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Exhaustive weakly wandering sequences

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ABSTRACT

A sequence of integers $\{n_i : i = 0, 1, \ldots\}$ is an *exhaustive weakly wandering* sequence for a transformation T if for some measurable set W, $X = \bigcup_{i=0}^{\infty} T^{n_i} W(disj)$. We introduce a hereditary *Property* (H) for a sequence of integers associated with an infinite ergodic transformation T, and show that it is a sufficient condition for the sequence to be an exhaustive weakly wandering sequence for T. We then show that every infinite ergodic transformation admits sequences that possess Property (H), and observe that Property (H) is inherited by all subsequences of a sequence that possess it. As a corollary, we obtain an application to tiling the set of integers $\mathbb{Z}$ with infinite subsets.

1. INTRODUCTION

Let T be a measurable transformation defined on a non-atomic, σ-finite measure space $(X, \mathcal{B}, m)$. In the sequel, all sets mentioned will be assumed to be measurable (belong to $\mathcal{B}$), and statements will be understood to mean "up to sets of measure zero" even when not explicitly stated. All transformations $T : X \to X$ are 1-1 and onto. We say that T preserves the measure m, or m is an invariant measure for T, if $m(TA) = m(T^{-1}A) = m(A)$ for every $A \in \mathcal{B}$, and the transformation T is ergodic if $TA = A$ implies $m(A) = 0$ or $m(X - A) = 0$. When T is ergodic, measure preserving, and $m(X) = \infty$, we shall refer to it as an *infinite ergodic transformation.*

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Definition 1. A sequence of integers $\mathbb{A} = \{a_i : i = 0, 1, 2, \ldots\}$ is an *exhaustive weakly wandering* (*eww*) *sequence* for the transformation T if there exists a set W, $m(W) > 0$, whose images under T fill up the space X and are mutually disjoint under the sequence $\mathbb{A}$, i.e.

$$(1) \qquad \bigcup_{i=0}^{\infty} T^{a_i} W = X,$$

$$(2) \qquad T^{a_i} W \cap T^{a_j} W = \emptyset \quad \text{for } i \neq j.$$

A sequence which satisfies only (2) is called *weakly wandering* (*ww*) *sequence.* (1) is the exhaustive condition. The set A is referred to as the *exhaustive weakly wandering* (*eww*) *set.* It need not be unique. An example of an eww set for an infinite ergodic transformation, in fact the first such example, may be found in [2].

It was shown in [6] that aperiodic nonsingular transformations without an absolutely continuous invariant probability measure have eww sets. In this article we concentrate on infinite ergodic transformations, and introduce a hereditary property for non-negative sequences of integers associated with an infinite ergodic transformation T; see Property (H) in Definition 2 below. Then, we prove in Theorem 3 that Property (H) is sufficient for a sequence of integers to be an eww sequence for the transformation T. In the following section, using elementary arguments, we show that for every infinite ergodic transformation T there exists an abundance of sequences possessing Property (H). Using the hereditary property of the sequences, we conclude with an application, which says that there exists a tiling of the integers $\mathbb{Z}$ by an infinite subset $\mathbb{A} \subset \mathbb{Z}$ with the property that every infinite subset $\mathbb{A}'$ of $\mathbb{A}$ again tiles $\mathbb{Z}$.

2. HEREDITARY PROPERTY

In this section we present Property (H). It follows that Property (H) is inherited by every infinite subsequence of a sequence that possesses it, while in Theorem 3 we show that every sequence that possesses Property (H) is an eww sequence.

Definition 2. Let T be an infinite ergodic transformation and A_0 be a set with $m(A_0) > 0$. We shall say that an increasing sequence $\{n_i : 0 = n_0 < n_1 < \cdots\}$ of non-negative integers possesses Property (H) for the transformation T with the set A_0 if it satisfies the following:

$$(\mathrm{H}) \qquad \begin{cases} \text{for } i, j, k, l = 0, 1, 2, \ldots, \ i \neq j, \ 0 \leqslant k' \leqslant k, \ 0 \leqslant l' \leqslant l, \text{ we have} \\ T^{n_i - n_k + k'} A_0 \cap T^{n_j - n_l + l'} A_0 = \emptyset \text{ if one of the following holds:} \\ \bullet \text{ any one of the indices } \{i, j, k, l\} \text{ is greater than all the others;} \\ \bullet \ i = l \text{ and } i > \{j, k\}; \\ \bullet \ j = k \text{ and } j > \{i, l\}. \end{cases}$$

Remark 2-1. The conditions $0 \leqslant k' \leqslant k$, $0 \leqslant l' < l$ make the Property (H) hereditary. That is, any subsequence also possesses Property (H), after re-indexing and shifting if necessary to begin at 0.

