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Note

Graphs with chromatic polynomial $\sum_{l\leqslant m_0}\binom{l}{m_0-l}(\lambda)_l$ ☆

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Abstract

In this paper, using the properties of chromatic polynomial and adjoint polynomial, we characterize all graphs having chromatic polynomial $\sum_{l\leqslant m_0}\binom{l}{m_0-l}(\lambda)_l$.

Keywords: Chromatic polynomial; Adjoint polynomial

1. Introduction

The graphs considered here are finite, undirected and simple. Let G be a graph and $P(G,\lambda)$ its chromatic polynomial, $h(G,x)$ be its adjoint polynomial. Two graphs G and H are *chromatically equivalent* if $P(G,\lambda)=P(H,\lambda)$, and *adjointly equivalent* if $h(G,x)=h(H,x)$. A graph G is *chromatically unique* if $P(G,\lambda)=P(H,\lambda)$ implies that H is isomorphic to G. Similarly, a graph G is *adjointly unique* if $h(G,x)=h(H,x)$ implies that H is isomorphic to G. By $\bar{G}$ we denote the complement of G. It is obvious that a graph G is chromatically unique if and only if $\bar{G}$ is adjointly unique.

To compute chromatic polynomial for a given graph is well known, but to determine the graphs with a given chromatic polynomial is not easy. In this paper, using the

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properties of adjoint polynomial and the relations of adjoint and chromatical polynomial, we shall give all graphs determined by the polynomial $\sum_{l\leqslant m_0}\binom{l}{m_0-l}(\lambda)_l$.

Let G be a graph, $p(G)$ and $q(G)$ be its order and size, respectively. The symbols P_n, C_n and D_n stand for the following graphs of order n: the path, the cycle and the graph obtained by identifying a vertex of K_3 with one endvertex of P_{n-2}. Let $T(l_1,l_2,l_3)$, $(l_1\leqslant l_2\leqslant l_3)$ be a tree with one vertex of degree 3 and three vertices of degree 1 in which the distances from the vertex of degree 3 to the vertices of degree 1 are l_1, l_2 and l_3, respectively. Let $h(G,x)=x^{\alpha(G)}h_1(G,x)$, where $h_1(G,x)$ is a polynomial with a nonzero constant term.

For convenience, let $h(G)$ stand for $h(G,x)$, and $h_1(G)$ for $h_1(G,x)$. We will write $h(l_1,l_2,h_3)$ for $h(T(l_1,l_2,l_3),x)$, and $h_1(l_1,l_2,l_3)$ for $h_1(T(l_1,l_2,l_3),x)$. Let $\beta(G)$ denote the least root of $h_1(G)$. The reader may refer to [[5,2]] for all notations and terminology not explained here.

2. Preliminaries

Let $b_i(G)$ denote the number of ideal subgraphs with $p-i$ components (see [[5]]), then

$$P(\bar{G},\lambda)=\sum_{i=0}^{p-1} b_i(G)(\lambda)_{p-i},$$

where $p=|V(G)|$ and $(\lambda)_k=\lambda(\lambda-1)\cdots(\lambda-k+1)$.

Definition 1 (Liu [[5]]). If G is a graph with p vertices, then the polynomial

$$h(G,x)=\sum_{i=0}^{p-1} b_i(G)x^{p-i}$$

is called *the adjoint polynomial* of G.

Definition 2 (Liu [[5]]).

$$R(G)=\begin{cases} b_2(G)-\binom{q(G)-1}{2}+1 & \text{if } q(G)>0,\\ 0 & \text{if } q(G)=0.\end{cases}$$

is said to be *the character of a graph G*.

Lemma 1 (Liu [[5]]). *If G has k connected components $G_1,G_2,\ldots,G_k$, then*

$$h(G,x)=\prod_{i=1}^{k} h(G_i.x),\quad R(G)=\sum_{i=1}^{k} R(G_i).$$

Lemma 2 (Liu [[5]]). (1) (Liu [[1]]) $h(P_{2k+1}) = h(C_{k+1})h(P_k)$ $(k \geqslant 3)$,

(2) (Liu [[4]]) $h_1(C_n) = h_1(1,1,n-2)$, $h_1(D_n) = h_1(1,2,n-3)$,
(3) $h(P_2)h(C_6) = h(P_3)h(D_5)$,
(4) $h(P_2)h(C_9) = h(P_5)h(D_6)$,
(5) $h(P_2)h(C_{15}) = h(P_5)h(C_5)h(D_7)$.

