Note

Graphs with chromatic polynomial
\[ \sum_{l \leq m_0} \binom{l}{m_0 - l} (\lambda)^l \]

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Abstract

In this paper, using the properties of chromatic polynomial and adjoint polynomial, we characterize all graphs having chromatic polynomial \[ \sum_{l \leq m_0} \binom{l}{m_0 - l} (\lambda)^l \].

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1. Introduction

The graphs considered here are finite, undirected and simple. Let \( G \) be a graph and \( P(G, \lambda) \) its chromatic polynomial, \( h(G, x) \) be its adjoint polynomial. Two graphs \( G \) and \( H \) are chromatically equivalent if \( P(G, \lambda) = P(H, \lambda) \), and adjointly equivalent if \( h(G, x) = h(H, x) \). A graph \( G \) is chromatically unique if \( P(G, \lambda) = P(H, \lambda) \) implies that \( H \) is isomorphic to \( G \). Similarly, a graph \( G \) is adjointly unique if \( h(G, x) = h(H, x) \) implies that \( H \) is isomorphic to \( G \). By \( \bar{G} \) we denote the complement of \( G \). It is obvious that a graph \( G \) is chromatically unique if and only if \( \bar{G} \) is adjointly unique.

To compute chromatic polynomial for a given graph is well known, but to determine the graphs with a given chromatic polynomial is not easy. In this paper, using the

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properties of adjoint polynomial and the relations of adjoint and chromatical polynomial, we shall give all graphs determined by the polynomial \( \sum_{l \leq m_0} \binom{m}{m_0-1} \lambda^l \).

Let \( G \) be a graph, \( p(G) \) and \( q(G) \) be its order and size, respectively. The symbols \( P_n, C_n \) and \( D_n \) stand for the following graphs of order \( n \): the path, the cycle and the graph obtained by identifying a vertex of \( K_3 \) with one endvertex of \( P_{n-2} \). Let \( T(l_1, l_2, l_3) \), \( (l_1 \leq l_2 \leq l_3) \) be a tree with one vertex of degree 3 and three vertices of degree 1 in which the distances from the vertex of degree 3 to the vertices of degree 1 are \( l_1, l_2 \) and \( l_3 \), respectively. Let \( h(G, x) = x^{q(G)} h_1(G, x) \), where \( h_1(G, x) \) is a polynomial with a nonzero constant term.

For convenience, let \( h(G) \) stand for \( h(G, x) \), and \( h_1(G) \) for \( h_1(G, x) \). We will write \( h(l_1, l_2, l_3) \) for \( h(T(l_1, l_2, l_3), x) \), and \( h_1(l_1, l_2, l_3) \) for \( h_1(T(l_1, l_2, l_3), x) \). Let \( \beta(G) \) denote the least root of \( h_1(G) \). The reader may refer to [[5,2]] for all notations and terminology not explained here.

2. Preliminaries

Let \( b_i(G) \) denote the number of ideal subgraphs with \( p-i \) components (see [[5]]), then
\[
P(G, \lambda) = \sum_{i=0}^{p-1} b_i(G) \lambda^{p-i},
\]
where \( p = |V(G)| \) and \( (\lambda)^k = \lambda(\lambda - 1) \cdots (\lambda - k + 1) \).

**Definition 1** (Liu [[5]]). If \( G \) is a graph with \( p \) vertices, then the polynomial
\[
h(G, x) = \sum_{i=0}^{p-1} b_i(G) x^{p-i}
\]
is called the adjoint polynomial of \( G \).

**Definition 2** (Liu [[5]]).
\[
R(G) = \begin{cases} 
    b_2(G) - \binom{q(G) - 1}{2} + 1 & \text{if } q(G) > 0, \\
    0 & \text{if } q(G) = 0.
\end{cases}
\]
is said to be the character of a graph \( G \).

**Lemma 1** (Liu [[5]]). If \( G \) has \( k \) connected components \( G_1, G_2, \ldots, G_k \), then
\[
h(G, x) = \prod_{i=1}^{k} h(G_i, x), \quad R(G) = \sum_{i=1}^{k} R(G_i).
\]
Lemma 2 (Liu [[5]]). (1) (Liu [[1]]) \( h(P_{2k+1}) = h(C_{k+1})h(P_k) \) \( (k \geq 3) \),

(2) (Liu [[4]]) \( h_1(C_n) = h_1(1, 1, n - 2), \) \( h_1(D_n) = h_1(1, 2, n - 3) \),

(3) \( h(P_2)h(C_0) = h(P_3)h(D_5) \),

(4) \( h(P_2)h(C_0) = h(P_3)h(D_6) \),

(5) \( h(P_2)h(C_{15}) = h(P_5)h(C_5)h(D_7) \).