Remark 2-2. Observe that, the set A_0 is already weakly wandering. That is, setting $k, l = 0$ shows that $\{n_i\}$ is a weakly wandering sequence for T with the set A_0. Setting $k = j,\ l = i$ and $k' = l' = 0$ shows that $\{2n_i\}$ is also a weakly wandering sequence for T with the set A_0, while setting $i = l$ and $k, l' = 0$ gives $T^{2n_i} A_0 \cap T^{n_j} A_0 = \emptyset$ for $i \neq j$. Since $T^{2n_i} A_0 \cap T^{n_i} A_0 = T^{n_i} A_0 \cap A_0 = \emptyset$ it follows that the combined sequence $\{n_i\} \cup \{2n_i\}$ is also weakly wandering.

Remark 2-3. The definition is expressly designed for the set $W \supset A_0$ constructed below, (3), to be exhaustive weakly wandering. When applying the definition to the cases for disjointness in Property (H) the following are not covered explicitly: (i) $i = k > \{j, l\}$ and the symmetric $j = l > \{i, k\}$, (ii) $k = l > \{i, j\}$, and (iii) $i = k = l > j$ and the symmetric $j = k = l > i$.

Theorem 3. *Let $\{n_i : 0 = n_0 < n_1 < n_2 < \cdots\}$ be an increasing sequence of non-negative integers. If the sequence $\{n_i\}$ possesses Property* (H) *for an infinite ergodic transformation T with a set A_0, $m(A_0) > 0$, then, $\{n_i : i = 0, 1, 2, \ldots\}$ is an exhaustive weakly wandering sequence for the transformation T with a set $W \supset A_0$.*

Proof. Let the set

$$
\begin{aligned}
(3) \qquad W &= \bigcup_{p=0}^{\infty} T^{-n_p+p} A_p \\
&= A_0 \cup T^{-n_1+1} A_1 \cup T^{-n_2+2} A_2 \cup \cdots \cup T^{-n_p+p} A_p \cup \cdots
\end{aligned}
$$

where the sets $A_p \subset A_0$ are chosen inductively as follows.

$$\text{Let} \quad C_1 = T A_0 \backslash A_0 \quad \text{and put} \quad A_1 = T^{-1} C_1 \subset A_0.$$

$$\text{Let} \quad C_2 = T^2 A_0 \Big\backslash \bigcup_{r=0}^{1} \bigcup_{s=0}^{1} T^{n_r - n_s + s} A_s \quad \text{and put} \quad A_2 = T^{-2} C_2 \subset A_0.$$

Having chosen the sets $A_1, A_2, \ldots, A_{p-1}$,

$$\text{let} \quad C_p = T^p A_0 \Big\backslash \bigcup_{r=0}^{p-1} \bigcup_{s=0}^{p-1} T^{n_r - n_s + s} A_s \quad \text{and put} \quad A_p = T^{-p} C_p \subset A_0.$$

First we show the exhaustive property, i.e. $X = \bigcup_{r=0}^{\infty} T^{n_r} W$. For $p > 0$,

$$
\begin{aligned}
\bigcup_{r=0}^{p} T^{n_r} W &\supset \bigcup_{r=0}^{p} T^r A_r \cup \bigcup_{r=0}^{p-1} \bigcup_{s=0}^{p-1} T^{n_r - n_s + s} A_s \\
&= \bigcup_{r=0}^{p} C_r \cup \bigcup_{r=0}^{p-1} \bigcup_{s=0}^{p-1} T^{n_r - n_s + s} A_s \\
&= \bigcup_{r=0}^{p} T^r A_0.
\end{aligned}
$$

The assumption that T is ergodic implies $\bigcup_{r=0}^{\infty} T^{n_r} W \supset \bigcup_{r=0}^{\infty} T^r A_0 = X$.

It remains to show the weakly wandering property, i.e. $T^{n_i} W \cap T^{n_j} W = \emptyset$ for $i \neq j$; $i, j = 0, 1, 2, \dots$. Breaking this down to individual (non-symmetrical) cases yields:

(a) $T^{n_i} A_0 \cap T^{n_j} A_0 = \emptyset$;
(b) $T^{n_i} A_0 \cap T^{n_j - n_l + l} A_l = \emptyset$;
(c) $T^{n_i - n_k + k} A_k \cap T^{n_j - n_l + l} A_l = \emptyset$.