Proof. Conditions (3)–(5) can be directly verified.

Lemma 3 (Liu [[5]]). *Let G be a connected graph, then*

(1) $R(G) \leqslant 1$, *and the equality holds if and only if* $G \cong P_n$ $(n \geqslant 2)$ *or* $G \cong K_3$,
(2) $R(G) = 0$ *if and only if G is one of the graphs* K_1, C_n, D_n *and* $T(l_1, l_2, l_3)$.

Lemma 4 (Liu [[6]]). *Let T be a tree, and* $f(T,\mu)$ *be the characteristic polynomial of T. If*

$$f(T,\mu) = \mu^{\theta(T)} f_1(T,\mu), \quad h(T,x) = x^{\alpha(T)} h_1(T,x)$$

and $x = -\mu^2$, *then*

$$h_1(T,x) = (-1)^k f_1(T,\mu),$$

where $\theta(T)$ *and* $\alpha(T)$ *are the degrees of the lowest terms of* $f(T,\mu)$ *and* $h(T,x)$, *respectively, and k is the number of edges in a maximum matching.*

Lemma 5 (Biggs [[1]]). *If* λ *is m-fold eigenvalue of tree T, then* $-\lambda$ *is, too.*

Lemma 6 (Cvetkovic et al. [[3]]). (1) *The eigenvalues of the T-shape tree* $T(1,1,n-1)$ *are* 0 *and*

$$2\cos\frac{2i-1}{2n+2}\pi, \quad 1 \leqslant i \leqslant n+1.$$

(2) *The eigenvalues of* P_n *are*

$$2\cos\frac{i}{n+1}\pi, \quad 1 \leqslant i \leqslant n.$$

Proposition 1. (1) *The root-set of* $h_1(C_n)$ *is*

$$\left\{-2\left(1+\cos\frac{2i-1}{n}\pi\right) \middle| 1 \leqslant i \leqslant \left[\frac{n}{2}\right]\right\}.$$

(2) *The root-set of* $h_1(P_n)$ *is*

$$\left\{-2\left(1+\cos\frac{2i}{n+1}\pi\right) \middle| 1 \leqslant i \leqslant \left[\frac{n}{2}\right]\right\}.$$

Proof. (1) Since $h_1(C_n,x)=h_1(1,1,n-2)$, by Lemma 6, we know that the eigenvalues of $T(1,1,n-2)$ are 0 and $2\cos((2i-1)/2n)\pi, 1\leqslant i\leqslant n$. From Lemmas 2 and 4, we have

$$h_1(C_n,x)=h_1(1,1,n-2)=(-1)^k f_1(T(1,1,n-2),\mu),$$

where $x=-\mu^2$ and k is the number of edges in a maximum matching of $T(1,1,n-2)$. Since the degree of $h_1(C_n)$ equals one-half of the degree of $f_1(T(1,1,n-2),\mu)$, by Lemma 5, the roots of $f_1(T(1,1,n-2),\mu)$ are symmetric about the origin. Thus the root of $h_1(C_n)$ is opposite to positive roots of square. By Lemma 6 and the trigonometric formula, we get (1). Similarly, we can show that (2) is true. □

Lemma 7 (Wang and Liu [[7]]). (1) *For* $n\geqslant 4$, $\beta(D_n)\leqslant\beta(C_n)\leqslant\beta(P_n)$ *and equality holds if and only if* $n=4$,

(2) (Wang and Liu [[7]]) *for* $n\geqslant 4$, $\beta(D_{n+1})<\beta(D_n)$ *and* $\beta(C_{n+1})<\beta(C_n)$; *for* $\geqslant 2$, $\beta(P_{n+1})<\beta(P_n)$,
(3) *for* $m>3$ *and* $n\geqslant 1$, $(h_1(C_m),h_1(P_{2n}))=1,(m>3,n\geqslant 1)$,
(4) *for* $m\geqslant 4$ *and* $n\geqslant 1$, $h_1(P_n)$ *and* $h_1(C_m)$ *have no multiple root*,
(5) *for* $n\geqslant 4$, $\beta(C_n)>-4$.

Proof. By Proposition 1, we have (3)–(5).