Proof. Conditions (3)–(5) can be directly verified.

Lemma 3 (Liu [[5]]). Let \( G \) be a connected graph, then

(1) \( R(G) \leq 1 \), and the equality holds if and only if \( G \cong P_n \) \( (n \geq 2) \) or \( G \cong K_3 \),

(2) \( R(G) = 0 \) if and only if \( G \) is one of the graphs \( K_1, C_n, D_n \) and \( T(l_1, l_2, l_3) \).

Lemma 4 (Liu [[6]]). Let \( T \) be a tree, and \( f(T, \mu) \) be the characteristic polynomial of \( T \). If

\[
f(T, \mu) = \mu^{\theta(T)} f_1(T, \mu), \quad h(T, x) = x^{\alpha(T)} h_1(T, x)
\]

and \( x = -\mu^2 \), then

\[
h_1(T, x) = (-1)^k f_1(T, \mu),
\]

where \( \theta(T) \) and \( \alpha(T) \) are the degrees of the lowest terms of \( f(T, \mu) \) and \( h(T, x) \), respectively, and \( k \) is the number of edges in a maximum matching.

Lemma 5 (Biggs [[1]]). If \( \lambda \) is \( m \)-fold eigenvalue of tree \( T \), then \( -\lambda \) is, too.

Lemma 6 (Cvetkovic et al. [[3]]). (1) The eigenvalues of the T-shape tree \( T(1, 1, n - 1) \) are 0 and

\[
2 \cos \frac{2i - 1}{2n+2} \pi, \quad 1 \leq i \leq n + 1.
\]

(2) The eigenvalues of \( P_n \) are

\[
2 \cos \frac{i}{n+1} \pi, \quad 1 \leq i \leq n.
\]

Proposition 1. (1) The root-set of \( h_1(C_n) \) is

\[
\left\{-2 \left(1 + \cos \frac{2i - 1}{n} \pi\right) \mid 1 \leq i \leq \left[\frac{n}{2}\right]\right\}.
\]

(2) The root-set of \( h_1(P_n) \) is

\[
\left\{-2 \left(1 + \cos \frac{2i}{n + 1} \pi\right) \mid 1 \leq i \leq \left[\frac{n}{2}\right]\right\}.
\]
Proof. (1) Since \( h_1(C_n,x) = h_1(1,1,n-2) \), by Lemma 6, we know that the eigenvalues of \( T(1,1,n-2) \) are 0 and \( 2 \cos((2i-1)/2n)\pi, 1 \leq i \leq n \). From Lemmas 2 and 4, we have

\[
h_1(C_n,x) = h_1(1,1,n-2) = (-1)^{k} f_1(T(1,1,n-2),\mu),
\]

where \( x = -\mu^2 \) and \( k \) is the number of edges in a maximum matching of \( T(1,1,n-2) \). Since the degree of \( h_1(C_n) \) equals one-half of the degree of \( f_1(T(1,1,n-2),\mu) \), by Lemma 5, the roots of \( f_1(T(1,1,n-2),\mu) \) are symmetric about the origin. Thus the root of \( h_1(C_n) \) is opposite to positive roots of square. By Lemma 6 and the trigonometric formula, we get (1). Similarly, we can show that (2) is true. □

Lemma 7 (Wang and Liu [[7]]). (1) For \( n \geq 4 \), \( \beta(D_n) \leq \beta(C_n) \leq \beta(P_n) \) and equality holds if and only if \( n = 4 \),

(2) (Wang and Liu [[7]]) for \( n \geq 4 \), \( \beta(D_{n+1}) < \beta(D_n) \) and \( \beta(C_{n+1}) < \beta(C_n) \); for \( n \geq 4 \), \( \beta(P_{n+1}) < \beta(P_n) \).

(3) for \( m > 3 \) and \( n \geq 1 \), \( (h_1(C_m), h_1(P_{2n})) = 1, (m > 3, n \geq 1) \),

(4) for \( m \geq 4 \) and \( n \geq 1 \), \( h_1(P_n) \) and \( h_1(C_m) \) have no multiple root,

(5) for \( n \geq 4 \), \( \beta(C_n) > -4 \).

Proof. By Proposition 1, we have (3)–(5).