Case (a) is true because $\{n_i\}$ is weakly wandering for the set A_0 (see remark immediately after Definition 2). Case (b) can be rolled into case (c) by putting $k = 0$. We analyze (c) as follows.

From the definition of the sets C_p and A_p, it is immediate that

$$(4) \qquad T^{n_r - n_s + s} A_s \cap T^p A_p = \emptyset \quad \text{if } p > \{r, s\}.$$

If $j = l$ and $l > \{i, k\}$ then (c) follows from (4); similarly if $i = k$ and $k > \{j, l\}$. If $k = l > \{i, j\}$ or $k = l = i > j$ then (c) holds since $A_i \subset A_0$ and $T^{n_i} A_0 \cap T^{n_j} A_0 = \emptyset$; similarly if $k = l = j > i$. In all the other cases (c) follows from Property (H) (see remarks after Definition 2). □

Corollary 4. *Let $\{n_i : 0 = n_0 < n_1 < n_2 < \cdots\}$ be an increasing sequence of non-negative integers. If the sequence $\{n_i\}$ possesses Property* (H) *for T with a set A_0, then, any subsequence $\{n_i'\}$ of the sequence $\{n_i\}$ is an exhaustive weakly wandering sequence for T with a set $W \supset A_0$.*

Proof. We note that Property (H) is a hereditary property when we consider a subsequence $\{n_i'\}$ of the sequence $\{n_i\}$. This is because $n_i' = n_{k_i}$, where $i \leqslant k_i$ for $i \geqslant 0$. We note however, that the set $W \supset A_0$ depends on the subsequence $n_i' = n_{k_i}$ for $i = 0, 1, 2, \dots$, and is different for different subsequences. □

3. INFINITE ERGODIC TRANSFORMATIONS

We begin with the following important Property (I) for infinite ergodic transformations.

Definition 5. A measure preserving transformation satisfies Property (I) if

$$(\mathrm{I}) \qquad m(A) < \infty, \ m(B) < \infty \implies \lim_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} m(T^i A \cap B) = 0.$$

We prove, in Theorem 6 below, that every ergodic, infinite measure preserving transformation possesses Property (I). It is a simple result, yet it incorporates the significant features of infinite ergodic transformations that distinguish them from ergodic transformations that preserve a finite measure. Most of the results we prove are a consequence of Property (I). The property is well known and follows easily from Birkhoff's Ergodic Theorem; however, we choose to prove it directly, using elementary arguments. We believe this is of separate interest.

Theorem 6. *Every infinite ergodic transformation T satisfies Property* (I).

For the proof of Theorem 6 we introduce some notation, define the countable equivalence of two sets, $E \sim F$, and prove two lemmas.

Consider a set B with $m(B) < \infty$. For any measurable set $E \in \mathcal{B}$ denote

$$\begin{aligned}
&m_B(E) = m(E \cap B),\\
&\sigma_{(B,n)}(E) = \frac{1}{n}\sum_{i=0}^{n-1} m_B(T^i E) = \frac{1}{n}\sum_{i=0}^{n-1} m(T^i E \cap B),\\
&\bar{\sigma}_B(E) = \limsup_{n\to\infty} \sigma_{(B,n)}(E).
\end{aligned}$$

It is easy to see $m_B(E) \leqslant m(B)$ and $m_B(X) = \sigma_{B,n}(X) = \bar{\sigma}_B(X) = m(B)$. On the other hand, $m(E) = m(F) \not\Rightarrow m_B(E) = m_B(F)$ and $m(E) = m(F) \not\Rightarrow \sigma_{B,n}(E) \neq \sigma_{B,n}(F)$. However, the following definition and the next two lemmas will give $\bar{\sigma}_B(E) = \bar{\sigma}_B(F)$ when $m(E) = m(F) < \infty$. Finally, in the proof of Theorem 6, all this will be put together to show that $\bar{\sigma}_B(A) = 0$ for finite sets A and B – which is Property (I). See [3,4], and [5].

