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J. Math. Anal. Appl. 315 (2006) 132–143

Journal of MATHEMATICAL ANALYSIS AND APPLICATIONS

www.elsevier.com/locate/jmaa

Error estimates for discontinuous Galerkin method for nonlinear parabolic equations

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Received 7 March 2004

Available online 8 September 2005

Submitted by William F. Ames

Abstract

We consider the nonlinear parabolic partial differential equations. We construct a discontinuous Galerkin approximation using a penalty term and obtain an optimal $L^\infty(L^2)$ error estimate.

Keywords: Discontinuous Galerkin method; Error estimate; Nonlinear parabolic equation

1. Introduction

Discontinuous Galerkin methods with interior penalties for elliptic and parabolic equations were introduced by several authors [1,2,7]. They generalized Nitsche method in [3] to treat the Dirichlet boundary condition with penalty terms on the boundary of the domain. These methods referred to as interior penalty Galerkin schemes are not locally mass conservative.

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1 This work was supported by Korea Research Foundation Grant (KRF-2003-015-C00070).

0022-247X/$ – see front matter
doi:10.1016/j.jmaa.2005.07.027

A new type of elementwise conservative discontinuous Galerkin method for diffusion problem was introduced and analyzed by Oden et al. [4]. Recently, Riviere and Wheeler [5] introduced a locally conservative discontinuous Galerkin formulation for nonlinear parabolic equations and derived a priori $L^\infty(L^2)$ and $L^2(H^1)$ error estimates. However, the error estimate in the $L^\infty(L^2)$ norm is not optimal.

The objective of this paper is to obtain an optimal error estimate in the $L^\infty(L^2)$ norm, which improves the result of [5]. A model problem and some assumptions are introduced in Section 2. In Section 3, we describe the definitions and the formulation of the discontinuous Galerkin method. And the optimal error estimates are obtained in Section 4.

2. A model problem

Consider the following nonlinear parabolic partial differential equation:

$$u_t - \nabla \cdot \big(a(x,u)\nabla u\big) = f(x,u), \quad (x,t) \in \Omega \times (0,\ T], \tag{2.1}$$

with the boundary condition

$$a(x,u)\nabla u \cdot n = 0, \quad (x,t) \in \partial\Omega \times (0,T], \tag{2.2}$$

and the initial condition

$$u(x,0) = g(x), \quad x \in \Omega, \tag{2.3}$$

where Ω is a bounded convex domain in $\mathbf{R}^d$, $d = 1, 2$, and n is a unit outward normal vector to $\partial\Omega$.

Assume that the following conditions are satisfied.

1. For any bounded subset B of real numbers, there exist constants γ and γ^* such that

$$0 < \gamma \leqslant a(x,p) \leqslant \gamma^*, \quad 0 < \gamma \leqslant \frac{\partial}{\partial p} a(x,p) \leqslant \gamma^* \quad \text{for any } (x,p) \in \Omega \times B.$$

2. a and f are uniformly Lipschitz continuous with respect to their second variable.
3. The model problem has a unique solution satisfying the following regularity conditions:

$$u \in L^2\big([0,T], H^s(\Omega)\big), \quad u_t \in L^2\big([0,T], H^s(\Omega)\big), \quad \text{for } s \geqslant 2;$$
$$u_t \in L^\infty\big([0,T], L^\infty(\Omega)\big), \quad \nabla u \in L^\infty\big(\Omega \times [0,T]\big).$$

3. A discontinuous Galerkin method

Let $\mathcal{E}_h = \{E_1, E_2, \cdot, E_{N_h}\}$ be a subdivision of Ω, where E_j is a triangle or a quadrilateral. Let $h_j = \operatorname{diam}(E_j)$ be the diameter of E_j and $h = \max\{h_j\colon\ j = 1, 2, \ldots, N_h\}$. We denote the edges of the elements by $\{e_1, e_2, \ldots, e_{P_h}, e_{P_h+1}, \ldots, e_{M_h}\}$, where $e_k \subset \Omega$, $1 \leqslant k \leqslant P_h$, and $e_k \subset \partial\Omega$, $P_h + 1 \leqslant k \leqslant M_h$. For each edge e_k, $P_h + 1 \leqslant k \leqslant M_h$, we take n_k the unit outward normal vector to $\partial\Omega$. And if $e_k = \partial E_i \cap \partial E_j$ for $i < j$ and $1 \leqslant k \leqslant P_h$ then we take n_k the unit outward normal vector to E_i.