Lemma 8 (Cvetkovic et al. [[3]]). Let \( T \) be a tree and \( \lambda_1(T) \) the maximum eigenvalue of \( T \). Then \( \lambda_1(T) < 2 \) if and only if

\[
T \in \{P_n, T(1,1,n), T(1,2,2), T(1,2,3), T(1,2,4)\}.
\]

Proposition 2. Let \( T \) be a tree, then \( \beta(T) > -4 \) if and only if

\[
T \in \{P_n, T(1,1,n), T(1,2,2), T(1,2,3), T(1,2,4)\}.
\]

Proof. It follows directly from Lemmas 4 and 8.

Lemma 9 (Zhao et al. [[8]]). (1) Let \( G \) be a connected graph such that \( R(G) = -k \) and \( q(G) \geq p(G) + k - 1 \). Then \( \beta(G) \leq -4 \), where \( k = 1,2,3 \),

(2) Let \( G \) be a connected graph such that \( k \geq 4 \) and \( R(G) = -k \). Then \( q(G) < p(G) + k - 1 \).

Lemma 10. (1) \( \beta(D_3) = \beta(C_6) = \beta(P_{11}) \),

(2) \( \beta(D_6) = \beta(C_6) = \beta(P_{17}) \),

(3) \( \beta(D_7) = \beta(C_{15}) = \beta(P_{20}) \).
Proof. These are direct results of Lemmas 2 and 7.

Lemma 11 (Liu [[5]]).

\[ h(P_n, x) = \sum_{k \leq n} \left( \binom{n}{k} x^k \right) \]

Lemma 12 (Zhao et al. [[8]]). Let \( n \geq 2 \). Then \( \bar{P}_n \) is chromatically unique if and only if \( n = 3, 5 \) or \( n \neq 4 \) is even.

Proposition 3. Let \( m_0 \) be odd, then the adjoint equivalent graphs of \( P_{m_0} \) can only be

\[ rK_1 \cup fK_3 \cup \left( \bigcup_{i=1}^{l} P_i \right) \cup \left( \bigcup_{j=1}^{m} C_i \right) \cup \left( \bigcup_{l=1}^{n} \mathbb{T}(l_1^{(i)}, l_2^{(i)}, l_3^{(i)}) \right) \cup \left( \bigcup_{i \in B} D_i \right), \]

where \( f = 0, 1; \ f + l = 1; \ r + n \leq f, \ B \subset \{4, 5, 6, 7\}, \) and \( l_1^{(i)} = 1, l_2^{(i)} = 2, l_3^{(i)} \leq 4, \) or \( l_1^{(i)} = l_2^{(i)} = 1, \ i = 1, 2, \ldots, n. \)

Proof. Let \( m_i = (m_i - 1)/2 \) \((i = 1, 2, \ldots, k)\) be positive integers. By Lemma 2(1), it follows that

\[ h_1(P_{m_0}) = \prod_{i=1}^{k'} h_1(C_{m_i+1})h_1(P_{m_i}), \]

where if \( m_k = 1, 2, \) then \( k' = k = 1, \) else \( k' = k. \)

Let \( H \) be the adjoinly equivalent graph of \( P_{m_0}, \) by Lemma 7 we know that \( h_1(P_{m_0}) \) has no multiple root. Thus \( h_1(H) \) at most has one \( h_1(K_3) \). So by Lemma 2(2), \( h_1(D_{l'}) = h_1(1, 2, l' - 3). \) If \( l' \geq 8, \) by Lemma 7 and Proposition 2 we have that \( \beta(D_{l'}) \leq -4 < \beta(P_{m_0}). \) So \( h_1(P_{m_0}) \) does not contain \( h_1(D_{l'})(l' \geq 8). \) If \( G \cong T(l_1, l_2, l_3), \) and \( l_1 = 1, l_2 = 2, l_3 \geq 5 or l_1 \neq 1, l_2 \neq 1, 2, \) according to Proposition 2, we know that \( h_1(P_{m_0}) \) does not include \( h_1(G). \) Hence we can assert that

\[ H = rK_1 \cup fK_3 \cup \left( \bigcup_{i=1}^{l} P_i \right) \cup \left( \bigcup_{j=1}^{m} C_i \right) \cup \left( \bigcup_{l=1}^{n} \mathbb{T}(l_1^{(i)}, l_2^{(i)}, l_3^{(i)}) \right) \cup \left( \bigcup_{i \in B} D_i \right) \]

