local $\pi$-base of an ultrafilter $\mathcal{U}$ if and only if $\{ w_P : P \in \text{Part}_{\text{fin}}(B) \} \subseteq \mathcal{U}$.

**Proof.** If $X$ is a local $\pi$-base of an ultrafilter $\mathcal{U}$, then for every finite partition $P$ there is a $p \in \mathcal{U} \cap P$ and hence we have an $x \in X^+$ such that $x \leq p$. Since $x \leq w_P$, $w_P$ is nonzero.

On the other hand, assume that $w_P \neq \emptyset$ for every $P \in \text{Part}_{\text{fin}}(B)$. Since $w_{P_1 \land P_2} \leq w_{P_1} \land w_{P_2}$, where $P_1 \land P_2$ denotes the coarsest refinement of both $P_1$ and $P_2$, the family $\mathcal{F} = \{ w_P : P \in \text{Part}_{\text{fin}}(B) \}$ has the finite intersection property. Let $\mathcal{U}$ be an arbitrary ultrafilter extending $\mathcal{F}$. Consider an arbitrary $u \in \mathcal{U}$. The element $w_{\{ u, -u \}}$ cannot be disjoint with $u$, which means that for some $x \in X^+$ we have $x \leq u$. We have proved that $X$ is a local $\pi$-base of $\mathcal{U}$.

It remains to show that if for some partition $P$ we have $w_P \notin \mathcal{V} \in \text{Ult}(B)$, then $X$ is not a local $\pi$-base of $\mathcal{V}$. In that case there is a unique $p \in P \cap \mathcal{V}$ and there is no element in $X^+$, which is below $p \land -w_P \in \mathcal{V}$. \(\Box\)

The statement admits the following strengthening for the case of complete Boolean algebras.

**Theorem 1.4.** Let $B$ be a complete Boolean algebra, let $X \subseteq B$ be closed under finite meets. Then $X$ is a local $\pi$-base of some ultrafilter if and only if no element of $B$ is independent with respect to $X$.

**Proof.** The case of a finite Boolean algebra is trivial. Assume that $B$ is an infinite complete Boolean algebra. If $X$ is a local $\pi$-base of some ultrafilter $\mathcal{U}$, then it is easy to see that there is no element of $B$, which is independent with respect to $X$. In this implication, the completeness of $B$ is not needed.

Assume now that $X$ is not a local $\pi$-base of any ultrafilter and that there is no element of $B$ independent with respect to $X$. By the previous lemma, there is some finite partition $P = \{ p_0, p_1, \ldots, p_n \}$ such that no $x \in X^+$ is below any $p_i$. Necessarily $n \geq 2$ and every member of $X$ meets at least two members of the partition $P$. Set

$$Y = \{ y \in X : (\forall z \in X) \ (z \leq y \rightarrow \{ p \in P : p \land z \neq \emptyset \}) = \{ p \in P : p \land y \neq \emptyset \} \}. $$
The set \( Y \) is dense in \( X \). Let us choose a maximal disjoint subset \( D \subseteq Y \). For an element \( d \in D \), let us denote \( \varphi(d) \) the smallest number in the set \( \{0, 1, \ldots, n\} \) for which \( d \land p_{\varphi(d)} \neq 0 \). Now, consider \( u_0 = \bigvee \{d \land p_{\varphi(d)} : d \in D\} \) and \( u_1 = -u_0 \). We shall verify that \( u_0 \) is independent with respect to \( X \), which will be the desired contradiction.

Let \( y_0 \in X \) be arbitrary. By the maximality, there is some \( d \in D \) such that \( y_1 = y_0 \land d \neq 0 \). Since the set \( X \) is closed under finite meets, the element \( y_1 \) meets the same members of the partition \( P \) as \( d \) does. Thus \( y_0 \land u_0 \supseteq y_0 \land d \land p_{\varphi(d)} \neq 0 \). However, there is also some \( i \neq \varphi(d) \), \( i \in \{0, 1, \ldots, n\} \), such that \( p_i \land d \neq 0 \). We have \( 0 \neq y_1 \land p_i \leq p_i \land d \leq u_1 \), therefore \( y_0 \land u_1 \neq 0 \), too. \( \square \)

**Notation 1.5.**

\[
\pi_X(R) = \min\{\pi_X(U) : U \in \text{Ult}(R)\},
\]

\[
\pi_R(R) = \min\{\pi_R(R \upharpoonright u) : u \in R^+\}.
\]

**Proposition 1.6.** The following two equalities hold for an arbitrary Boolean algebra \( B \):

(i) \( r_{\text{fin}}(B) = \pi_X(B) \);

(ii) \( r_\kappa(B) = \pi_R(B) \).

**Proof.** (i) follows from Lemma 1.3.

Let us show (ii). To verify \( \leq \), take some \( u \in R^+ \) with \( \pi_R(R \upharpoonright u) \) minimal possible, then choose some set \( X \), which is dense in the algebra \( R \upharpoonright u \). For an arbitrary partition \( P \in \text{Part}(R) \) there is some \( p \in P \) such that \( p \land u \neq 0 \). Hence \( p \land u \in R \upharpoonright u \), therefore there is some \( x \in X \) satisfying \( x \leq p \land u \leq p \).