Definition 7. Two sets E and F are said to be *countably equivalent*, denoted $E \sim F$, if there are partitions of E and F and powers of T, $\{T^{p_i}\}$ such that

$$\begin{aligned}
&E = \bigcup_{i=1}^{\infty} E_i \text{ (disjoint)}, \qquad F = \bigcup_{i=1}^{\infty} F_i \text{ (disjoint)},\\
&\text{and} \quad T^{p_i} E_i = F_i \quad \text{for } i = 1, 2, \ldots.
\end{aligned}$$

Lemma 8. *Let T be an infinite ergodic transformation. Consider a set $B \in \mathcal{B}$ of finite measure, and let E and F be two sets of finite measure satisfying $E \sim F$. Then,*

(a) $\qquad \bar{\sigma}_B(E) = \bar{\sigma}_B(F)$.

Moreover, suppose $\{E_0, E_1, \ldots, E_r\}$ is a finite collection of sets of finite measure satisfying $E_i \sim E_j$, $E_i \cap E_j = \emptyset$, for $i \neq j$, $i, j = 0, 1, 2, \ldots, r$. *Then*

(b) $\qquad \bar{\sigma}_B\left(\bigcup_{i=1}^{r} E_i\right) = r\bar{\sigma}_B(E_0)$.

Proof. We first note that for any set $E \in \mathcal{B}$

$$|\sigma_{(B,n)}(E) - \sigma_{(B,n)}(T^p E)| = \left|\frac{1}{n}\sum_{i=0}^{n-1} m(T^i E \cap B) - \frac{1}{n}\sum_{i=0}^{n-1} m(T^{i+p} E \cap B)\right|$$

$$= \frac{1}{n}\left|\sum_{i=0}^{p-1} m(T^i E \cap B) - \sum_{i=n}^{n+p-1} m(T^i E \cap B)\right|$$

$$\leqslant \frac{2|p|}{n} m(B) \quad \text{for any } n = 1, 2, \ldots \text{ and } p = 0, \pm 1, \pm 2, \ldots.$$

Since E and F have finite measure and $E \sim F$, for any $\epsilon > 0$, choose an integer $s > 0$ such that the tail is small, i.e. $\bigcup_{i=s}^{\infty} E_i < \epsilon$ and $\bigcup_{i=s}^{\infty} F_i < \epsilon$. Then

$$\begin{aligned}
&|\sigma_{(B,n)}(E) - \sigma_{(B,n)}(F)| \\
&\quad = \left|\sigma_{(B,n)}\left(\bigcup_{i=1}^{\infty} E_i\right) - \sigma_{(B,n)}\left(\bigcup_{i=1}^{\infty} F_i\right)\right| \\
&\quad = \left|\sigma_{(B,n)}\left(\bigcup_{i=1}^{s-1} E_i\right) + \sigma_{(B,n)}\left(\bigcup_{i=s}^{\infty} E_i\right)\right. \\
&\qquad \left. - \sigma_{(B,n)}\left(\bigcup_{i=1}^{s-1} F_i\right) - \sigma_{(B,n)}\left(\bigcup_{i=s}^{\infty} F_i\right)\right| \\
&\quad \leqslant \sum_{i=1}^{s-1} \left|\sigma_{(B,n)}(E_i) - \sigma_{(B,n)}(T^{p_i} E_i)\right| + 2\epsilon \\
&\quad \leqslant \sum_{i=1}^{s-1} \frac{2|p_i|}{n} \cdot m(B) + 2\epsilon.
\end{aligned}$$

Since $\epsilon > 0$ is arbitrary, we conclude

$$\left|\sigma_{(B,n)}(E) - \sigma_{(B,n)}(F)\right| \to 0 \quad \text{as } n \to \infty, \tag{5}$$

and this proves (a), i.e. $\bar{\sigma}_B(E) = \bar{\sigma}_B(F)$.

To prove (b), we again apply (5), getting

$$\begin{aligned}
&\left|\sigma_{(B,n)}\left(\bigcup_{i=1}^{r} E_i\right) - r\sigma_{(B,n)}(E_0)\right| \\
&\quad \leqslant \sum_{i=1}^{r} \left|\sigma_{(B,n)}(E_i) - \sigma_{(B,n)}(E_0)\right| \to 0 \quad \text{as } n \to \infty,
\end{aligned}$$

and this proves $\bar{\sigma}_B(\bigcup_{i=1}^{r} E_i) = r\bar{\sigma}_B(E_0)$. □

Lemma 9. *Let T be an infinite ergodic transformation, and let E and F be two sets such that $0 < m(E) = m(F) < \infty$. Then, $E \sim F$.*