\[ \cup \left( \bigcup_{i=1}^{s_2} H_i \right) \cup \left( \bigcup_{i=s_2+1}^{s_1} H_i \right) \cup \cdots \cup \left( \bigcup_{i=s_{t-1}+1}^{s_t} H_i \right), \]

where \( R(H_i) = -j \) and \( H_i \) is connected if \( s_j - 1 + 1 \leq i \leq s_j, \) \( s_0 = 0, \) \( j = 1, 2, \ldots, t; \) \( B \subset \{4, 5, 6, 7\}; \) \( l_1^{(i)} = 1, l_2^{(i)} = 2; \) \( l_3^{(i)} \leq 4 \) or \( l_1^{(i)} = l_2^{(i)} = 1; \) \( f = 0, 1. \)
By Lemma 1, it follows that

\[ R(H) = f R(K_3) + \sum_{i=1}^{l} R(P_{n_i}) + \sum_{i=1}^{m} R(C_{v_i}) + \sum_{i=1}^{n} R(T(l_1^{(i)}, l_2^{(i)}, l_3^{(i)})) \]

\[ + \sum_{i \in B} R(D_i) + \sum_{i=1}^{s_l} R(H_i). \]

From Lemma 3,

\[ \sum_{i=1}^{s_l} R(H_i) = 1 - l - f, \quad \sum_{i=1}^{s_l} |R(H_i)| = l + f - 1. \]

From (1), we know that

\[ q(H) = f q(K_3) + \sum_{i=1}^{l} q(P_{n_i}) + \sum_{i=1}^{m} q(C_{v_i}) + \sum_{i=1}^{n} q(T(l_1^{(i)}, l_2^{(i)}, l_3^{(i)})) \]

\[ + \sum_{i \in B} q(D_i) + \sum_{i=1}^{s_l} q(H_i). \]

Since \( \beta(H_i) \geq \beta(P_{m_0}) > -4 \) and \( H_i \ (1 \leq i \leq s_l) \) is connected, by Lemma 9 we have

\[ q(H_i) \leq p(H_i) + |R(H_i)| - 2, \quad 1 \leq i \leq s_l. \]

So, we can get that

\[ q(H) \leq f p(K_3) + \sum_{i=1}^{l} p(P_{n_i}) + \sum_{i=1}^{m} p(C_{v_i}) + \sum_{i=1}^{n} p(T(l_1^{(i)}, l_2^{(i)}, l_3^{(i)})) \]

\[ + \sum_{i \in B} p(D_i) + \sum_{i=1}^{s_l} (p(H_i) + |R(H_i)| - 2) - l - n \]

\[ = p(H) - 2s_l - r - n + f - 1 \]

and

\[ q(H) = q(P_{m_0}) = p(P_{m_0}) - 1 = p(H) - 1. \]

Thus \( 2s_l + r + n \leq f, \quad f = 0, 1 \). Clearly, \( s_l = 0 \). Recalling

\[ \sum_{i=1}^{s_l} |R(H_i)| = l + f - 1, \]

we get that \( f + l = 1 \) and \( r + n \leq f \). Hence Proposition 3 holds. \( \square \)
**Proposition 4.** Let $m_0$ be odd. If
\[ h_1(P_{m_0}) = h_1(P_n) \prod_{i=1}^{m} h_1(C_{v_i}) \prod_{i \in B} h_1(D_i); \quad \text{and} \quad B \subset \{4, 5, 6, 7\} \] (2)
or
\[ h_1(P_{m_0}) = h_1(P_4) \prod_{i=1}^{m} h_1(C_{v_i}) \prod_{i \in B} h_1(D_i) h_1(1, 1, l_3), \quad \text{and} \quad B \subset \{4, 5, 6, 7\}, \] (3)
then $5, 6, 7 \notin B$.

**Proof.** Let $m_i = (m_{i-1} - 1)/2$ ($i = 1, 2, \ldots, k$) be positive integers. We need only prove the fact that $h_1(D_i)$ cannot divide $h_1(P_{m_0})$ for $i = 5, 6, 7$. We prove the fact by induction on $m_0$.

We first show that $h_1(D_7)$ cannot divide $h_1(P_{m_0})$.

If $m_0 < 29$, by Lemma 10 we see that $h_1(D_7)$ cannot divide $h_1(P_{m_0})$. If $m_0 = 29$, assume $h_1(D_7) | h_1(P_{29})$. Since $h_1(P_{m_0})$ is without multiple root, by Lemma 2 and (2), we have
\[ h_1(P_5) h_1(C_5) h_1(P_{14}) h_1(C_{15}) = h_1(P_5) h_1(C_5) h_1(D_7) h_1(P_n) \prod_{i=1}^{m} h_1(C_{v_i}) \prod_{i \in B'} h_1(D_i), \]
where $B' = B \setminus \{7\}$.