Finally, suppose that \( X \) is a collection showing \( r_\kappa(B) \). Aiming for a contradiction, assume \( |X| < \pi_R(R) \). Then the set \( \{v \in R^+ : (\exists x \subseteq X)(x < v)\} \) is dense in \( R \). If we consider any partition consisting from members of this dense set, we reach a contradiction to the assumption that \( X \) guarantees \( r_\kappa(B) \). \( \square \)

**Theorem 1.7.** If a Boolean algebra \( B \) is homogeneous or complete, then \( r(B) = r_{\text{fin}}(B) \).

In other words, the refinement number of \( B \) equals to the minimal \( \kappa \) character of an ultrafilter on \( B \).

**Proof.** Denote \( \kappa = r(B) \). We may assume that there are no atoms in \( B \), because for any Boolean algebra with atoms the equality \( r = r_\kappa \) holds.

(a) Suppose \( B \) is homogeneous. Then \( B \) is infinite and for every \( u \in B^+ \) it is true that \( r(B \upharpoonright u) = r(B) = \kappa \geq \omega \). For every \( u \in B^+ \) select a family \( X_u \subseteq (B \upharpoonright u)^+ \), which witnesses to \( r(B \upharpoonright u) \). Define by an induction \( Y_0 = X_1, Y_{n+1} = \bigcup \{X_u : u \in Y_n\} \). Since \( |Y_n| \leq \kappa \cdot \kappa = \kappa \), the set \( Y = \bigcup \{Y_n : n \in \omega\} \) is of size \( \kappa \).

It remains to verify that \( Y \) witnesses to \( r_{\text{fin}}(B) \). Suppose that the following holds for some positive integer \( k \):

\[
(\forall P \in \text{Part}_k(B)) \left( \exists p \in P \left( \exists y \in Y \right) (y \leq p) \right).
\]
Let \( Q = \{q_0, q_1, \ldots, q_k\} \) be a \((k+1)\)-partition of unity. Then for the \(k\)-partition \( \{q_0 \lor q_1, q_2, \ldots, q_k\} \) there exists a member \( y \in Y \) such that either \( y \leq q_j \) for some \( j = 2, \ldots, k \) and we are done, or \( y \leq q_0 \lor q_1 \). In the case that neither \( y \leq q_0 \) nor \( y \leq q_1 \) is true, then \( \{q_0 \land y, q_1 \land y\} \in \text{Part}_2(\mathcal{B} \upharpoonright y) \) and then there exists some \( z \in X_y \subset Y \), which is below some element of this 2-partition.

(b) follows from Theorem 1.4.

The first author was kindly informed by Bill Weiss that he had proved case (b) using a different technique several years ago (unpublished).

The equality between refinement number and the minimal \( \pi \)-character does not hold in the general case of Boolean algebras. We shall present elsewhere an example of a Boolean algebra \( \mathcal{B} \) with \( r_\mathcal{B} < r_\mathcal{B} \).

The main point of our interest is the relation between characteristics \( \pi_\mathcal{B}(\mathcal{B}) = r_\mathcal{B} \) and \( \pi_\mathcal{B}(\mathcal{B}) = r_\mathcal{B} \) for complete Boolean algebras. The inequality \( \pi_\mathcal{B}(\mathcal{B}) \leq \pi_\mathcal{B}(\mathcal{B}) \) is trivial. It is also easy to find an example of an incomplete Boolean algebra, where the sharp inequality holds: Consider for example an algebra of all clopen subsets of a one-point compactification of \( \omega \times \{0, 1\}^\omega \).

We must confess that we know no example of a complete Boolean algebra, for which the sharp inequality \( r_\mathcal{B} < r_\mathcal{B} \) is true.

In the sequel we shall show first some combinatorial consequences of the equality \( r_\mathcal{B} = r_\mathcal{B} \) and then we shall prove that GCH implies \( r_\mathcal{B} = r_\mathcal{B} \) for all complete Boolean algebras with ccc.

The next observation says that we can restrict our considerations to the algebras, which are homogeneous in \( \pi \)-weight.

**Proposition 1.8.** Let \( \mathcal{B} \) be a complete Boolean algebra and suppose that \( X \subset \mathcal{B} \) is closed under finite meets. If \( X \) is a local \( \pi \)-base of some ultrafilter, then there is some element \( u \in \mathcal{B}^+ \) such that for every \( v \in \mathcal{B}^+ \), \( v \leq u \), the set \( X \upharpoonright v \) is a local \( \pi \)-base of some ultrafilter.

**Proof.** We shall make use of Theorem 1.4. Put \( I = \{a \in \mathcal{B} : X \upharpoonright a \) is not a local \( \pi \)-base of any ultrafilter}. If \( \{a_j : j \in J\} \subset I \) is a disjoint system, then for every \( j \in J \) there is some \( c_j < a_j \) which is independent with respect to \( X \upharpoonright a_j \). The element \( c = \bigvee \{c_j : j \in J\} \) is independent with respect to \( X \upharpoonright \bigvee \{a_j : j \in J\} \), which means that \( \bigvee \{a_j : j \in J\} \notin I \). It is enough to set \( u = \neg \bigvee A \), where \( A \) is a maximal disjoint subset of \( I \). Notice that \( u \neq \emptyset \), because \( X \) is a local \( \pi \)-base of some ultrafilter. \( \square \)

**Corollary 1.9.** If \( \mathcal{B} \) is a complete Boolean algebra, then there is some \( u \in \mathcal{B}^+ \) homogeneous in \( \pi \)-weight such that \( \pi_\mathcal{B}(\mathcal{B}) = \pi_\mathcal{B}(\mathcal{B} \upharpoonright u) \).