For $s \geqslant 0$, let

$$H^s(\mathcal{E}_h) = \{ v \in L^2(\Omega) \mid v|_{E_j} \in H^s(E_j),\ j = 1, 2, \ldots, N_h \}.$$

We now define the average and the jump for $\phi \in H^s(\mathcal{E}_h), s > \frac{1}{2}$. If $e_k = \partial E_i \cap \partial E_j$ for $i < j$ and $1 \leqslant k \leqslant P_h$, we set

$$\{\phi\} = \frac{1}{2}(\phi|_{E_i})\big|_{e_k} + \frac{1}{2}(\phi|_{E_j})\big|_{e_k}, \qquad [\phi] = (\phi|_{E_i})\big|_{e_k} - (\phi|_{E_j})\big|_{e_k}.$$

The L^2 inner product is denoted by $(\cdot\,,\cdot)$. The usual Sobolev norm on $E \subset \mathbf{R}^d$ is denoted by $\|\cdot\|_{m,E}$ for a nonnegative integer m. If $E = \Omega$, we simply denote it by $\|\cdot\|_m$ and if $m = 0$, denote it by $\|\cdot\|$.

We define the following broken norm:

$$|||\phi|||^2 = \sum_{j=1}^{N_h} \bigl(\|\phi\|_{1,E_j}^2 + h_j^2 |\phi|_{2,E_j}^2 \bigr) + J^\sigma(\phi, \phi),$$

where

$$J^\sigma(\phi, \psi) = \sum_{k=1}^{P_h} \frac{\sigma}{|e_k|} \int_{e_k} [\phi][\psi]\, ds$$

denotes the interior penalty term. Here, $|e_k|$ denotes the length of e_k and σ is a positive real number.

Let r be a positive integer. The finite element subspace is taken to be

$$D_r(\mathcal{E}_h) = \prod_{j=1}^{N_h} P_r(E_j),$$

where $P_r(E_j)$ denotes the set of all polynomials of total degree less than or equal to r on E_j, even if E_j is a quadrilateral.

The following lemma is given in [5,6].

Lemma 3.1. *Let $u \in H^s(\Omega)$ for $s \geqslant 2$ and let $r \geqslant 2$. Let $\bar{a}$ be a positive constant. Then there is $\hat{u} \in D_r(\mathcal{E}_h)$, an interpolant of u such that*

$$\int_{e_k} \{ \bar{a} \nabla(\hat{u} - u) \cdot n_k \}\, ds = 0, \quad k = 1, \ldots, P_h, \tag{3.1}$$

$$\|\hat{u} - u\|_{\infty, E_j} \leqslant c \frac{h^\mu}{r^{s-1}} \|u\|_{s,E_j}, \quad j = 1, 2, \ldots, N_h, \tag{3.2}$$

$$\bigl\| \nabla(\hat{u} - u) \bigr\|_{0,E_j} \leqslant c \frac{h^{\mu-1}}{r^{s-1}} \|u\|_{s,E_j}, \quad j = 1, 2, \ldots, N_h, \tag{3.3}$$

$$\bigl\| \nabla^2(\hat{u} - u) \bigr\|_{0,E_j} \leqslant c \frac{h^{\mu-2}}{r^{s-2}} \|u\|_{s,E_j}, \quad j = 1, 2, \ldots, N_h, \tag{3.4}$$

and

$$\|\hat{u} - u\|_{0,E_j} \leqslant c \frac{h^\mu}{r^{s-1}} \|u\|_{s,E_j}, \quad j = 1, 2, \ldots, N_h, \tag{3.5}$$

where $\mu = \min(r+1, s)$. *Moreover, for* $e_k = \partial E_i \cap \partial E_j$,

$$\|\nabla \hat{u}\|_{\infty,e_k} \leqslant c\|\nabla u\|_{\infty,E_i \cup E_j}. \tag{3.6}$$

The discontinuous Galerkin approximation $u_h(\cdot\,,t) \in D_r(\mathcal{E}_h)$ to the solution u of (2.1)–(2.3) is defined by

$$\begin{aligned}
&\left(\frac{\partial u_h}{\partial t}, v\right) + \sum_{j=1}^{N_h} \int_{E_j} a(u_h)\nabla u_h \cdot \nabla v\,dx - \sum_{k=1}^{P_h} \int_{e_k} \big\{a(u_h)\nabla u_h \cdot n_k\big\}[v]\,ds \\
&\quad - \sum_{k=1}^{P_h} \int_{e_k} \big\{a(u_h)\nabla v \cdot n_k\big\}[u_h]\,ds + J^\sigma(u_h, v) \\
&= \big(f(u_h), v\big), \quad t > 0,\ v \in D_r(\mathcal{E}_h),
\end{aligned} \tag{3.7}$$

and

$$u_h(\cdot\,,0) = P_h g, \tag{3.8}$$

where $P_h g$ is an appropriate projection of g to be defined later.