We denote by $\beta$(left) the minimum root of the left-hand side and by $\beta$(right) the minimum root of the right-hand side. Since
\[ h_1(P_2) h_1(C_{15}) = h_1(P_3) h_1(C_5) h_1(D_7), \]
then
\[ h_1(P_5) h_1(C_5) h_1(P_{14}) = h_1(P_2) h_1(P_n) \prod_{i=1}^{m} h_1(C_{v_i}) \prod_{i \in B'} h_1(D_i), \]
where $\beta$(left) $\in \{\beta(D_4), \beta(D_5), \beta(D_6)\}$.

According to Lemma 4, the roots of $h_1(C_m)$ and $h_1(P_n)$ are real numbers, then $\beta$(right) $\neq \beta(\bigcup_{i=1}^{m} C_{v_i})$. So $\beta(P_{14}) = \beta(P_n)$ and $n = 14$. Eliminating $h_1(P_{14})$ from both sides of the above equality, we get that
\[ h_1(P_5) h_1(C_5) = h_1(P_2) \prod_{i=1}^{m} h_1(C_{v_i}) \prod_{i \in B'} h_1(D_i). \]

So, $\beta$(left) $\in \{\beta(D_4), \beta(D_5), \beta(D_6)\}$ and there exists $v_i$ such that $\beta(C_5) = \beta(C_{v_i})$. We may assume $v_1 = 5$, then
\[ h_1(P_5) = h_1(P_2) \prod_{i=2}^{m} h_1(C_{v_i}) \prod_{i \in B'} h_1(D_i). \]
But then we have $\beta(\text{left}) = \beta(P_5) = -3$ and $\beta(\text{right}) = \beta(D_6 \bigcup_{i=2}^{m} C_{v_i})$. However, this is a contradiction because $\beta(D_6) = \beta(P_{17}) < \beta(P_5)$ and $\beta(C_{v_i}) \neq \beta(P_5)$ for every $v_i \geq 4$. So when $m_0 = 29$, $h_1(D_7)$ cannot divide $h_1(P_{m_0})$.

We assume for the moment that $29 < k \leq m_0$, then $h_1(D_7)$ cannot divide $h_1(P_k)$.

If $k = m_0$, by the conduction of proposition and Lemma 2(1), we have $B \subset \{4, 5, 6, 7\}$ such that

$$h_1(C_{m_i+1})h_1(P_{m_i}) = h_1(P_n) \prod_{i=1}^{m_0} h_1(C_{v_i}) \prod_{i \in B} h_1(D_i),$$

then $\beta(\text{left}) = \beta(C_{m_i+1})$. Since there is $n < m_0$ such that $\beta(P_n) \neq \beta(P_{m_0})$ and $m_0 > 29$, by Lemma 2, 7, we know that

$$\beta(\text{right}) < \beta(P_{29}) = \beta(C_{15}) \leq \beta \left( \bigcup_{i \in B} D_i \right).$$

Thus, from $\prod_{i=1}^{m_0} h_1(C_{v_i})$ we can get $\beta(\text{right})$.

Similar to $m_0 = 29$, we may assume that $C_{m_i+1} \simeq C_{v_1}$. In this case, we have

$$h_1(P_{m_i}) = h_1(P_n) \prod_{i=2}^{m} h_1(C_{v_i}) \prod_{i \in B} h_1(D_i).$$

If $m_1$ is odd, by the induction hypothesis, we know $7 \notin B$.

If $m_1$ is even, since $m_1 \geq 14$, we know $P_{m_1}$ is adjointly unique by Lemma 12, so $7 \notin B$. Similarly, we can show that $5, 6 \notin B$ in (2).

Suppose $l_3 > 1$. Since $h_1(1, 1, l_3) = h_1(C_{l_3+2})$, we know that (3) equals (2).