Therefore when examining the minimal \( \pi \)-character of ultrafilters in complete Boolean algebras we may consider only algebras, which are homogeneous in \( \pi \)-weight and cellularity.
Definition 1.10. We say that a Boolean algebra \( B \) has an increasing chain interleaved with independent elements, if there is some cardinal number \( \tau \), an increasing chain \( \{ H_\alpha : \alpha < \tau \} \) of subsets of \( B \) and a sequence of elements \( \{ p_\alpha : \alpha < \tau \} \subseteq B^+ \) such that the following are satisfied:

(i) if \( \alpha < \beta < \tau \), then \( H_\alpha \subseteq H_\beta \);
(ii) each \( H_\alpha \subseteq B^+ \) is closed under finite meets;
(iii) \( \bigcup \{ H_\alpha : \alpha < \tau \} \) is a dense subset of \( B \);
(iv) for each \( \alpha < \tau \), both \( p_\alpha, -p_\alpha \in H_{\alpha + 1} \);
(v) an element \( p_\alpha \) is independent with respect to \( H_\alpha \).

Theorem 1.11. Let \( B \) be an infinite complete Boolean algebra which is homogeneous in \( \pi \)-weight. Whenever \( \pi_X(B) = \pi(B) \), then \( B \) has an increasing chain interleaved with independent elements of length \( \pi_X(B) \).

Proof. Denote \( \pi_X(B) = \tau \). Choose a dense set \( H = \{ h_\alpha : \alpha < \tau \} \) in \( B \). By Theorem 1.4, we know that for every set \( X \subseteq B^+ \) of size less than \( \tau \) there is an element \( p \in B \), which is independent w.r.t. \( X \). From that it is obvious, how to get independent elements during a transfinite induction. Suppose we have already found \( H_\alpha \) and \( p_\alpha \) for all \( \alpha < \beta \). Let \( H_\beta \) be the closure under finite meets of the set \( \bigcup_{\alpha < \beta} H_\alpha \cup \{ p_\alpha, -p_\alpha : \alpha < \beta \} \cup \{ h_\beta \} \). Then \( |H_\beta| \leq |\beta \cdot \omega| < \tau \), thus we can find an element \( p_\beta \in B \) independent with respect to \( H_\beta \). Since \( H \subseteq \bigcup_{\alpha < \tau} H_\alpha \), the condition (iii) from the definition is also satisfied. \( \square \)

Example 1.12. The algebra \( B_0 = \mathcal{P}(\omega)/\text{fin} \) versus its completion \( B_1 = \text{Compl}(\mathcal{P}(\omega)/\text{fin}) \).

(i) The algebra \( B_0 \) is not complete, but it is homogeneous and \( \piw(B_0) = \piw(B_1) = 2^\omega \). Therefore refinement number of \( B_0 \) equals to the minimal \( \pi \)-character in \( B_0 \), i.e. \( r(B_0) = \pi_X(B_0) \).

It is well known that \( \pi_X(B_0) \) can be less than \( 2^\omega \) [12]. Recently, Shelah and Goldstern proved that even the inequality \( \pi_X(B_0) < \chi(B_0) \) is consistent [11].

Let us remark that CH implies that \( B_0 \) has an increasing chain interleaved with independent elements.

(ii) For the complete Boolean algebra \( B_1 \), one has the equality \( \pi_X(B_1) = \piw(B_1) = 2^\omega \). This follows from the fact that every system \( \{ u_\alpha : \alpha < \kappa \} \) has a disjoint refinement, provided \( \kappa < 2^\omega \), i.e. there is a disjoint system \( \{ d_\alpha : \alpha < \kappa \} \subseteq B_1^+ \) such that \( d_\alpha \leq u_\alpha \) for every \( \alpha < \kappa \).

From Theorem 1.11, it follows that the algebra \( B_1 \) has always an increasing chain interleaved with independent elements of length \( 2^\omega \). Let us remark that the cardinal characteristics \( h \) is the smallest cardinal \( \tau \) such that the algebra \( B_0 \) (or, equivalently, \( B_1 \)) is not \( (\tau, \cdot, 2) \)-distributive [1, 2]. It is not difficult to see that \( h \) is the smallest cardinal number such that \( B_1 \) has an increasing chain interleaved with independent elements of length \( h \).

Our aim now is to prove the following statement.
Theorem 1.13. Suppose that every cardinal number \( \lambda \) with uncountable cofinality satisfies \( \lambda^\omega = \lambda \). Then for every complete ccc Boolean algebra \( B \), \( r_{\text{fin}}(B) = r_{\infty}(B) \), or equivalently, \( \pi_X(B) = \pi_W(B) \).

First, we shall give a couple of general lemmas.

Lemma 1.14. Let \( B \) be an infinite Boolean algebra homogeneous in \( \pi \)-weight. Then for an arbitrary non-void \( X \subseteq B^+ \), \( |X| < \pi_W(B) \), there is a partition \( P \) of the unity such that every \( x \in X \) meets infinitely many members of \( P \).

Proof. The partition \( P \) will be constructed by induction. As \( |X| < \pi_W(B) \), there is some \( p_0 \in B^+ \) such that \( X \cap B \upharpoonright p_0 = \emptyset \). Necessarily \( p_0 \not\in B \). Let us choose some nonzero \( w_1 \) disjoint with \( p_0 \) and consider \( X \upharpoonright w_1 \). From the homogeneity of \( B \) in \( \pi \)-weight it follows that there is some \( 0 < p_1 < w_1 \) such that \( X \upharpoonright w_1 \cap B \upharpoonright p_1 \subseteq \{0\} \).