4. A priori error estimate

Define

$$\begin{aligned}
B(\rho : v, w) = &\sum_{j=1}^{N_h} \int_{E_j} a(\rho)\nabla v \cdot \nabla w\,dx - \sum_{k=1}^{P_h} \int_{e_k} \big\{a(\rho)\nabla v \cdot n_k\big\}[w]\,ds \\
&- \sum_{k=1}^{P_h} \int_{e_k} \big\{a(\rho)\nabla w \cdot n_k\big\}[v]\,ds + J^\sigma(v, w)
\end{aligned} \tag{4.1}$$

and

$$B_\lambda(\rho : v, w) = B(\rho : v, w) + \lambda(v, w). \tag{4.2}$$

Then we obtain the following lemmas.

Lemma 4.1. *There exists a constant* c *such that*

$$\big|B_\lambda(\rho : v, w)\big| \leqslant c|||v|||\,|||w|||, \quad v, w \in H^2(\mathcal{E}_h).$$

Proof. Let $v, w \in H^2(\mathcal{E}_h)$. Note that we have

$$\big|B_\lambda(\rho : v, w)\big| \leqslant \left|\sum_{j=1}^{N_h} \int_{E_j} a(\rho)\nabla v \cdot \nabla w\,dx\right| + \left|\sum_{k=1}^{P_h} \int_{e_k} \big\{a(\rho)\nabla v \cdot n_k\big\}[w]\,ds\right|$$

$$+\left|\sum_{k=1}^{P_h}\int_{e_k}\{a(\rho)\nabla w\cdot n_k\}[v]\,ds\right|+\left|J^\sigma(v,w)\right|+\left|\lambda(v,w)\right|$$
$$\equiv I_1+I_2+I_3+I_4+I_5. \tag{4.3}$$

Now we need to obtain the bounds for each $I_1, I_2, \dots, I_5$. Clearly, we have

$$I_1 \leqslant \gamma^* \sum_{j=1}^{N_h} \|\nabla v\|_{0,E_j}\|\nabla w\|_{0,E_j} \leqslant \gamma^* |||v|||\,|||w||| \tag{4.4}$$

and by the trace theorem, we get

$$\begin{aligned} I_2 &\leqslant \gamma^* \sum_{k=1}^{P_h}\left|\int_{e_k}\{\nabla v\cdot n_k\}[w]\,ds\right| \\ &\leqslant \gamma^*\left(\sum_{k=1}^{P_h}\frac{\sigma}{|e_k|}\int_{e_k}[w]^2\,ds\right)^{1/2}\left(\sum_{k=1}^{P_h}\frac{|e_k|}{\sigma}\int_{e_k}\{\nabla v\cdot n_k\}^2\,ds\right)^{1/2} \\ &\leqslant c|||v|||\,|||w|||. \end{aligned} \tag{4.5}$$

Similarly, we can get

$$I_3 \leqslant c|||v|||\,|||w|||. \tag{4.6}$$

By the definition of $|||\cdot|||$, we easily obtain

$$I_4 \leqslant |||v|||\,|||w||| \tag{4.7}$$

and

$$I_5 \leqslant \lambda|||v|||\,|||w|||. \tag{4.8}$$

Therefore, substituting the bounds (4.4)–(4.8) into (4.3), we obtain

$$\left|B_\lambda(\rho:v,w)\right| \leqslant c|||v|||\,|||w|||. \qquad \square$$

Lemma 4.2. *For a sufficiently large σ, there exists a constant c satisfying*

$$B_\lambda(\rho:v,v) \geqslant c|||v|||^2, \quad v\in D_r(\mathcal{E}_h).$$

Proof. Let $v\in D_r(\mathcal{E}_h)$. Then we get

$$\begin{aligned} B_\lambda(\rho:v,v) &= \sum_{j=1}^{N_h}\int_{E_j} a(\rho)|\nabla v|^2\,dx - 2\sum_{k=1}^{P_h}\int_{e_k}\{a(\rho)\nabla v\cdot n_k\}[v]\,ds \\ &\quad + J^\sigma(v,v)+\lambda(v,v) \\ &\geqslant \gamma\sum_{j=1}^{N_h}\|\nabla v\|_{0,E_j}^2 - \delta\sum_{k=1}^{P_h}|e_k|\left\|\{\nabla v\}\right\|_{0,e_k}^2 - c\delta^{-1}\sum_{k=1}^{P_h}\frac{1}{|e_k|}\left\|[v]\right\|_{0,e_k}^2 \\ &\quad + J^\sigma(v,v)+\lambda(v,v) \end{aligned}$$