If $l_3 = 1$, then

$$h_1(P_{m_0}) = (x + 3) h_1(P_4) \prod_{i=1}^{m} h_1(C_{v_i}) \prod_{i \in B} h_1(D_i).$$

Note that $h_1(P_5) = h_1(P_2)(x + 3)$. Similar to the proof of (2), we can show that $5, 6, 7 \notin B$ in (3). □

3. Main results and proofs

Theorem 1. Let $m_0$ be a positive integer. Then the graph $G$ has chromatic polynomial of the form

$$\sum_{l \leq m_0} \binom{l}{m_0 - l} (x)_l.$$
if and only if $G$ satisfies one of the following:

(a) For even $m_0 \geq 4$, either $\tilde{G} \cong P_{m_0}$ or else $\tilde{G} \cong K_1 \cup K_3$.
(b) For odd $m_0$ with $m_i = (m_i - 1)/2$ a positive integer for $i = 1, 2, \ldots, k$, either $\tilde{G} \cong P_{m_0}$ or else

(i) $\tilde{G} \in \{K_1 \cup K_3 \cup \bigcup_{j=1}^{k} C_{m_1+1}, K_3 \cup \bigcup_{j \neq j}^{k} C_{m_1+1} \cup T(1, 1, m_j - 1), P_{m_j} \cup \bigcup_{i=1}^{l} C_{m_i+1} \mid j = 1, 2, \ldots, k\}$ if $m_k = 4$,
(ii) $\tilde{G} \in \{P_{m_j} \cup \bigcup_{j=1}^{l} C_{m_i+1}, (j = 1, 2, \ldots, k)\}$ if $m_k \neq 4$ is even,
(iii) $\tilde{G} \in \{P_3 \cup D_4 \cup \bigcup_{j=1}^{k-2} C_{m_1+1}, P_{m_j} \cup \bigcup_{i=1}^{l} C_{m_i+1} \mid j = 1, 2, \ldots, k - 1\}$, if $m_k = 1$.

**Proof.** Sufficiency: If $P(\tilde{G}, \lambda) = \sum_{i=1}^{p-1} b_i(G)(\lambda)^{p-i}$, then $h(G, x) = \sum_{i=1}^{p-1} b_i(G)x^{p-i}$.

Since $h(P_4) = h(K_1 \cup K_3)$, $h(C_4) = h(D_4)$ and $h(1, 1, l) = h(C_{l+2} \cup K_1)$, by Lemma 2(1) and Lemma 11 sufficiency is obvious.

Necessity: We need only prove that the adjointly equivalent graphs of $P_{m_0}$ belong to the class of graphs described in this theorem.

If $m_0 \neq 4$ is an even, by Lemma 12, obviously conclusion holds.

If $m_0 = 4$, we can directly prove that $K_1 \cup K_3$ is only adjoint equivalent graph of $P_4$.

If $m_0$ is odd, by Proposition 3, the adjointly set of $P_{m_0}$ is

$$rK_1 \cup fK_3 \cup \left( \bigcup_{i=1}^{l} P_{n_i} \right) \cup \left( \bigcup_{i=1}^{m} C_{v_i} \right) \cup \left( \bigcup_{i=1}^{n} T(l^{(i)}_1, l^{(i)}_2, l^{(i)}_3) \right) \cup \left( \bigcup_{i \in B} D_i \right),$$

where $f = 0, 1; f + l = 1; r + n \leq f; l^{(i)}_1 = l^{(i)}_2 = 1$ or $l^{(i)}_1 = 1, l^{(i)}_2 = 2, l^{(i)}_3 \leq 4, B \subset \{4, 5, 6, 7\}$.

We discuss each case in the following.

**Case 1:** $f = 1, l = 0, r = 1$ and $n = 0$. Then

$$H \cong K_1 \cup K_3 \cup \left( \bigcup_{i=1}^{m} C_{v_i} \right) \cup \left( \bigcup_{i \in B} D_i \right).$$

By Proposition 4, we get $B \subset \{4\}$. Since $h_1(H) = h_1(P_{m_0})$ and $h_1(K_1 \cup K_3) = h_1(P_4)$, we have

$$\prod_{i=1}^{k} h_1(C_{m_i+1})h_1(P_{m_i}) = h_1(P_4) \prod_{i=1}^{m} h_1(C_{v_i}) \prod_{i \in B} h_1(D_i) \quad (m_k = 2N) \quad (4)$$

or

$$\prod_{i=1}^{k-1} h_1(C_{m_i+1})h_1(P_{m_k-1}) = h_1(P_4) \prod_{i=1}^{m} h_1(C_{v_i}) \prod_{i \in B} h_1(D_i) \quad (m_k = 1, 2). \quad (5)$$