Lemma 8 (Cvetkovic et al. [[3]]). *Let* T *be a tree and* $\lambda_1(T)$ *the maximum eigenvalue of* T. *Then* $\lambda_1(T)<2$ *if and only if*

$$T\in\{P_n,T(1,1,n),T(1,2,2),T(1,2,3),T(1,2,4)\}.$$

Proposition 2. *Let* T *be a tree, then* $\beta(T)>-4$ *if and only if*

$$T\in\{P_n,T(1,1,n),T(1,2,2),T(1,2,3),T(1,2,4)\}.$$

Proof. It follows directly from Lemmas 4 and 8.

Lemma 9 (Zhao et al. [[8]]). (1) *Let* G *be a connected graph such that* $R(G)=-k$ *and* $q(G)\geqslant p(G)+k-1$. *Then* $\beta(G)\leqslant -4$, *where* $k=1,2,3$,

(2) *Let* G *be a connected graph such that* $k\geqslant 4$ *and* $R(G)=-k$. *Then* $q(G)<p(G)+k-1$.

Lemma 10. (1) $\beta(D_5)=\beta(C_6)=\beta(P_{11})$,

(2) $\beta(D_6)=\beta(C_9)=\beta(P_{17})$,
(3) $\beta(D_7)=\beta(C_{15})=\beta(P_{29})$.

Proof. These are direct results of Lemmas 2 and 7.

Lemma 11 (Liu [[5]]).

$$h(P_n,x)=\sum_{k\leqslant n}\binom{k}{n-k}x^k.$$

Lemma 12 (Zhao et al. [[8]]). *Let $n\geqslant 2$. Then $\bar{P}_n$ is chromatically unique if and only if $n=3,5$ or $n\neq 4$ is even.*

Proposition 3. *Let m_0 be odd, then the adjoint equivalent graphs of P_{m_0} can only be*

$$rK_1\cup fK_3\cup\left(\bigcup_{i=1}^{l}P_{u_i}\right)\cup\left(\bigcup_{i=1}^{m}C_{v_i}\right)\cup\left(\bigcup_{i=1}^{n}T(l_1^{(i)},l_2^{(i)},l_3^{(i)})\right)\cup\left(\bigcup_{i\in B}D_i\right),$$

where $f=0,1$; $f+l=1$; $r+n\leqslant f$, $B\subset\{4,5,6,7\}$, and $l_1^{(i)}=1,l_2^{(i)}=2,l_3^{(i)}\leqslant 4$, or $l_1^{(i)}=l_2^{(i)}=1$, $i=1,2,\ldots,n$.

Proof. Let $m_i=(m_{i-1}-1)/2$ $(i=1,2,\ldots,k)$ be positive integers. By Lemma 2(1), it follows that

$$h_1(P_{m_0})=\prod_{i=1}^{k'}h_1(C_{m_i+1})h_1(P_{m_{k'}}),$$

where if $m_k=1,2$, then $k'=k-1$, else $k'=k$.

Let H be the adjointly equivalent graph of P_{m_0}, by Lemma 7 we know that $h_1(P_{m_0})$ has no multiple root. Thus $h_1(H)$ at most has one $h_1(K_3)$. So by Lemma 2(2), $h_1(D_{l'})=h_1(1,2,l'-3)$. If $l'\geqslant 8$, by Lemma 7 and Proposition 2 we have that $\beta(D_{l'})\leqslant -4<\beta(P_{m_0})$. So $h_1(P_{m_0})$ does not contain $h_1(D_{l'})(l'\geqslant 8)$. If $G\cong T(l_1,l_2,l_3)$, and $l_1=1$, $l_2=2,l_3\geqslant 5$ or $l_1\neq 1$, $l_2\neq 1$, 2, according to Proposition 2, we know that $h_1(P_{m_0})$ does not include $h_1(G)$. Hence we can assert that

$$\begin{aligned}H=rK_1\cup fK_3\cup&\left(\bigcup_{i=1}^{l}P_{u_i}\right)\cup\left(\bigcup_{i=1}^{m}C_{v_i}\right)\cup\left(\bigcup_{i=1}^{n}T(l_1^{(i)},l_2^{(i)},l_3^{(i)})\right)\cup\left(\bigcup_{i\in B}D_i\right)\\ &\cup\left(\bigcup_{i=1}^{s_2}H_i\right)\cup\left(\bigcup_{i=s_2+1}^{s_3}H_i\right)\cup\cdots\cup\left(\bigcup_{i=s_{t-1}+1}^{s_t}H_i\right),\end{aligned}\qquad(1)$$

where $R(H_i)=-j$ and H_i is connected if $s_{j-1}+1\leqslant i\leqslant s_j$, $s_0=0$, $j=1,2,\ldots,t$; $B\subset\{4,5,6,7\}$; $l_1^{(i)}=1,l_2^{(i)}=2$; $l_3^{(i)}\leqslant 4$ or $l_1^{(i)}=l_2^{(i)}=1$; $f=0,1$.