$$\geqslant c_1 \sum_{j=1}^{N_h} \|\nabla v\|_{0,E_j}^2 + \left(1 - \frac{c\delta^{-1}}{\sigma}\right) J^\sigma(v,v) + \lambda(v,v)$$
$$\geqslant c |||v|||^2$$

for a sufficiently large σ. $\square$

Lemma 4.3. *Let $t \in [0,T]$ be fixed. Suppose that $\phi \in H^2(\mathcal{E}_h)$ satisfies*

$$B_\lambda(u:\phi, v) = F(v), \quad v \in D_r(\mathcal{E}_h),$$

where $F : H^2(\mathcal{E}_h) \to \mathbb{R}$ is a linear map. Let M_1 and M_2 be constants for which

$$|F(w)| \leqslant M_1 |||w|||, \quad w \in H^2(\mathcal{E}_h)$$

and

$$|F(\psi)| \leqslant M_2 \|\psi\|_2, \quad \psi \in H^2 \cap H_0^1.$$

Then

$$\|\phi\| \leqslant c\big(|||\phi||| + M_1\big)h + M_2.$$

Proof. Let $L(u)$ be the elliptic operator defined by

$$L(u)w = -\nabla \cdot \big(a(u)\nabla w\big) + \lambda w.$$

For $\phi \in L^2(\Omega)$, let $\psi \in H^2 \cap H_0^1$ be the solution of

$$-\nabla \cdot \big(a(u)\nabla \psi\big) + \lambda \psi = \phi \quad \text{in } \Omega.$$

By the standard regularity result of the Dirichlet problem for the operator L, we get

$$\|\psi\|_2 \leqslant c\|\phi\|.$$

Let ψ_τ be the interpolant of ψ satisfying $|||\psi - \psi_\tau||| \leqslant ch\|\psi\|_2$. Since

$$\|\phi\|^2 = \big(\phi, -\nabla \cdot \big(a(u)\nabla\psi\big)\big) + \lambda(\phi,\psi) = B_\lambda(u:\phi,\psi),$$

we obtain

$$\begin{aligned}\|\phi\|^2 &= B_\lambda(u:\phi, \psi - \psi_\tau) + B_\lambda(u:\phi,\psi_\tau)\\ &\leqslant c|||\phi|||\,|||\psi - \psi_\tau||| + F(\psi) - F(\psi - \psi_\tau)\\ &\leqslant c\big[\big(|||\phi||| + M_1\big)h + M_2\big]\|\psi\|_2\\ &\leqslant c\big[\big(|||\phi||| + M_1\big)h + M_2\big]\|\phi\|,\end{aligned}$$

which implies

$$\|\phi\| \leqslant c\big[\big(|||\phi||| + M_1\big)h + M_2\big]. \qquad \square$$

By Lemmas 4.1 and 4.2, there exists a unique $\tilde{u} \in D_r(\mathcal{E}_h)$ satisfying

$$B_\lambda(u : u - \tilde{u}, v) = 0, \quad v \in D_r(\mathcal{E}_h).$$

To estimate the error of $u - u_h$, we let $\eta = u - \tilde{u}$, $\theta = \tilde{u} - \hat{u}$ and $\xi = \tilde{u} - u_h$.

Theorem 4.1. *For $r, s \geqslant 2$, there exists a constant c satisfying*

$$|||u-\tilde{u}||| \leqslant c\frac{h^{\mu-1}}{r^{s-2}}\|u\|_s,$$

$$\|u-\tilde{u}\| \leqslant c\frac{h^{\mu}}{r^{s-2}}\|u\|_s,$$

$$|||u_t-\tilde{u}_t||| \leqslant c\frac{h^{\mu-1}}{r^{s-2}}\big(\|u\|_s+\|u_t\|_s\big),$$

and

$$\|u_t-\tilde{u}_t\| \leqslant c\frac{h^{\mu}}{r^{s-2}}\big(\|u\|_s+\|u_t\|_s\big),$$

where $\mu = \min(r+1, s)$.