In (4) and (5),

$$\beta(\text{left}) = \beta(C_{m_1+1}), \quad \beta(\text{right}) = \beta \left( \left( \bigcup_{i=1}^{m} C_{v_i} \right) \cup \left( \bigcup_{i \in B} D_i \right) \right).$$
If $\beta(\text{right}) = \beta(C_4) = \beta(D_4)$, then (4) does not hold. From (5) we get $k = 2, m_2 = 1$ and $h_1(P_3) = h_1(P_4)$, which is a contradiction. Hence $\beta(\text{right}) = \beta(\bigcup_{i=1}^{m} C_i)$. Suppose $\beta(\text{right}) = \beta(C_{v_1})$, by symmetry, we have that $C_{v_1+1} \cong C_{v_1}$. Clearly, $m_1 + 1 = v_1$. Eliminating $h_1(C_{m_1+1})$ from both sides of (4) and (5), we obtain that

$$\prod_{i=2}^{k} h_1(C_{m_{i+1}})h_1(P_{m_i}) = h_1(P_4) \prod_{i=2}^{m} h_1(C_{v_i}) \prod_{i \in B} h_1(D_i) \quad (m_k = 2N)$$

or

$$\prod_{i=2}^{k-1} h_1(C_{m_{i+1}})h_1(P_{m_{i-1}}) = h_1(P_4) \prod_{i=2}^{m} h_1(C_{v_i}) \prod_{i \in B} h_1(D_i) \quad (m_k = 1, 2).$$

Next we continue to proceed this step, there is $i = 2, 3, \ldots, k$ or $i = 2, 3, \ldots, k - 1$ such that $m_i + 1 = v_i$. So

$$h_1(P_{m_k}) = h_1(P_4) \prod_{i=k+1}^{m} h_1(C_{v_i}) \prod_{i \in B} h_1(D_i) \quad (m_k = 2N)$$

or

$$h_1(P_{m_{k-1}}) = h_1(P_4) \prod_{i=k}^{m} h_1(C_{v_i}) \prod_{i \in B} h_1(D_i) \quad (m_k = 1, 2).$$

Since $m_{k-1} = 3, 5$ or $m_k$ is an even number greater than or equal to 4, by Lemma 12, we have that $m_k = 4, \bigcup_{i=1}^{m} C_{v_i} \cong \bigcup_{i=1}^{m} C_{v_{m+1}}$ and $|B| = 0$. Hence,

$$H \cong K_1 \cup K_3 \cup \left( \bigcup_{i=1}^{k} C_{m_{i+1}} \right).$$

**Case 2:** $f = 1, \quad l = 0, \quad r = 0$ and $n = 1$. So

$$H \cong K_3 \cup \left( \bigcup_{i=1}^{m} C_{v_i} \right) \cup \left( \bigcup_{i \in B} D_i \right) \cup T(l_1^{(1)}, l_2^{(1)}, l_3^{(1)}).$$

From Propositions 2 and 4, it is clear that $B \subset \{4\}$ and $l_1^{(1)} = 1, l_2^{(1)} = 2$ and $l_3^{(1)} \leq 4$. We replace $l_3^{(1)}$ by $l_3$. For $h_1(1, 2, l_3) = h_1(D_{h_3+3})$, we have that $h_1(P_{m_0})$ has no multiple root. So,

$$h_1(P_{m_0}) = h_1(P_4) \prod_{i=1}^{m} h_1(C_{v_i}) h(1, 1, l_3) \prod_{i \in B'} h_1(D_i),$$

where $B' \subset \{4\}$.

Suppose $l_3 \geq 2$. Since $h_1(1, 1, l_3) = h_1(C_{l_3+2})$, similar to case 1, there are $i, j = 1, 2, \ldots, k; \quad i \neq j$ such that

$$v_i = m_i + 1, \quad l_3 = m_j - 1.$$
So

\[ m_k = 4, \quad |B| = 0, \]

namely, there is a \( j = 1, 2, \ldots, k \) with

\[ H \cong K_3 \cup \left( \bigcup_{i \neq j}^k C_{m_i+1} \right) \cup T(1, 1, m_j - 1). \]

Suppose \( l_3 = 1 \). For \( h_1(1, 1, 1) = x + 3 \), similar to the discussion of (4) and (5), there is \( m_k = 2N \) such that

\[ h_1(P_{m_k}) = (x + 3)h_1(P_4) \prod_{i \in A} h_1(C_{v_i}) \prod_{i \in B} h_1(D_i) \]

or \( m_k = 1, 2 \) such that

\[ h_1(P_{m_k-1}) = (x + 3)h_1(P_4) \prod_{i \in A} h_1(C_{v_i}) \prod_{i \in B} h_1(D_i), \]

where \( A \subset \{1, 2, \ldots, m\} \), \( B \subset \{4\} \). When \( m_k = 4 \), the formula above is false. When \( m_k = 1, 2 \) or \( m_k = 2l > 4 \), both formulae above contradict Lemma 12.