By Lemma 1, it follows that

$$R(H) = fR(K_3) + \sum_{i=1}^{l} R(P_{u_i}) + \sum_{i=1}^{m} R(C_{v_i}) + \sum_{i=1}^{n} R(T(l_1^{(i)}, l_2^{(i)}, l_3^{(i)}))$$

$$+ \sum_{i \in B} R(D_i) + \sum_{i=1}^{s_t} R(H_i).$$

From Lemma 3,

$$\sum_{i=1}^{s_t} R(H_i) = 1 - l - f, \quad \sum_{i=1}^{s_t} |R(H_i)| = l + f - 1.$$

From (1), we know that

$$q(H) = fq(K_3) + \sum_{i=1}^{l} q(P_{u_i}) + \sum_{i=1}^{m} q(C_{v_i}) + \sum_{i=1}^{n} q(T(l_1^{(i)}, l_2^{(i)}, l_3^{(i)}))$$

$$+ \sum_{i \in B} q(D_i) + \sum_{i=1}^{s_t} q(H_i).$$

Since $\beta(H_i) \geqslant \beta(P_{m_0}) > -4$ and H_i $(1 \leqslant i \leqslant s_t)$ is connected, by Lemma 9 we have

$$q(H_i) \leqslant p(H_i) + |R(H_i)| - 2, \quad 1 \leqslant i \leqslant s_t.$$

So, we can get that

$$q(H) \leqslant fp(K_3) + \sum_{i=1}^{l} p(P_{u_i}) + \sum_{i=1}^{m} p(C_{v_i}) + \sum_{i=1}^{n} p(T(l_1^{(i)}, l_2^{(i)}, l_3^{(i)}))$$

$$+ \sum_{i \in B} p(D_i) + \sum_{i=1}^{s_t} (p(H_i) + |R(H_i)| - 2) - l - n$$

$$= p(H) - 2s_t - r - n + f - 1$$

and

$$q(H) = q(P_{m_0}) = p(P_{m_0}) - 1 = p(H) - 1.$$

Thus $2s_t + r + n \leqslant f$, $f = 0, 1$. Clearly, $s_t = 0$. Recalling

$$\sum_{i=1}^{s_t} |R(H_i)| = l + f - 1,$$

we get that $f + l = 1$ and $r + n \leqslant f$. Hence Proposition 3 holds. □

Proposition 4. *Let m_0 be odd. If*

$$h_1(P_{m_0}) = h_1(P_n)\prod_{i=1}^{m} h_1(C_{v_i})\prod_{i\in B} h_1(D_i); \quad and \quad B\subset\{4,5,6,7\} \tag{2}$$

or

$$h_1(P_{m_0}) = h_1(P_4)\prod_{i=1}^{m} h_1(C_{v_i})\prod_{i\in B} h_1(D_i)h_1(1,1,l_3), \quad and \quad B\subset\{4,5,6,7\}, \tag{3}$$

then $5,6,7\notin B$.

Proof. Let $m_i = (m_{i-1}-1)/2$ $(i=1,2,\ldots,k)$ be positive integers. We need only prove the fact that $h_1(D_i)$ cannot divide $h_1(P_{m_0})$ for $i=5,6,7$. We prove the fact by induction on m_0.

We first show that $h_1(D_7)$ cannot divide $h_1(P_{m_0})$.

If $m_0<29$, by Lemma 10 we see that $h_1(D_7)$ cannot divide $h_1(P_{m_0})$. If $m_0=29$, assume $h_1(D_7)|h_1(P_{29})$. Since $h_1(P_{m_0})$ is without multiple root, by Lemma 2 and (2), we have

$$h_1(P_5)h_1(C_5)h_1(P_{14})h_1(C_{15}) = h_1(P_5)h_1(C_5)h_1(D_7)h_1(P_n)\prod_{i=1}^{m} h_1(C_{v_i})\prod_{i\in B'} h_1(D_i),$$

where $B' = B-\{7\}$.