Suppose we have constructed a disjoint system \( \{ p_\alpha : \alpha < \beta \} \). If it is maximal, then it is the desired partition of unity, \( P \). If it is not maximal, then there is some nonzero \( w_\beta \), which is disjoint with all the \( p_\alpha \). Then for \( p_\beta \) we can choose an arbitrary nonzero element satisfying \( p_\beta < w_\beta \), \( X \upharpoonright w_\beta \cap B \upharpoonright p_\beta \subseteq \{0\} \). Finally we must get a maximal disjoint system \( P = \{ p_\alpha : \alpha < \gamma \} \).

If there were some \( x \in X \) meeting only finitely many elements of the partition, then there would exist the biggest \( \alpha \) with \( x \wedge p_\alpha \neq 0 \). For this \( \alpha \), \( 0 \neq x \wedge w_\alpha = x \wedge p_\alpha \), but this means that \( x \wedge w_\alpha \) is a nonzero element from \( X \upharpoonright w_\alpha \), which is below \( p_\alpha \)—a contradiction with our choice of the element \( p_\alpha \).

Lemma 1.15. Let \( B \) be an infinite complete ccc Boolean algebra homogeneous in \( \pi \)-weight. If \( \pi_X(B) < \pi_W(B) \), then the cofinality of \( \pi_X(B) \) is uncountable.

Proof. Let us denote \( \lambda = \pi_X(B) \). The algebra \( B \) has no atoms, thus \( \lambda \geq \omega \). Choose a local \( \pi \)-base \( X \) of some suitable ultrafilter, \( |X| = \lambda \). We may assume that \( X \) is closed under finite meets. Since \( |X| < \pi_W(B) \), then by Lemma 1.14 there is a countable partition \( \{ p_n : n \in \omega \} \) of unity, such that every \( x \in X \) meets infinitely many \( p_n \)'s.

Suppose now that \( \text{cf}(\lambda) = \omega \). Then we may write \( X = \bigcup_{n \in \omega} X_n \), where \( X_n \subseteq X_{n+1} \), every \( X_n \) is closed under finite meets and \( |X_n| < \lambda \). Then the family \( X_n \upharpoonright p_n \) is a local \( \pi \)-base of no ultrafilter, because \( |X_n \upharpoonright p_n| < \lambda = \pi_X(B) \). Therefore there is some \( u_n \leq p_n \) independent with respect to \( X_n \upharpoonright p_n \). Now it is clear that the element \( u = \bigvee_{n \in \omega} u_n \) is independent with respect to \( X \), which is a contradiction.

Lemma 1.16. Let \( B \) be a complete Boolean algebra, let \( X \subseteq B^+ \) be a local \( \pi \)-base of some ultrafilter in \( B \). Then there is a nonzero element \( a \in B^+ \) such that the family \( \mathcal{F} = \{ x - \bigvee Y : x \in X, Y \subseteq X \} \) is dense below \( B \upharpoonright a \).

Proof. Assume the contrary. Proceeding by a transfinite induction, we shall find an element \( u \in B^+ \), which is independent with respect to \( X \). This will contradict our assumption that \( X \) is a local \( \pi \)-base of some ultrafilter.
Put $X_0 = X$. Since $\mathcal{F}$ is not dense below any element of $\mathcal{B}^+$, there is some $u_0 \neq 0$ such that $s - u_0 \neq 0$ for all nonzero $s \in \mathcal{F}$. Let $X_1 = \{x \in X_0; x \wedge u_0 = 0\}$ and let $w_0 = 1 - \bigvee X_1$.

Let $\alpha$ be an ordinal and suppose that for all $\beta < \alpha$, $u_\beta$, $w_\beta$ and $X_\beta$ are known. If $\bigcap_{\beta < \alpha} X_\beta \subset \{0\}$, then the induction stops here. Otherwise let $X_\alpha = \bigcap_{\beta < \alpha} X_\beta$ and choose a nonzero $u_\alpha \leq 1 - \bigvee_{\beta < \alpha} w_\beta$ such that $s - u_\alpha \neq 0$ for all nonzero $s \in \mathcal{F}$. Then define $X_{\alpha + 1}$ to be the set $\{x \in X_\alpha; x \wedge u_\alpha = 0\}$ and let

$$w_\alpha = 1 - (\bigvee X_{\alpha + 1} \lor \bigvee \{w_\beta; \beta < \alpha\}).$$

This completes the inductive definitions.

Denote by $\xi$ the first ordinal with $\bigcap_{\alpha < \xi} X_\alpha \subset \{0\}$ and let $u = \bigvee_{\alpha < \xi} u_\alpha$. We need to show that $u$ is independent with respect to $X$.

Let $x \in X^+$ be arbitrary. Clearly, there is some $\alpha < \xi$ with $x \wedge u_\alpha \neq 0$, otherwise $x \in \bigcap_{\alpha < \xi} X_\alpha$.

We may assume that $\alpha$ is the first such. It remains to show that $x \wedge (w_\alpha - u_\alpha) \neq 0$, too. Since $x \in X_\beta$ for all $\beta \leq \alpha$, we have $x \wedge w_\beta = 0$ for all $\beta < \alpha$. Thus $x - \bigvee X_{\alpha + 1} \leq w_\alpha$. By the choice of $u_\alpha$, $x - \bigvee X_{\alpha + 1} - u_\alpha = 0$, therefore $x \wedge (w_\alpha - u_\alpha) \neq 0$. □

**Proof of Theorem 1.13.** We may consider only complete Boolean algebras which are homogeneous in $\pi$-weight. The proof will go by an induction on the size of a $\pi$-basis.