**Case 3:** \( f = 1 \), \( l = 0 \), \( r = 0 \) and \( n = 0 \). There is \( B \subset \{4\} \) such that

\[ H \cong K_3 \cup \left( \bigcup_{i = 1}^m C_{v_i} \right) \cup \left( \bigcup_{i \in B} D_i \right). \]

Since \( |V(H)| = |E(H)| \) and \( |V(P_{m_k})| \neq |E(P_{m_k})| \), we know that \( H \) and \( P_{m_k} \) are not adjointly equivalent, this is a contradiction.

**Case 4:** \( f = 0 \), \( l = 1 \) and \( n = r = 0 \). There is \( B \subset \{4, 5, 6, 7\} \) such that

\[ H \cong P_u \cup \left( \bigcup_{i = 1}^m C_{v_i} \right) \cup \left( \bigcup_{i \in B} D_i \right). \]

Here set \( u_1 = u \), by Proposition 4, we know that \( B \subset \{4\} \). Thus

\[ \prod_{i = 1}^k h_1(C_{m_i+1})h_1(P_{m_k}) = h_1(P_u) \prod_{i \in A} h_1(C_{v_i}) \prod_{i \in B} h_1(D_i) \quad (m_k = 2N) \quad (6) \]

and

\[ \prod_{i = 1}^k h_1(C_{m_i+1})h_1(P_{m_k-1}) = h_1(P_u) \prod_{i = 1}^m h_1(C_{v_i}) \prod_{i \in B} h_1(D_i) \quad (m_k = 1, 2). \quad (7) \]

In (6), by Lemma 7(3), we have

\[ \left( \prod_{i = 1}^m h_1(C_{v_i}) \prod_{i \in B} h_1(D_i), h_1(P_{m_k}) \right) = 1, \]
thus \( h_1(P_{m_k}) \mid h_1(P_u) \). If \( u \) is an even, for the same reason, we have that \( h_1(P_{m_k}) \mid h_1(P_u) \), so \( u = m_k \). If we eliminate \( h_1(P_{m_k}) \) from both sides of (6), there is \( B \subset \{4\} \) with

\[
\prod_{i=1}^{k} h_1(C_{m_i+1}) = \prod_{i=1}^{m} h_1(C_{v_i}) \prod_{i \in B} h_1(D_i).
\]

Since \( m_k \geq 4 \), comparing the least roots of both sides, in the above formula, we get that \( |B| = 0 \) and there is a \( j = 1, 2, \ldots, k \) such that

\[
H \cong P_j \cup \left( \bigcup_{i=1}^{j} C_{m_i+1} \right).
\]

If \( u \) is odd, by Lemma 2(1), we have \( u = 3, 5 \) by symmetry. If \( u = 3 \), similar to (4), we have

\[
h_1(P_{m_k}) = h_1(P_3) \prod_{i \in A} h_1(C_{v_i}) \prod_{i \in B} h_1(D_i), \quad A \subset \{1, 2, \ldots, m\}.
\]

This contradicts Lemma 12.

A similar contradiction occurs when \( u = 5 \).

In (7), according to three cases, we have that \( u \) is even, \( u = 3 \) and \( 5 \). Similar to the discussion of (5), we can show that

\[
h_1(P_{m_{k-1}}) = h_1(P_u) \prod_{i \in A} h_1(C_{v_i}) \prod_{i \in B'} h_1(D_i),
\]

where \( B' \subset B, A \subset \{1, 2, \ldots, m\} \). By Lemma 12, we know \( |A| = 0, |B'| = 0 \) and there is \( u = 3 \) with \( m_k = 1 \), \( u = 5 \) with \( m_k = 2 \). Namely, if \( m_k = 1 \), then

\[
H \cong P_j \cup \left( \bigcup_{i=1}^{j} C_{m_i+1} \right) \quad (j = 1, 2, \ldots, k-1),
\]

or

\[
H \cong P_3 \cup D_4 \cup \left( \bigcup_{i=1}^{k-2} C_{m_i+1} \right).
\]

If \( m_k = 2 \), then

\[
H \cong P_j \cup \left( \bigcup_{i=1}^{j} C_{m_i+1} \right) \quad (j = 1, 2, \ldots, k-1).
\]

The proof is completed. \( \square \)
References