Proof. Since T is ergodic, we choose the smallest positive integer $n_1 > 0$ such that $m(T^{n_1} E \cap F) > 0$. Put $F_1 = T^{n_1} E \cap F$ and $E_1 = T^{-n_1} F_1$. If $m(E \backslash E_1) = 0$ then we are done. If not, choose the smallest positive integer n_2 such that

$m(T^{n_2}(E\backslash E_1) \cap (F\backslash F_1)) > 0$. Continuing in this manner, suppose the mutually disjoint sets $\{E_1, E_2, \ldots, E_k\}$ and $\{F_1, F_2, \ldots, F_k\}$ have been chosen, satisfying:

$$E \supset \bigcup_{i=1}^{k} E_i \,(\text{disj}), \qquad F \supset \bigcup_{i=1}^{k} F_i \,(\text{disj}),$$
$$\text{and} \quad T^{n_i} E_i = F_i \quad \text{for } i = 1, 2, \ldots, k.$$

If $m(E - \bigcup_{i=1}^{k} E_i) > 0$, then let $n_{k+1} > 0$ be the smallest positive integer such that $m[T^{n_{k+1}}(E - \bigcup_{i=1}^{k} E_i) \cap (F - \bigcup_{i=1}^{k} F_i)] > 0$, and let $F_{k+1} = T^{n_{k+1}}(E - \bigcup_{i=1}^{k} E_i) \cap (F - \bigcup_{i=1}^{k} F_i)$ and $E_{k+1} = T^{-n_{k+1}} F_{k+1}$.

By the assumption that T is ergodic and measure preserving, and the sets E and F are of finite measure, we continue inductively until either $m(E - \bigcup_{i=1}^{s} E_i) = 0$ for some $s > 0$, or $m(E - \bigcup_{i=1}^{\infty} E_i) = 0$. This shows that $E \sim F$. □

Proof of Theorem 6. Let us consider two sets A and B of finite measure. Since $m(X) = \infty$, using Lemma 9, it is possible to get an infinite collection of sets of finite measure, $\{A_0, A_1, A_2, \ldots\}$, satisfying $A_0 = A$, $A_i \sim A_j$, and $A_i \cap A_j = \emptyset$ for $i \neq j$, $i, j = 0, 1, 2, \ldots$.

From Lemma 8, we conclude that for any integer $r > 0$, $m(B) = \bar{\sigma}_B(X) > \bar{\sigma}_B(\bigcup_{i=1}^{r} A_i) = r\bar{\sigma}_B(A)$. Since this is true for $r = 1, 2, \ldots$ and $m(B) < \infty$, we conclude $\bar{\sigma}_B(A) = 0$ – which is Property (I). □

Corollary 10. *Let T be an infinite ergodic transformation. Then,*

$$(6) \qquad m(A) < \infty \implies \liminf_{n\to\infty}\big[m(T^n A \cap A) + m(T^{2n} A \cap A)\big] = 0.$$

Proof. The following inequality is straightforward.

$$\frac{1}{n}\sum_{i=0}^{n-1}\big[m(T^i A \cap A) + m(T^{2i} A \cap A)\big] \leqslant 2\left\{\frac{1}{2n}\sum_{i=0}^{2n-1} 2m(T^i A \cap A)\right\}.$$

Apply Theorem 6 to the right-hand-side and (6) follows. □

The limit in (6) need not exist. However, if one restricts to a weakly wandering sequence, then the limit will exist.

Proposition 11. *Let $\{n_i\}$ be a weakly wandering sequence for the infinite ergodic transformation T. Then,*

$$m(A) < \infty \implies \lim_{i\to\infty} m(T^{n_i} A \cap A) = 0.$$

Proof. Let W, $m(W) > 0$, be a weakly wandering set for the transformation T with the sequence $\{n_i\}$. Then for any set of finite measure A

$$m(A) \geqslant m\left(\bigcup_{i=1}^{\infty} T^{n_i} W \cap A\right) = \sum_{i=1}^{\infty} m(T^{n_i} W \cap A)$$
$$\implies \quad \lim_{i\to\infty} m(T^{n_i} W \cap A) = 0.$$

For $k = \pm 1, \pm 2, \ldots$, the sets $T^k W$ are also weakly wandering for the sequence $\{n_i\}$, hence

$$\lim_{i\to\infty} m(T^{n_i+k} W \cap A) = 0.$$

If we denote $W_N = \bigcup_{i=0}^{N} T^i W$, for fixed $N > 0$ we conclude

$$\lim_{i\to\infty} m(T^{n_i} W_N \cap A) = 0.$$

Since T is ergodic, given $\epsilon > 0$ there is an $N > 0$, so that $A = (W_N \cap A) \cup (A \backslash W_N)$, disjoint, with $m(A \backslash W_N) < \epsilon$.