If $\pi w(\mathcal{B}) = 1$, then the algebra in question is a trivial one and there is nothing to prove.

For an algebra $\mathcal{B}$ with $\pi w(\mathcal{B}) = \omega$, we have $\pi \chi(\mathcal{B}) = \pi w(\mathcal{B})$, because $\mathcal{B}$ has no atoms.

Let $\kappa = \pi w(\mathcal{B}) > \omega$ and suppose that $\lambda - \pi \chi(\mathcal{B}) < \kappa$. Necessarily $\lambda \geq \omega$, because there are no atoms in $\mathcal{B}$. By Lemma 1.15, $\text{cf}(\lambda) \neq \omega$. Our assumption on cardinal exponentiation implies $\lambda^\omega = \lambda$. Take an arbitrary set $X \subset \mathcal{B}$, $|X| = \lambda$, which is a local $\pi$-base of some ultrafilter in $\mathcal{B}$. By Lemma 1.16, the family $\mathcal{F} = \{x - \bigvee Y; x \in X, Y \subset X\}$ is dense in $\mathcal{B} \upharpoonright a$ for some nonzero $a$. Since $\mathcal{B}$ is ccc, $\mathcal{F} = \{x - \bigvee Y; x \in X, Y \in [X]^\omega\}$. Consequently, $|\mathcal{F}| \leq \lambda^\omega = \lambda$. We have $\pi w(\mathcal{B} \upharpoonright a) \leq \lambda < \kappa = \pi w(\mathcal{B})$, though $\mathcal{B}$ is homogeneous in $\pi$-weight, a contradiction. □

Now we shall consider complete Boolean algebras of size $2^\omega$, which satisfy ccc. The previous theorem says very few for these algebras, namely $\pi \chi(\mathcal{B}) = \pi w(\mathcal{B})$ provided CH holds. This fact is well known.

What if CH does not hold? As we have already mentioned, we do not know any example of a Boolean algebra $\mathcal{B}$ with $\pi \chi(\mathcal{B}) < \pi w(\mathcal{B})$. It would be the most exciting to find a consistent example of such an algebra with ccc and of size $2^\omega$, that means, of $\pi$-weight at most $2^\omega$.

The forthcoming statement shows, that when looking for such an algebra, one must have in mind also cardinal characteristics which concern functions and sets of natural numbers.
Let us remind that the refinement number $r$ equals, by definition, to $\pi\chi(\mathcal{P}(\omega)/\text{fin})$, equivalently,

\[ r = \min\{|\mathcal{L}| : \mathcal{L} \subseteq [\omega]^\omega \text{ and } (\forall M \subseteq [\omega]^\omega)(\exists T \in \mathcal{L})(T \subseteq M \lor T \subseteq \omega \setminus M)\}. \]

The dominating number is

\[ d = \min\{|\mathcal{F}| : \mathcal{F} \subseteq [\omega]^\omega \text{ and } (\forall g \in [\omega]^\omega)(\exists f \in \mathcal{F})(f \text{ eventually dominates } g)\}. \]

Both of $r < d$ and $d < r$ are consistent, for the first inequality, see [11], for the second one it suffices to add many random reals.

**Proposition 1.17.** Let $\mathcal{B}$ be a complete Boolean algebra, satisfying ccc. Then

\[ \pi\chi(\mathcal{B}) \geq \min\{\pi\omega(\mathcal{B}), \max(r, d)\}. \]

**Proof.** By Proposition 1.8, we are allowed to assume that $\mathcal{B}$ is homogeneous in $\pi$-weight. If there is an atom in $\mathcal{B}$, then there is nothing to prove, so assume that $\kappa = \pi\omega(\mathcal{B}) \geq \omega$. Let $\mathcal{U}$ be an arbitrary ultrafilter in $\mathcal{B}$, let $X \subseteq \mathcal{B}^+$ be a local $\pi$-base of $\mathcal{U}$.

(i) We prove that $|X| \geq \min(\kappa, r)$. Suppose the contrary, which means that $|X| < \kappa$ and $|X| < r$. Hence the set $X$ is not dense in any $\mathcal{B} \upharpoonright u$. By Lemma 1.14 and from $c(\mathcal{B}) = \omega$ we conclude that there is a countable partition of unity $P = \{p_n : n \in \omega\}$ such that every $x \in X$ is compatible with infinitely many $p_n$'s. Set

\[ \mathcal{U}_P = \{Y \subseteq \omega : \forall \{p_n : n \in Y\} \in \mathcal{U}\}, \]

\[ X_P = \{\{n \in \omega : p_n \wedge x \neq 0\} : x \in X\}. \]

Then $\mathcal{U}_P$ is a uniform ultrafilter in $\mathcal{P}(\omega)$ and for every $Y \in \mathcal{U}_P$ there is an $A \in X_P$ such that $A \subseteq Y$. Therefore $r \leq |X_P|$. However, by the assumption, $|X_P| \leq |X| < r$, which is a contradiction. We proved that $\pi\chi(\mathcal{U}) \geq \min(\kappa, r)$.