Proof. By Lemmas 4.1 and 4.2, we have

$$|||\theta|||^2 \leqslant cB_\lambda(u:\theta,\theta) = cB_\lambda(u:u-\hat{u},\theta) \leqslant c|||u-\hat{u}|||\,|||\theta|||$$

and so $|||\theta||| \leqslant c|||u-\hat{u}|||$. From Lemma 3.1, we have

$$\begin{aligned}
|||u-\hat{u}|||^2 &= \sum_{j=1}^{N_h}\big(\|u-\hat{u}\|_{1,E_j}^2 + h_j^2|u-\hat{u}|_{2,E_j}^2\big) + J^\sigma(u-\hat{u},u-\hat{u}) \\
&\leqslant \sum_{j=1}^{N_h}\left(\frac{h_j^{2(\mu-1)}}{r^{2(s-1)}}\|u\|_{s,E_j}^2 + \frac{h_j^{2(\mu-1)}}{r^{2(s-2)}}\|u\|_{s,E_j}^2\right) + \sum_{k=1}^{P_h}\frac{\sigma}{|e_k|}\int_{e_k}[u-\hat{u}]^2\,ds \\
&\leqslant \sum_{j=1}^{N_h}\frac{h_j^{2(\mu-1)}}{r^{2(s-2)}}\|u\|_{s,E_j}^2 + \sum_{j=1}^{N_h}h_j^2\big(\|u-\hat{u}\|_{0,E_j}^2 + h_j^2\big\|\nabla(u-\hat{u})\big\|_{0,E_j}^2\big) \\
&\leqslant \sum_{j=1}^{N_h}\frac{h_j^{2(\mu-1)}}{r^{2(s-2)}}\|u\|_{s,E_j}^2 \\
&\leqslant \frac{h^{2(\mu-1)}}{r^{2(s-2)}}\|u\|_s^2,
\end{aligned}$$

which implies $|||u-\hat{u}||| \leqslant c\frac{h^{(\mu-1)}}{r^{(s-2)}}\|u\|_s$. Therefore, we get

$$|||\eta||| \leqslant |||\theta||| + |||u-\hat{u}||| \leqslant c|||u-\hat{u}||| \leqslant c\frac{h^{\mu-1}}{r^{s-2}}\|u\|_s. \tag{4.9}$$

By Lemma 4.3, we have

$$\|\eta\| \leqslant ch|||\eta||| \leqslant c\frac{h^{\mu}}{r^{s-2}}\|u\|_s. \tag{4.10}$$

Differentiating $B_\lambda(u:\eta,v)=0$ with respect to t, we obtain

$$B_\lambda(u:\eta_t,v) = G(v),$$

where

$$G(v) = -\sum_{j=1}^{N_h} \int_{E_j} \left(\frac{d}{dt}a(u)\right)\nabla\eta \cdot \nabla v\,dx + \sum_{k=1}^{P_h} \int_{e_k} \left\{\left(\frac{d}{dt}a(u)\right)\nabla\eta \cdot n_k\right\}[v]\,ds$$
$$+ \sum_{k=1}^{P_h} \int_{e_k} \left\{\left(\frac{d}{dt}a(u)\right)\nabla v \cdot n_k\right\}[\eta]\,ds.$$

It is easy to show that the linear map G satisfies the following inequalities:

$$\left|G(v)\right| \leqslant c|||\eta|||\,|||v|||, \quad v \in H^2(\mathcal{E}_h)$$

and

$$\left|G(v)\right| \leqslant c\|\eta\|\|v\|_2, \quad v \in H^2 \cap H_0^1.$$

Therefore, by Lemma 4.3, we get

$$\|\eta_t\| \leqslant c\left[\left(|||\eta_t||| + |||\eta|||\right)h + \|\eta\|\right]. \tag{4.11}$$

Hence we obtain

$$\begin{aligned}|||\theta_t|||^2 \leqslant cB_\lambda(u : \theta_t, \theta_t) &= cB_\lambda(u : \eta_t, \theta_t) + cB_\lambda(u : u_t - \hat{u}_t, \theta_t)\\ &= cG(\theta_t) + cB_\lambda(u : u_t - \hat{u}_t, \theta_t)\\ &\leqslant c\left(|||\eta|||\,|||\theta_t||| + |||u_t - \hat{u}_t|||\,|||\theta_t|||\right),\end{aligned}$$

which implies

$$|||\theta_t||| \leqslant c\left(|||\eta||| + |||u_t - \hat{u}_t|||\right).$$

Using Lemma 3.1 and (4.9), we obtain

$$|||\theta_t||| \leqslant c\sum_{j=1}^{N_h} \frac{h^{\mu-1}}{r^{s-2}}\left(\|u\|_{s,E_j} + \|u_t\|_{s,E_j}\right). \tag{4.12}$$