Therefore,

$$m(T^{n_i} A \cap A) = m\big(T^{n_i}(W_N \cap A) \cap A\big) + m\big(T^{n_i}(A \backslash W_N) \cap A\big)$$

which, for large enough N is less than 2ϵ and this completes the proof. □

4. EXISTENCE OF SEQUENCES POSSESSING PROPERTY (H)

In this section, we prove that sequences possessing Property (H) exist. Actually, we prove more. We show that such sequences exist for every infinite ergodic transformation. Furthermore, although we have no control on the size of the eww set W, we show that the set A_0 can be made arbitrarily close to any initial finite set of positive measure, Theorem 15.

Proposition 12. *Let T be an infinite ergodic transformation, and let the set A and $\epsilon > 0$ be such that $0 < \epsilon < m(A) < \infty$. Then, there exists a set $A_0 \subset A$ and a sequence $\{n_i : 0 = n_0 < n_1 < \cdots\}$ such that $m(A_0) > m(A) - \epsilon > 0$, and A_0 is weakly wandering for the sequence $\{n_i\}$ and also weakly wandering for the sequence $\{2n_i\}$.*

The idea is to use (6) to simultaneously remove small amounts from A while choosing the integers n_i so that the remainder of A is disjoint under the various images $\{T^{n_i}\}$.

Proof. Define $\epsilon_k = \epsilon/2^k$, for $k = 1, 2, \ldots$. Initialize $n_0 = 0$ and $B_1 = A$. Using (6) choose $n_1 > 0$ such that

$$m(T^{n_1} B_1 \cap B_1) + m(T^{2n_1} B_1 \cap B_1) < \epsilon_1.$$

Next, put $B_2 = T^{-2n_1} A \cup T^{-n_1} A \cup A$. Again, using (6), choose $n_2 > n_1$ such that

$$m(T^{n_2} B_2 \cap B_2) + m(T^{2n_2} B_2 \cap B_2) < \epsilon_2.$$

We continue by induction. Having chosen the integers $0 = n_0 < n_1 < \cdots < n_{k-1}$, we put

$$B_k = \bigcup_{j=0}^{k-1} T^{-n_j} A \cup \bigcup_{j=0}^{k-1} T^{-2n_j} A.$$

Using (6) we choose $n_k > n_{k-1}$ such that

$$m(T^{n_k} B_k \cap B_k) + m(T^{2n_k} B_k \cap B_k) < \epsilon_k.$$

Finally, define the set

$$A_0 = A \Big\backslash \bigcup_{i=1}^{\infty} T^{n_i} B_i \cup T^{2n_i} B_i.$$

First we observe that the amount removed from A is small—that is, the size of the set A_0 is close to the size of A.

$$m\left(A \cap \bigcup_{i=1}^{\infty} T^{n_i} B_i \cup T^{2n_i} B_i \right) < \sum_{i=1}^{\infty} \epsilon_i = \epsilon;$$

that is $m(A_0) > m(A) - \epsilon > 0$.

Next we need to show that A_0 is weakly wandering under the sequences $\{n_i\}$ and $\{2n_i\}$.

We examine $T^{n_i} A_0 \cap T^{n_j} A_0$, and without loss of generality, assume $i > j$. This means that $T^{-n_j} A \subset B_i$. Hence, $T^{n_i - n_j} A$ was removed from A in defining A_0, i.e. $T^{n_i - n_j} A_0 \cap A_0 = \emptyset \Longrightarrow T^{n_i} A_0 \cap T^{n_j} A_0 = \emptyset$. A similar argument shows that $\{2n_i\}$ is weakly wandering for the set A_0. □

Combining Proposition 11 and Proposition 12 we obtain

Corollary 13. *Let T be an infinite ergodic transformation. Then, there exists a sequence $\{c_i : i = 0, 1, 2, \ldots\}$, such that both $\{c_i\}$ and $\{2c_i\}$ are weakly wandering sequences for T. Moreover, for the sequence $\{c_i : i = 0, 1, 2, \ldots\}$, we have*

$$(7) \qquad m(A) < \infty \quad \Longrightarrow \quad \lim_{i \to \infty} \big[m(T^{c_i} A \cap A) + m(T^{2c_i} A \cap A) \big] = 0.$$