We denote by $\beta(\text{left})$ the minimum root of the left-hand side and by $\beta(\text{right})$ the minimum root of the right-hand side. Since

$$h_1(P_2)h_1(C_{15}) = h_1(P_5)h_1(C_5)h_1(D_7),$$

then

$$h_1(P_5)h_1(C_5)h_1(P_{14}) = h_1(P_2)h_1(P_n)\prod_{i=1}^{m} h_1(C_{v_i})\prod_{i\in B'} h_1(D_i),$$

where $\beta(\text{left}) = \beta(P_{14})\notin\{\beta(D_4),\beta(D_5),\beta(D_6)\}$.

According to Lemma 4, the roots of $h_1(C_m)$ and $h_1(P_n)$ are real numbers, then $\beta(\text{right})\neq\beta(\bigcup_{i=1}^{m} C_{v_i})$. So $\beta(P_{14})=\beta(P_n)$ and $n=14$. Eliminating $h_1(P_{14})$ from both sides of the above equality, we get that

$$h_1(P_5)h_1(C_5) = h_1(P_2)\prod_{i=1}^{m} h_1(C_{v_i})\prod_{i\in B'} h_1(D_i).$$

So, $\beta(\text{left})=\beta(C_5)\bar{\in}\{\beta(D_4),\beta(D_5),\beta(D_6)\}$ and there exists v_i such that $\beta(C_5)=\beta(C_{v_i})$. We may assume $v_1=5$, then

$$h_1(P_5) = h_1(P_2)\prod_{i=2}^{m} h_1(C_{v_i})\prod_{i\in B'} h_1(D_i).$$

But then we have $\beta(\text{left}) = \beta(P_5) = -3$ and $\beta(\text{right}) = \beta(D_6 \bigcup_{i=2}^{m} C_{v_i})$. However, this is a contradiction because $\beta(D_6) = \beta(P_{17}) < \beta(P_5)$ and $\beta(C_{v_i}) \neq \beta(P_5)$ for every $v_i \geqslant 4$. So when $m_0 = 29$, $h_1(D_7)$ cannot divide $h_1(P_{m_0})$.

We assume for the moment that $29 < k \leqslant m_0$, then $h_1(D_7)$ cannot divide $h_1(P_k)$.

If $k = m_0$, by the conduction of proposition and Lemma 2(1), we have $B \subset \{4, 5, 6, 7\}$ such that

$$h_1(C_{m_1+1})h_1(P_{m_1}) = h_1(P_n)\prod_{i=1}^{m} h_1(C_{v_i})\prod_{i\in B} h_1(D_i),$$

then $\beta(\text{left}) = \beta(C_{m_1+1})$. Since there is $n < m_0$ such that $\beta(P_n) \neq \beta(P_{m_0})$ and $m_0 > 29$, by Lemma 2, 7, we know that

$$\beta(\text{right}) < \beta(P_{29}) = \beta(C_{15}) \leqslant \beta\left(\bigcup_{i\in B} D_i\right).$$

Thus, from $\prod_{i=1}^{m} h_1(C_{v_i})$ we can get $\beta(\text{right})$.

Similar to $m_0 = 29$, we may assume that $C_{m_1+1} \cong C_{v_1}$. In this case, we have

$$h_1(P_{m_1}) = h_1(P_n)\prod_{i=2}^{m} h_1(C_{v_i})\prod_{i\in B} h_1(D_i).$$

If m_1 is odd, by the induction hypothesis, we know $7 \notin B$.

If m_1 is even, since $m_1 > 14$, we know P_{m_1} is adjointly unique by Lemma 12, so $7 \notin B$. Similarly, we can show that $5, 6 \notin B$ in (2).

Suppose $l_3 > 1$. Since $h_1(1, 1, l_3) = h_1(C_{l_3+2})$, we know that (3) equals (2).