From Lemma 3.1 and (4.12), we have

$$|||\eta_t||| \leqslant |||\theta_t||| + |||u_t - \hat{u}_t||| \leqslant c\frac{h^{\mu-1}}{r^{s-2}}\left(\|u\|_s + \|u_t\|_s\right). \tag{4.13}$$

Substituting (4.9), (4.10) and (4.13) into (4.11), we get

$$\begin{aligned}\|\eta_t\| &\leqslant c\left\{\left[\frac{h^{\mu-1}}{r^{s-2}}\left(\|u\|_s + \|u_t\|_s\right) + \frac{h^{\mu-1}}{r^{s-2}}\|u\|_s\right]h + \frac{h^\mu}{r^{s-2}}\|u\|_s\right\}\\ &\leqslant c\frac{h^\mu}{r^{s-2}}\left(\|u\|_s + \|u_t\|_s\right). \qquad \square\end{aligned}$$

Theorem 4.2. *Let u be the solution of* (2.1)–(2.3) *and let u_h be the solution of* (3.7) *and* (3.8). *Then there exists a constant C such that*

$$\|u - u_h\| \leqslant C\frac{h^\mu}{r^{s-2}}\left(\|u\|_{L^2(H^s)} + \|u_t\|_{L^2(H^s)}\right),$$

where $\mu = \min(r+1, s)$ and $r, s \geqslant 2$.

Proof. For $v \in D_r(\mathcal{E}_h)$, we obtain from (2.1) and (3.7)

$$\left(\frac{\partial(u-u_h)}{\partial t}, v\right) + B_\lambda(u:u,v) - B_\lambda(u_h:u_h,v) = \big(f(u)-f(u_h),v\big) + \lambda(u-u_h,v).$$

Hence, for $v \in D_r(\mathcal{E}_h)$ we get

$$\left(\frac{\partial\xi}{\partial t}, v\right) + B_\lambda(u_h:\xi,v) = -\left(\frac{\partial\eta}{\partial t}, v\right) - B_\lambda(u_h:\eta,v) + B_\lambda(u_h:u,v) - B_\lambda(u:u,v) + \big(f(u)-f(u_h),v\big) + \lambda(u-u_h,v). \tag{4.14}$$

Notice that

$$\begin{aligned} &-B_\lambda(u_h:\eta,v) + B_\lambda(u_h:u,v) - B_\lambda(u:u,v) \\ &\quad = B_\lambda(u_h:\tilde{u},v) - B_\lambda(u:\tilde{u},v) - B_\lambda(u:\eta,v) \\ &\quad = \sum_{j=1}^{N_h}\int_{E_j}\big(a(u_h)-a(u)\big)\nabla\tilde{u}\cdot\nabla v\,dx - \sum_{k=1}^{P_h}\int_{e_k}\big\{\big(a(u_h)-a(u)\big)\nabla\tilde{u}\cdot n_k\big\}[v]\,ds \\ &\qquad - \sum_{k=1}^{P_h}\int_{e_k}\big\{\big(a(u_h)-a(u)\big)\nabla v\cdot n_k\big\}[\tilde{u}]\,ds. \end{aligned} \tag{4.15}$$

Therefore substituting (4.15) into (4.14), we obtain

$$\begin{aligned} &\left(\frac{\partial\xi}{\partial t}, \xi\right) + B_\lambda(u_h:\xi,\xi) \\ &\quad = -\left(\frac{\partial\eta}{\partial t}, \xi\right) + \sum_{j=1}^{N_h}\int_{E_j}\big(a(u_h)-a(u)\big)\nabla\tilde{u}\cdot\nabla\xi\,dx \\ &\qquad - \sum_{k=1}^{P_h}\int_{e_k}\big\{\big(a(u_h)-a(u)\big)\nabla\tilde{u}\cdot n_k\big\}[\xi]\,ds \\ &\qquad - \sum_{k=1}^{P_h}\int_{e_k}\big\{\big(a(u_h)-a(u)\big)\nabla\xi\cdot n_k\big\}[\tilde{u}]\,ds + \big(f(u)-f(u_h),\xi\big) + \lambda(u-u_h,\xi) \\ &\quad \equiv I_1+I_2+I_3+I_4+I_5+I_6. \end{aligned} \tag{4.16}$$

Now we need to obtain the bounds for each $I_1, I_2, \ldots, I_6$. Clearly, we have

$$|I_1| \leqslant \left\|\frac{\partial\eta}{\partial t}\right\|\|\xi\| \leqslant c_1\left(\left\|\frac{\partial\eta}{\partial t}\right\|^2 + \|\xi\|^2\right). \tag{4.17}$$