Theorem 14. *Let T be an infinite ergodic transformation. Let the sequence $\{c_i\}$ be such that both $\{c_i\}$ and $\{2c_i\}$ are weakly wandering sequences for T, and let a set A and $\epsilon > 0$ be given, such that $0 < \epsilon < m(A) < \infty$. Then, there exists a subset $A_0 \subset A$, $m(A_0) > m(A) - \epsilon > 0$, and a subsequence $\{n_i\}$ of the sequence $\{c_i\}$, such that the sequence $\{n_i\}$ satisfies Property* (H) *for the transformation T with the set A_0.*

The proof is similar to the proof of Proposition 12. In this case however we will remove from A a bit more than necessary in order to simplify the notation and indices.

Proof. Put $\epsilon_k = \epsilon/2^k$, for $k \geqslant 1$ and initialize $n_0 = 0$, $p_1 = 1$, and

$$B_1 = \bigcup_{j=-p_1}^{p_1} T^j A = T^{-1} A \cup A \cup TA.$$

Using (7), choose $n_1 = c_{i_1}$ such that

$$m(T^{n_1} B_1 \cap B_1) + m(T^{2n_1} B_1 \cap B_1) < \epsilon_1.$$

Next, put $p_2 = 2n_1 + p_1 + 2$,

$$B_2 = \bigcup_{j=-p_2}^{p_2} T^j A$$

and using (7) choose $n_2 = c_{i_2} > n_1$ such that

$$m(T^{n_2} B_2 \cap B_2) + m(T^{2n_2} B_2 \cap B_2) < \epsilon_2.$$

We continue by induction.
Having chosen $p_1, p_2, \ldots, p_{k-1}$, and $0 = n_0 < n_1 < n_2 < \cdots < n_{k-1}$; we let

$$p_k = 2n_{k-1} + p_{k-1} + k = \sum_{s=1}^{k-1} 2n_s + \sum_{s=1}^{k} s,$$

$$B_k = \bigcup_{j=-p_k}^{p_k} T^j A.$$

Using (7), we choose $n_k = c_{i_k} > n_{k-1}$ such that

$$m(T^{n_k} B_k \cap B_k) + m(T^{2n_k} B_k \cap B_k) < \epsilon_k.$$

Finally, we let

$$A_0 = A \Big\backslash \bigcup_{i=1}^{\infty} T^{n_i} B_i \cup T^{2n_i} B_i.$$

First, observe that the set A_0 is close in size to the size of A. That is, the amount we have removed is

$$m\left(\bigcup_{i=1}^{\infty} (T^{n_i} B_i \cup T^{2n_i} B_i) \cap A \right) < \sum_{i=1}^{\infty} \epsilon_i = \epsilon,$$

and $m(A_0) > m(A) - \epsilon > 0$.

Now we need to show the Property (H) conditions. For this we analyze when the following is true.

$$T^{n_i - n_k + k'} A_0 \cap T^{n_j - n_l + l'} A_0 = \emptyset. \tag{8}$$

First we examine the cases when there is a unique maximum index.

Suppose $i > \{j, l, k\}$. Then

$$-p_i \leqslant -n_k + k' - n_j + n_l - l' \leqslant p_i = \sum_{s=1}^{i-1} 2n_s + \sum_{s=1}^{i} s$$

which means that $T^{-n_k+k'-n_j+n_l-l'} A \subset B_i$. Hence, $T^{n_i-n_k+k'-n_j+n_l-l'} A$ was removed, $T^{n_i-n_k+k'-n_j+n_l-l'} A_0 \cap A_0 = \emptyset$ and (8) holds. Similarly for the symmetric case $j > \{i, l, k\}$. If $k > \{i, j, l\}$, then $-p_k < -n_i + n_j - n_l + l' - k' < p_k$ and (8) holds. The case $l > \{i, j, k\}$ is symmetric and this completes the cases when there is a unique maximum index.