If $l_3 = 1$, then

$$h_1(P_{m_0}) = (x+3)h_1(P_4)\prod_{i=1}^{m} h_1(C_{v_i})\prod_{i\in B} h_1(D_i).$$

Note that $h_1(P_5) = h_1(P_2)(x+3)$. Similar to the proof of (2), we can show that $5, 6, 7 \notin B$ in (3). □

3. Main results and proofs

Theorem 1. *Let m_0 be a positive integer. Then the graph G has chromatic polynomial of the form*

$$\sum_{l \leqslant m_0} \binom{l}{m_0 - l} (\lambda)_l$$

if and only if G satisfies one of the following:

(a) *For even* $m_0 \geqslant 4$, *either* $\bar{G} \cong P_{m_0}$ *or else* $\bar{G} \cong K_1 \cup K_3$.
(b) *For odd* m_0 *with* $m_i = (m_{i-1} - 1)/2$ *a positive integer for* $i = 1, 2, \ldots, k$, *either* $\bar{G} \cong P_{m_0}$ *or else*

 (i) $\bar{G} \in \{K_1 \cup K_3 \bigcup_{i=1}^{k} C_{m_i+1}, K_3 \bigcup_{i \neq j}^{k} C_{m_i+1} \cup T(1,1,m_j - 1),\ P_{m_j} \bigcup_{i=1}^{j} C_{m_i+1}\ (j = 1, 2, \ldots, k)\}$ *if* $m_k = 4$,
 (ii) $\bar{G} \in \{P_{m_j} \bigcup_{i=1}^{j} C_{m_i+1},\ (j = 1, 2, \ldots, k)\}$ *if* $m_k \neq 4$ *is even*,
 (iii) $\bar{G} \in \{P_3 \cup D_4 \bigcup_{i=1}^{k-2} C_{m_i+1}, P_{m_j} \bigcup_{i=1}^{j} C_{m_i+1}\ (j = 1, 2, \ldots, k - 1)\}$ *if* $m_k = 1$.

Proof. *Sufficiency*: If $P(\bar{G}, \lambda) = \sum_{i=1}^{p-1} b_i(G)(\lambda)_{p-i}$, then $h(G, x) = \sum_{i=1}^{p-1} b_i(G) x^{p-i}$. Since $h(P_4) = h(K_1 \cup K_3)$, $h(C_4) = h(D_4)$ and $h(1,1,l) = h(C_{l+2} \cup K_1)$, by Lemma 2(1) and Lemma 11 sufficiency is obvious.

Necessity: We need only prove that the adjointly equivalent graphs of P_{m_0} belong to the class of graphs described in this theorem.

If $m_0 \neq 4$ is an even, by Lemma 12, obviously conclusion holds.

If $m_0 = 4$, we can directly prove that $K_1 \cup K_3$ is only adjoint equivalent graph of P_4.

If m_0 is odd, by Proposition 3, the adjointly set of P_{m_0} is

$$rK_1 \cup fK_3 \cup \left(\bigcup_{i=1}^{l} P_{u_i} \right) \cup \left(\bigcup_{i=1}^{m} C_{v_i} \right) \cup \left(\bigcup_{i=1}^{n} T(l_1^{(i)}, l_2^{(i)}, l_3^{(i)}) \right) \cup \left(\bigcup_{i \in B} D_i \right),$$

where $f = 0, 1;\ f + l = 1;\ r + n \leqslant f;\ l_1^{(i)} = l_2^{(i)} = 1$ or $l_1^{(i)} = 1, l_2^{(i)} = 2, l_3^{(i)} \leqslant 4, B \subset \{4, 5, 6, 7\}$.

We discuss each case in the following.

Case 1: $f = 1, l = 0, r = 1$ and $n = 0$. Then

$$H \cong K_1 \cup K_3 \cup \left(\bigcup_{i=1}^{m} C_{v_i} \right) \cup \left(\bigcup_{i \in B} D_i \right).$$

By Proposition 4, we get $B \subset \{4\}$. Since $h_1(H) = h_1(P_{m_0})$ and $h_1(K_1 \cup K_3) = h_1(P_4)$, we have

$$\prod_{i=1}^{k} h_1(C_{m_i+1}) h_1(P_{m_k}) = h_1(P_4) \prod_{i=1}^{m} h_1(C_{v_i}) \prod_{i \in B} h_1(D_i) \quad (m_k = 2N) \tag{4}$$

or

$$\prod_{i=1}^{k-1} h_1(C_{m_i+1}) h_1(P_{m_{k-1}}) = h_1(P_4) \prod_{i=1}^{m} h_1(C_{v_i}) \prod_{i \in B} h_1(D_i) \quad (m_k = 1, 2). \tag{5}$$