And we obtain

$$|I_2| \leqslant c\sum_{j=1}^{N_h}\int_{E_j}|u_h-u||\nabla\tilde{u}\cdot\nabla\xi|\,dx$$
$$\leqslant c\|\nabla\tilde{u}\|_\infty\sum_{j=1}^{N_h}\bigl(\|\eta\|_{0,E_j}+\|\xi\|_{0,E_j}\bigr)\|\nabla\xi\|_{0,E_j}$$
$$\leqslant c_2\bigl(\|\eta\|^2+\|\xi\|^2\bigr)+\epsilon_1\|\nabla\xi\|^2. \tag{4.18}$$

Note that for $e_k=\partial E_i\cap\partial E_j$ and $E^{ij}=E_i\cup E_j$

$$\|\nabla\tilde{u}\|_{\infty,e_k} \leqslant \|\nabla\hat{u}\|_{\infty,e_k}+\|\nabla\theta\|_{\infty,e_k}$$
$$\leqslant c\|\nabla u\|_{\infty,E^{ij}}+ch^{-1/2}\|\nabla\theta\|_{0,e_k}$$
$$\leqslant c\|\nabla u\|_{\infty,E^{ij}}+ch^{-1}\bigl(\|\nabla\theta\|_{0,E^{ij}}+h\|\nabla^2\theta\|_{0,E^{ij}}\bigr)$$
$$\leqslant c\|\nabla u\|_{\infty,E^{ij}}+ch^{-1}\|\nabla\theta\|$$
$$\leqslant c\|\nabla u\|_{\infty,E^{ij}}+c\frac{h^{\mu-2}}{r^{s-2}}\|u\|_s<\infty \tag{4.19}$$

and by condition 2, we get

$$\left|\int_{e_k}\bigl\{\bigl(a(u_h)-a(u)\bigr)\nabla\tilde{u}\cdot n_k\bigr\}[\xi]\,ds\right|$$
$$\leqslant c\|\nabla\tilde{u}\|_{\infty,e_k}\bigl\|\{u-u_h\}\bigr\|_{0,e_k}\bigl\|[\xi]\bigr\|_{0,e_k}$$
$$\leqslant c\bigl(\bigl\|\{\eta\}\bigr\|_{0,e_k}+\bigl\|\{\xi\}\bigr\|_{0,e_k}\bigr)\bigl\|[\xi]\bigr\|_{0,e_k}$$
$$\leqslant \epsilon_2\frac{\sigma}{|e_k|}\bigl\|[\xi]\bigr\|_{0,e_k}^2+c_3h\bigl(h^{-1}\|\eta\|_{0,E^{ij}}^2+h\|\nabla\eta\|_{0,E^{ij}}^2+h^{-1}\|\xi\|_{0,E^{ij}}^2\bigr)$$
$$\leqslant \epsilon_2\frac{\sigma}{|\eta_k|}\bigl\|[\xi]\bigr\|_{0,e_k}^2+c_3\bigl(\|\eta\|_{0,E^{ij}}^2+h^2\|\nabla\eta\|_{0,E^{ij}}^2+\|\xi\|_{0,E^{ij}}^2\bigr). \tag{4.20}$$

Therefore using the results (4.19) and (4.20), we obtain

$$|I_3|=\left|\sum_{k=1}^{P_h}\int_{e_k}\bigl\{\bigl(a(u_h)-a(u)\bigr)\nabla\tilde{u}\cdot n_k\bigr\}[\xi]\,ds\right|$$
$$\leqslant \epsilon_2J^\sigma(\xi,\xi)+c_3\sum_{j=1}^{N_h}\bigl(\|\eta\|_{0,E_j}^2+h^2\|\nabla\eta\|_{0,E_j}^2+\|\xi\|_{0,E_j}^2\bigr)$$
$$\leqslant \epsilon_2|||\xi|||^2+c_3\bigl(\|\eta\|^2+h^2\|\nabla\eta\|^2+\|\xi\|^2\bigr).$$