Now consider the case $i = l > \{j, k\}$ (and the symmetric case $j = k > \{i, l\}$). We have $-p_i < -n_j + n_k - k' + l' < p_i$ which gives $T^{-n_j+n_k-k'+l'} A \subset B_i$. Then $T^{2n_i-n_j+n_k-k'+l'} A \subset T^{2n_i} B_i$ was removed in the creation of A_0 and again (8) holds. □

Theorem 15. *Let T be an infinite ergodic transformation; then, for any set A and $\epsilon > 0$ with $0 < \epsilon < m(A) < \infty$, there exists a subset $A_0 \subset A$, $m(A_0) > m(A) - \epsilon > 0$, and a sequence of integers $\{n_i\}$ with $0 = n_0 < n_1 < n_2 < \cdots$, such that the sequence $\{n_i : i = 0, 1, 2, \ldots\}$ satisfies Property* (H) *for the transformation T with the set A_0.*

Proof. The existence of the set A_0 and the sequence $\{n_i : i = 0, 1, 2, \ldots\}$ satisfying Property (H) follow from Proposition 12 and Theorem 14. □

We recall that Property (H) holds for the same set A_0 and for any subsequence $\{n'_i\}$ of the sequence $\{n_i\}$, since $n'_i = n_{k_i}$, where $i \leqslant k_i$ for $i \geqslant 0$.

Corollary 16. *Let T be an infinite ergodic transformation, and let both $\{n_i\}$ and $\{2n_i\}$ be weakly wandering sequences for T; then, there exists an eww subsequence $\{n'_i\}$ of the sequence $\{n_i\}$ with the property that any subsequence of the sequence $\{n'_i\}$ is again an eww subsequence for T.*

Proof: Follows from Theorem 14 and the Corollary to Theorem 3.

5. TILING $\mathbb{Z}$

Next, we show how exhaustive weakly wandering sequences may be used to tile the set of all integers $\mathbb{Z}$. See [1] for further details on the connection of exhaustive weakly wandering sequences to integer tilings.

For simplicity, let us assume that all subsets $\mathbb{A} \subset \mathbb{Z}$ that we mention are infinite, and $0 \in \mathbb{A}$. We say that a subset $\mathbb{A} \subset \mathbb{Z}$ *tiles* $\mathbb{Z}$ if there exists another subset $\mathbb{B} \subset \mathbb{Z}$, called a *complementing subset*, if every integer $n \in \mathbb{Z}$ can be uniquely written as a sum $n = a + b$ with $a \in \mathbb{A}$ and $b \in \mathbb{B}$, denoted $\mathbb{A} \oplus \mathbb{B} = \mathbb{Z}$.

An exhaustive weakly wandering sequence $\mathbb{A}$ will be a tiling set for $\mathbb{Z}$. Using the transformation T and a point a in the set A, we can obtain a complementing sequence $\mathbb{B}$.

Definition 17. The *hitting sequence* $\mathbb{B}$ for the point $x \in A$ is the sequence of integers $\{b : T^b(x) \in A\}$.

The sequence $\mathbb{B}$ is naturally indexed. Since $x \in A$, we put $b_0 = 0$ and have $x = T^0 x = T^{b_0} x \in A$. Thus the sequence is two-sided, $\mathbb{B} = \{\cdots < b_{-1} < 0 = b_0 < b_1 < \cdots\}$.

Theorem 18. *The sequences $\mathbb{A}$ and $\mathbb{B}$ obtained from an infinite ergodic transformation T as described above form a complementing pair for the integers, that is $\mathbb{A} \oplus \mathbb{B} = \mathbb{Z}$.*

Proof. Let $n \in \mathbb{Z}$ and $x \in A$. Since $X = \bigcup T^a(A)$, disjoint, the image $T^n(x) \in T^a(A)$ for exactly one $a \in \mathbb{A}$. Let $y = T^{-a}(T^n x) \in A$. By the definition of the hitting sequence there must be a (necessarily unique) $b \in \mathbb{B}$, with $T^n(x) = y$. Hence, $T^n(x) = T^a(T^b x) = T^{a+b}(x)$ and $n = a + b$. $\square$

In point of fact, each point $x \in A$ gives a different sequence $\mathbb{B}$, which means that the exhaustive weakly wandering sequence has an uncountable number of complements.

Finally, combining the above discussion with Theorem 15 and its Corollary, we conclude with the following:

Corollary 19. *There exists an infinite subset $\mathbb{A} \subset \mathbb{Z}$ that tiles $\mathbb{Z}$ with the property that every infinite subset $\mathbb{A}' \subset \mathbb{A}$ also tiles $\mathbb{Z}$.*

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