(ii) We prove now that $|X| \geq \min(\kappa, d)$. Let $|X| < \min(\kappa, d)$. We have a partition $P = \{p_n : n \in \omega\}$, the same as in (i). For every $n \in \omega$ let us set $X_n = (X \upharpoonright p_n)^\ast$. Since $|X_n| \sim \kappa$ and since $\mathcal{B}$ is homogeneous in $\pi$-weight, there is a partition $Q_n = \{v_{n,i} : i \in \omega\}$ of an element $p_n$ such that every $x \in X_n$ meets infinitely many members of $Q_n$.

Next, for every $x \in X$, let us define a function $f_x \in [\omega]^\omega$ as follows. For every $n \in \omega$ let $m_n = \min\{m \geq n : p_m \wedge x \neq 0\}$. Such a number must exist, because $x$ meets infinitely many $p_n$'s. Define

\[ f_x(n) = \min\{i \in \omega : v_{n,i} \wedge x \neq 0\}. \]

We assume that $|X| < d$, thus there is a strictly increasing function $f : \omega \rightarrow \omega$, which is not eventually dominated by any function $f_x$. Let

\[ u = \bigvee_{n < \omega} \bigvee \{v_{n,i} : i \leq f(n)\}. \]

\[ u = \bigvee_{n < \omega} \bigvee \{v_{n,i} : i \leq f(n)\}. \]
It remains to verify that the element $u$ is independent with respect to $X$ and we shall be done, because this will contradict to the fact that $X$ is a local $\pi$-base of an ultrafilter $\mathcal{U}$. Choose an arbitrary element $x \in X$. The function $f_x$ does not eventually dominate $f$, hence the set $A_x = \{n \in \omega : f_x(n) \leq f(x)\}$ is infinite. For $n \in A_x$, clearly $m_n \ni n$ and $f_x(n) = f_x(m_n)$. Since $f$ is increasing, we have $f_x(m_n) \leq f(m_n)$, too. Therefore $x \wedge u \ni x \wedge \bigvee \{v_{m_n,i} : i \leq f(m_n)\} \neq \emptyset$. The element $x$ is compatible with infinitely many elements of $Q_{m_n}$, but the element $u$ with finitely many only, therefore $x \wedge u \neq \emptyset$. We have shown that $u$ is independent with respect to $X$, which completes the proof. \[\Box\]

2. Discretely untouchable points

Here we apply the results of the previous section to a topological setting.

**Definition 2.1.** Let $X$ be a topological space. A point $x \in X$ is called *discretely untouchable*, if for every countable discrete set $C \subseteq X \setminus \{x\}$, $x \notin \bar{C}$.

A well-known theorem due to Frolik states that no infinite extremally disconnected compact space is homogeneous. However, Frolik's proof gives no idea, what topological property distinguishes the points witnessing the nonhomogeneity; besides that, it would be desirable if the property was a rather simple one.

By our opinion, the simplest possible property is to be discretely untouchable. Every infinite compact Hausdorff space must contain a point, which is *not* discretely untouchable—this follows immediately from compactness and Hausdorffness. Consequently, if one can prove that every infinite extremally disconnected compact space contains a discretely untouchable point, he would get the easiest and obvious argument for the nonhomogeneity.

Notice that every weak P-point as well as every $\omega_2$-good point is discretely untouchable. By [16], if $X$ is extremally disconnected compact space and $c(X) \geq \omega_1$, then there is a $c(X)^+$-good point in $X$; similarly, by [13], every extremally disconnected compact space $X$ with $\omega(X) > 2^\omega$ has weak P-point. Therefore in the sequel, we restrict our attention to the small infinite extremally disconnected compact spaces, i.e. to those $X$ with $c(X) = \omega$, $\omega(X) = 2^\omega$. The key tool is the proposition, which follows.

**Definition 2.2.** Let $X$ be a topological space, $n \in \mathbb{N}$. A family $\mathcal{L}$ of subsets of $X$ is called **$n$-linked**, if for each $\mathcal{H} \in [\mathcal{L}]^{<n}$, the intersection $\bigcap \mathcal{H}$ is nonempty. A family $\mathcal{L}$ is called **precisely $n$-linked**, if it is $n$-linked and if moreover there is some $\mathcal{M} \in [\mathcal{L}]^{n+1}$ with $\bigcap \mathcal{M} = \emptyset$.

A family $\mathcal{L}$ is called **discretely untouchable**, if for every countable discrete set $C \subseteq X$ there is some $L \in \mathcal{L}$ with $\bar{C} \cap L = \emptyset$. 
Proposition 2.3. Let \( X \) be an extremally disconnected compact space. Suppose that the Boolean algebra \( \text{Clop}(X) \) has an increasing chain interleaved with independent elements. Then for every \( n \in \mathbb{N} \) there is a precisely \( n \)-linked discretely untouchable collection of clopen subsets of \( X \).

Proof. Fix an \( n \in \mathbb{N} \). Notice first that there is some cardinal \( \tau \) and families \( \{ H_\alpha : \alpha < \tau \} \) such that \( H = \bigcup \{ H_\alpha : \alpha < \tau \} \) is a \( \pi \)-basis of \( X \), each \( P_\alpha \) is a partition of \( X \) into \( n + 1 \) non-void clopen subsets and the following holds:

(i) if \( \alpha < \beta < \tau \), then \( H_\alpha \subsetneq H_\beta \);
(ii) each \( H_\alpha \subset \mathcal{P}^+ \) is closed under finite intersections;
(iii) \( \bigcup \{ H_\alpha : \alpha < \tau \} \) is a \( \pi \)-basis of \( X \);
(iv) for each \( \alpha < \tau \), \( P_\alpha \subset H_{\alpha + 1} \);
(v) for every \( b \in H_\alpha \) and \( p \in P_\alpha \), \( b \cap p \neq \emptyset \).