In (4) and (5),

$$\beta(\text{left}) = \beta(C_{m_1+1}), \quad \beta(\text{right}) = \beta\left(\left(\bigcup_{i=1}^{m} C_{v_i} \right) \cup \left(\bigcup_{i \in B} D_i \right) \right).$$

If $\beta(\text{right})=\beta(C_4)=\beta(D_4)$, then (4) does not hold. From (5) we get $k=2, m_2=1$ and $h_1(P_3)=h_1(P_4)$, which is a contradiction. Hence $\beta(\text{right})=\beta(\bigcup_{i=1}^{m} C_{v_i})$. Suppose $\beta(\text{right})=\beta(C_{v_1})$, by symmetry, we have that $C_{m_1+1}\cong C_{v_1}$. Clearly, $m_1+1=v_1$. Eliminating $h_1(C_{m_1+1})$ from both sides of (4) and (5), we obtain that

$$\prod_{i=2}^{k} h_1(C_{m_i+1})h_1(P_{m_k})=h_1(P_4)\prod_{i=2}^{m} h_1(C_{v_i})\prod_{i\in B} h_1(D_i) \quad (m_k=2N)$$

or

$$\prod_{i=2}^{k-1} h_1(C_{m_i+1})h_1(P_{m_{k-1}})=h_1(P_4)\prod_{i=2}^{m} h_1(C_{v_i})\prod_{i\in B} h_1(D_i) \quad (m_k=1,2).$$

Next we continue to proceed this step, there is $i=2,3,\ldots,k$ or $i=2,3,\ldots,k-1$ such that $m_i+1=v_i$. So

$$h_1(P_{m_k})=h_1(P_4)\prod_{i=k+1}^{m} h_1(C_{v_i})\prod_{i\in B} h_1(D_i) \quad (m_k=2N)$$

or

$$h_1(P_{m_{k-1}})=h_1(P_4)\prod_{i=k}^{m} h_1(C_{v_i})\prod_{i\in B} h_1(D_i) \quad (m_k=1,2).$$

Since $m_{k-1}=3,5$ or m_k is an even number greater than or equal to 4, by Lemma 12, we have that $m_k=4, \bigcup_{i=1}^{m} C_{v_i}\cong\bigcup_{i=1}^{k} C_{m_i+1}$ and $|B|=0$. Hence,

$$H\cong K_1\cup K_3\cup\left(\bigcup_{i=1}^{k} C_{m_i+1}\right).$$

Case 2: $f=1,\ l=0,\ r=0$ and $n=1$. So

$$H\cong K_3\cup\left(\bigcup_{i=1}^{m} C_{v_i}\right)\cup\left(\bigcup_{i\in B} D_i\right)\cup T(l_1^{(1)},l_2^{(1)},l_3^{(1)}).$$

From Propositions 2 and 4, it is clear that $B\subset\{4\}$ and $l_1^{(1)}=1, l_2^{(1)}=2$ and $l_3^{(1)}\leqslant 4$. We replace $l_3^{(1)}$ by l_3. For $h_1(1,2,l_3)=h_1(D_{l_3+3})$, we have that $h_1(P_{m_0})$ has no multiple root. So,

$$h_1(P_{m_0})=h_1(P_4)\prod_{i=1}^{m} h_1(C_{v_i})h(1,1,l_3)\prod_{i\in B'} h_1(D_i),$$

where $B'\subset\{4\}$.

Suppose $l_3\geqslant 2$. Since $h_1(1,1,l_3)=h_1(C_{l_3+2})$, similar to case 1, there are $i,j=1,2,\ldots,k;\ i\neq j$ such that

$$v_i=m_i+1,\quad l_3=m_j-1.$$

So

$$m_k = 4, \quad |B| = 0,$$

namely, there is a $j = 1, 2, \ldots, k$ with

$$H \cong K_3 \cup \left(\bigcup_{i \neq j}^{k} C_{m_i+1} \right) \cup T(1, 1, m_j - 1).$$

Suppose $l_3 = 1$. For $h_1(1,1,1) = x + 3$, similar to the discussion of (4) and (5), there is $m_k = 2N$ such that

$$h_1(P_{m_k}) = (x+3) h_1(P_4) \prod_{i \in A} h_1(C_{v_i}) \prod_{i \in B} h_1(D_i)$$

or $m_k = 1, 2$ such that

$$h_1(P_{m_{k-1}}) = (x+3) h_1(P_4) \prod_{i \in A} h_1(C_{v_i}) \prod_{i \in B} h_1(D_i),$$

where $A \subset \{1, 2, \ldots, m\}, B \subset \{4\}$. When $m_k = 4$, the formula above is false. When $m_k = 1, 2$ or $m_k = 2l > 4$, both formulae above contradict Lemma 12.