This observation is an obvious consequence of the fact that \( \text{Clop}(X) \) has an increasing chain interleaved with independent elements: Indeed, by Definition 1.10(i), (iv) and (v), every \( b \in H_\alpha \) meets every element of the partition

\[
\{ -p_\alpha, p_\alpha - p_{\alpha + 1}, p_\alpha \cap p_{\alpha + 1} - p_{\alpha + 2}, \ldots, p_\alpha \cap p_{\alpha + 1} \cap \cdots \\
\cap p_{\alpha + n - 1} - p_{\alpha + n}, p_\alpha \cap p_{\alpha + 1} \cap \cdots \cap p_{\alpha + n} \}
\]

and all the \( n + 1 \) members of this partition belong by Definition 1.10(ii) and (iv) to \( H_{\alpha + n + 1} \). Hence it is enough to pass to a suitable cofinal subset of indices.

For every \( \alpha < \tau \), fix some enumeration of \( P_\alpha = \{ p^\alpha_i : i < n + 1 \} \). For every \( b \in \bigcup_{\alpha < \tau} H_\alpha \), let \( \alpha(b) \) be the first \( \alpha < \tau \) with \( b \in H_\alpha \) and define \( b^\alpha = b \cap p^\alpha_{\alpha(b)} \) for \( i = 0, 1, \ldots, n \).

For a maximal disjoint family \( \mathcal{C} \subset H \) and for a function \( f: \mathcal{C} \to n + 1 \) let \( L(\mathcal{C}, f) = X \setminus \bigcup \{ b^\alpha : b \in \mathcal{C} \} \). We claim that the family

\[
\mathcal{L} = \{ L(\mathcal{C}, f) : \mathcal{C} \subset H \text{ is a maximal disjoint family}, f \in \mathcal{C}(n + 1) \}
\]

is the desired precisely \( n \)-linked discretely untouchable collection.

Let us show first that \( \mathcal{L} \) is \( n \)-linked. Choose arbitrary \( L_k = L(\mathcal{C}_k, f_k) \in \mathcal{L} \) for \( k < n \); we have to show that \( n \cdot \mathcal{L} \cup \mathcal{L} \cap \mathcal{L} \neq \emptyset \).

Let \( \alpha_0 = \min \{ \alpha(b) : b \in \bigcup_{k=0}^{n-1} \mathcal{C}_k \} \) and choose a maximal centered family \( \mathcal{D}_0 \subset \bigcup_{k=0}^{n-1} \mathcal{C}_k \) with \( \alpha(b) = \alpha_0 \) for all \( b \in \mathcal{D}_0 \). Notice that \( \mathcal{D}_0 \) is nonempty and that \( \mathcal{D}_0 \cap \mathcal{C}_k \neq \emptyset \) for all \( k < n \). We may assume that our enumeration was chosen in such a way that for some \( 0 < s_0 < n \), \( \mathcal{D}_0 \cap \mathcal{C}_k \neq \emptyset \) if and only if \( k < s_0 \). Then \( \mathcal{D}_0 \subset H_{s_0} \) and by Definition 1.10(ii), \( \bigcap \mathcal{D}_0 \subset H_{s_0} \), too. Since \( s_0 < n \), there is some \( j_0 < n + 1 \), \( j_0 \neq f_k(b) \) for all \( b \in \mathcal{D}_0 \). The partition \( P_{\alpha_0} \) is independent with respect to \( H_{s_0} \), thus our definitions imply that \( u_0 = \bigcap_{b \in \mathcal{D}_0} b^\alpha = (\bigcap \mathcal{D}_0)_b \neq \emptyset \) and \( u_0 \neq \emptyset \).

If \( s_0 = n \), we are done. Otherwise define \( \alpha_1 \) to be

\[
\min \left\{ \alpha(b) : b \in \bigcup_{k=s_0}^{n-1} \mathcal{C}_k \text{ and } b \cap u_0 \neq \emptyset \right\}
\]

and find a family \( \mathcal{D}_1 \subset \bigcup_{k=s_0}^{n-1} \mathcal{C}_k \) such that \( \alpha(b) = \alpha_1 \) for all \( b \in \mathcal{D}_1 \), \( \{ u_0 \} \cup \mathcal{D}_1 \) is centered and \( \mathcal{D}_1 \) is maximal with respect to these two properties.
Since every disjoint family $\mathcal{C}_k$ is maximal, $\mathcal{D}_1$ is non-void and we can again assume that for some $s_1 > s_0$, $s_1 < u$, $\mathcal{D}_1 \cap \mathcal{C}_k \neq \emptyset$ iff $s_0 < k < s_1$. As $\alpha_1 > \alpha_0$, $u_0 \in H_{\alpha_1}$ and $\mathcal{D}_1 \subset H_{\alpha_1}$, so $u_0 \cap \bigcap \mathcal{D}_1$. Again there is some $j_1 < n + 1$ with $j_1 \neq f_k(b)$ for all $s_0 < k < s_1$; so

$$\emptyset \neq (u_0 \cap \bigcap \mathcal{D}_1)_{j_1} \subset u_0 \cap \bigcap_{b \in \mathcal{D}_1} b_j \text{ and } u_1 \subset \bigcap_{k < s_1} L_k.$$

Now it should be clear how to proceed further. After at most $n$ steps we conclude that $\bigcap_{k < n} L_k$ is nonempty.