Case 3: $f = 1$, $l = 0$, $r = 0$ and $n = 0$. There is $B \subset \{4\}$ such that

$$H \cong K_3 \cup \left(\bigcup_{i=1}^{m} C_{v_i} \right) \cup \left(\bigcup_{i \in B} D_i \right).$$

Since $|V(H)| = |E(H)|$ and $|V(P_{m_0})| \neq |E(P_{m_0})|$, we know that H and P_{m_0} are not adjointly equivalent, this is a contradiction.

Case 4: $f = 0, l = 1$ and $n = r = 0$. There is $B \subset \{4, 5, 6, 7\}$ such that

$$H \cong P_u \cup \left(\bigcup_{i=1}^{m} C_{v_i} \right) \cup \left(\bigcup_{i \in B} D_i \right).$$

Here set $u_1 = u$, by Proposition 4, we know that $B \subset \{4\}$. Thus

$$\prod_{i=1}^{k} h_1(C_{m_i+1}) h_1(P_{m_k}) = h_1(P_u) \prod_{i=1}^{m} h_1(C_{v_i}) \prod_{i \in B} h_1(D_i) \quad (m_k = 2N) \tag{6}$$

and

$$\prod_{i=1}^{k} h_1(C_{m_i+1}) h_1(P_{m_{k-1}}) = h_1(P_u) \prod_{i=1}^{m} h_1(C_{v_i}) \prod_{i \in B} h_1(D_i) \quad (m_k = 1, 2). \tag{7}$$

In (6), by Lemma 7(3), we have

$$\left(\prod_{i=1}^{m} h_1(C_{v_i}) \prod_{i \in B} h_1(D_i), h_1(P_{m_k}) \right) = 1,$$

thus $h_1(P_{m_k}) \mid h_1(P_u)$. If u is an even, for the same reason, we have that $h_1(P_{m_k}) \mid h_1(P_u)$, so $u = m_k$. If we eliminate $h_1(P_{m_k})$ from both sides of (6), there is $B \subset \{4\}$ with

$$\prod_{i=1}^{k} h_1(C_{m_i+1}) = \prod_{i=1}^{m} h_1(C_{v_i}) \prod_{i\in B} h_1(D_i).$$

Since $m_k \geqslant 4$, comparing the least roots of both sides, in the above formula, we get that $|B| = 0$ and there is a $j = 1, 2, \ldots, k$ such that

$$H \cong P_{m_j} \cup \left(\bigcup_{i=1}^{j} C_{m_i+1} \right).$$

If u is odd, by Lemma 2(1), we have $u = 3, 5$ by symmetry. If $u = 3$, similar to (4), we have

$$h_1(P_{m_k}) = h_1(P_3) \prod_{i\in A} h_1(C_{v_i}) \prod_{i\in B} h_1(D_i), \quad A \subset \{1, 2, \ldots, m\}.$$

This contradicts Lemma 12.

A similar contradiction occurs when $u = 5$.

In (7), according to three cases, we have that u is even, $u = 3$ and 5. Similar to the discussion of (5), we can show that

$$h_1(P_{m_{k-1}}) = h_1(P_u) \prod_{i\in A} h_1(C_{v_i}) \prod_{i\in B'} h_1(D_i),$$

where $B' \subseteq B$, $A \subseteq \{1, 2, \ldots, m\}$. By Lemma 12, we know $|A| = 0$, $|B'| = 0$ and there is $u = 3$ with $m_k = 1$, $u = 5$ with $m_k = 2$. Namely, if $m_k = 1$, then

$$H \cong P_{m_j} \cup \left(\bigcup_{i=1}^{j} C_{m_i+1} \right) \quad (j = 1, 2, \ldots, k-1),$$

or

$$H \cong P_3 \cup D_4 \cup \left(\bigcup_{i=1}^{k-2} C_{m_i+1} \right).$$

If $m_k = 2$, then

$$H \cong P_{m_j} \cup \left(\bigcup_{i=1}^{j} C_{m_i+1} \right) \quad (j = 1, 2, \ldots, k-1).$$

The proof is completed. □

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