Next, let $\mathcal{C}$ be an arbitrary maximal disjoint family contained in $H$ and let $f_k(b) = k$ for every $b \in \mathcal{C}$. It is clear that the family $\{L_k = L(\mathcal{C}, f_k): k < n + 1\}$ consists of $n + 1$ members of $\mathcal{L}$ and its intersection is empty. So $\mathcal{L}$ is precisely $n$-linked.

It remains to show that $\mathcal{L}$ is discretely untouchable. Let $\{x_j: j \in \omega\} \subset X$ be a countable discrete set. Choose for each $x_j$ its clopen neighborhood $U_j$ such that $U_j \cap U_l = \emptyset$ whenever $j \neq l$. Since $H$ is a $\pi$-base of $X$, there is some maximal disjoint family $\mathcal{C} \subset H$ such that for every $b \in \mathcal{C}$, either $b \subset U_j$ for some $j \in \omega$ or $b \cap U_j = \emptyset$ for all $j \in \omega$. Let us define a mapping $f: \mathcal{C} \to n + 1$ as follows.

$$f(b) = 0 \text{ for all } b \in \mathcal{C} \text{ which do not meet any } U_j.$$

For $k < n + 1$ and $j \in \omega$, let $U_{j,k} = \bigcup\{b_k: b \subset U_j\}$. Since $\mathcal{C}$ is maximal, since all $b_k$'s are disjoint and since $X$ is extremally disconnected, there is a unique $k_j$ with $x_j \in U_{j,k_j}$. Define $f(b) = k_j$ for $b \in \mathcal{C}$, $b \subset U_j$.

For this mapping $f$, the set $L(\mathcal{C}, f)$ is clopen and disjoint with $\{x_j: j \in \omega\}$. Thus $\mathcal{L}$ is discretely untouchable.

Theorem 2.4. Let $X$ be a ccc extremally disconnected compact space such that $\pi_X(X) = \pi_w(X) \leq 2^\omega$. Then there is a discretely untouchable point in $X$.

Proof. If there is an isolated point in $X$, then it is discretely untouchable and we are done. So suppose that $X$ is dense-in-itself.

Let $U$ be a clopen subset of $X$ such that $\pi_w(U) = \pi_w(X)$. Hence for any point $x$ in this $U$ we have $\pi_X(x, X) = \pi_w(U)$. Choose a pairwise disjoint collection $\mathcal{U} \subset$ clopen non-void subsets of $U$, say $\{U_n: n \in \mathbb{N}\}$. By Theorem 1.11 and by Proposition 2.3, for each $n \in \mathbb{N}$ there is a precisely $n$-linked discretely untouchable collection $\mathcal{L}_n$ on $U_n$. Let $\mathcal{F}$ be a family of all subsets $F \subset \bigcup_{n \in \mathbb{N}} U_n$ such that for every $n \in \mathbb{N}$, $F \cap U_n \in \mathcal{L}_n$. Then $F$ is a subbasis of a nice filter on $U_n \in \mathcal{L}_n$. (Recall that a filter $\mathcal{G}$ on a disjoint union $\bigcup_{n \in \mathbb{N}} X_n$ of topological spaces is called nice, if for every $G \in \mathcal{G}$, the set $\{n \in \mathbb{N}: G \cap X_n = \emptyset\}$ is finite. This definition is due to van Mill [13].)

We shall make use of [13, Theorem 2.5]: There is a $2^\omega$-OK point

$$x \in \bigcap_{F \in \mathcal{F}} \text{cl } F \cap \beta \bigcup_{n \in \mathbb{N}} U_n \setminus \bigcup_{n \in \mathbb{N}} U_n,$$

where the closure is taken in $\beta \bigcup_{n \in \mathbb{N}} U_n$. Since the space $X$ is extremally disconnected, $\beta \bigcup_{n \in \mathbb{N}} U_n = \bigcup_{n \in \mathbb{N}} U_n$. 

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We claim that the point $x$ is as required. To this end, consider an arbitrary countable discrete set $C \subset X$. The set $C$ splits into three subsets

$$
C_0 = C \setminus \bigcup_{n \in \mathbb{N}} U_n,
$$

$$
C_1 = C \cap \left( \bigcup_{n \in \mathbb{N}} U_n \setminus \bigcup_{n \in \mathbb{N}} U_n \right),
$$

$$
C_2 = C \cap \bigcup_{n \in \mathbb{N}} U_n.
$$

The set $\bigcup_{n \in \mathbb{N}} U_n$ is a clopen neighborhood of a point $x$, which is disjoint with $C_0$, hence $x \not\in \overline{C_0}$. Further, $x \not\in \overline{C_1}$, because the point $x$ is a $2^\omega$-OK point in $\bigcup_{n \in \mathbb{N}} U_n \setminus \bigcup_{n \in \mathbb{N}} U_n$. Finally, every collection $\mathcal{L}_n$ is discretely untouchable, thus there is some clopen $L_n \subset U_n$ disjoint with $C_2$. The set $F = \bigcup_{n \in \mathbb{N}} L_n$ belongs to $\mathcal{F}$ and the extremal disconnectedness of $X$ implies that $\overline{F} \cap \overline{C_2} = \emptyset$. Since $x \in \overline{F}$, we have $x \not\in \overline{C_2}$.

We have proved that $x \not\in \overline{C}$. Since $C$ was arbitrary, we have verified that the point $x$ is discretely untouchable, which concludes the proof. $\square$

References