And note that

$$\left|\int_{e_k}\bigl\{\bigl(a(u_h)-a(u)\bigr)\nabla\xi\cdot n_k\bigr\}[\eta]\,ds\right|$$
$$\leqslant c\|\nabla\xi\|_{\infty,e_k}\bigl\|\{u_h-u\}\bigr\|_{0,e_k}\bigl\|[\eta]\bigr\|_{0,e_k}$$

$$\begin{aligned}
&\leqslant ch^{-1/2}\|\nabla\xi\|_{0,e_k}\big(\big\|\{\eta\}\big\|_{0,e_k}+\big\|\{\xi\}\big\|_{0,e_k}\big)\big(h^{-1/2}\|\eta\|_{0,E^{ij}}+h^{1/2}\|\nabla\eta\|_{0,E^{ij}}\big)\\
&\leqslant ch^{-1}\|\nabla\xi\|_{0,e_k}\big(h^{-1/2}\|\eta\|_{0,E^{ij}}+h^{1/2}\|\nabla\eta\|_{0,E^{ij}}+h^{-1/2}\|\xi\|_{0,E^{ij}}\big)\\
&\quad\times\big(\|\eta\|_{0,E^{ij}}+h\|\nabla\eta\|_{0,E^{ij}}\big)\\
&\leqslant ch^{-2}\|\nabla\xi\|_{0,E^{ij}}\big(\|\eta\|_{0,E^{ij}}+h\|\nabla\eta\|_{0,E^{ij}}+\|\xi\|_{0,E^{ij}}\big)h^2|||u|||_2\\
&\leqslant c\|\nabla\xi\|_{0,E^{ij}}\big(\|\eta\|_{0,E^{ij}}+h\|\nabla\eta\|_{0,E^{ij}}+\|\xi\|_{0,E^{ij}}\big).
\end{aligned}$$

Therefore we get

$$\begin{aligned}
|I_4|&=\left|\sum_{k=1}^{P_h}\int_{e_k}\big\{\big(a(u_h)-a(u)\big)\nabla\xi\cdot n_k\big\}[\eta]\right|\\
&\leqslant\epsilon_3|||\xi|||^2+c_4\big(\|\eta\|^2+h^2\|\nabla\eta\|^2+\|\xi\|^2\big).
\end{aligned}$$

And by condition 2, we easily get

$$\begin{aligned}
|I_5|&=\big|\big(f(u)-f(u_h),\xi\big)\big|\leqslant c\|u-u_h\|\|\xi\|\leqslant c_5\big(\|\eta\|^2+\|\xi\|^2\big),\\
|I_6|&=\big|\lambda(u-u_h,\xi)\big|\leqslant c_6\|u-u_h\|\|\xi\|\leqslant c_6\big(\|\eta\|^2+\|\xi\|^2\big).
\end{aligned}$$

Using the estimates for $I_1, I_2, \dots,$ and I_6, we obtain from (4.16)

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|\xi\|^2+c|||\xi|||^2&\leqslant\left(\frac{\partial\xi}{\partial t},\xi\right)+B_\lambda(u_h:\xi,\xi)\\
&\leqslant c\left(\left\|\frac{\partial\eta}{\partial t}\right\|^2+\|\xi\|^2+\|\eta\|^2+h^2\|\nabla\eta\|^2\right)+\epsilon|||\xi|||^2.
\end{aligned}$$

Therefore we get

$$\frac{d}{dt}\|\xi\|^2+\beta|||\xi|||^2\leqslant c\left(\left\|\frac{\partial\eta}{\partial t}\right\|^2+\|\eta\|^2+h^2\|\nabla\eta\|^2\right)+c\|\xi\|^2 \tag{4.21}$$

for a sufficiently small ϵ.

By applying Gronwall's lemma to (4.21), we have

$$\|\xi\|^2+\beta\int_0^t|||\xi|||^2d\tau\leqslant c\int_0^t\left(\left\|\frac{\partial\eta}{\partial t}\right\|^2+\|\eta\|^2+h^2\|\nabla\eta\|^2\right)d\tau+\big\|\xi(0)\big\|^2. \tag{4.22}$$

Assuming that the initial value $u_h(x,0)=P_hg\in D_r(\mathcal{E}_h)$ satisfies

$$\big\|u_h(0)-\tilde{u}(0)\big\|\leqslant c\frac{h^\mu}{r^{s-2}}\|g\|_s,\quad \mu=\min(r+1,s),$$

then from (4.22) and Lemma 3.1, we get

$$\|\xi\|^2+\beta\int_0^t|||\xi|||^2d\tau\leqslant c\frac{h^{2\mu}}{r^{2(s-2)}}\big(\|u\|^2_{L^2(H^s)}+\|u_t\|^2_{L^2(H^s)}\big),$$

where $\mu = \min(r+1, s)$. Thus we have

$$\|u - u_h\|^2 \leqslant C\big(\|u - \tilde{u}\|^2 + \|\tilde{u} - u_h\|^2\big) \leqslant C\frac{h^{2\mu}}{r^{2(s-2)}}\big(\|u\|^2_{L^2(H^s)} + \|u_t\|^2_{L^2(H^s)}\big),$$

where $\mu = \min(r+1, s)$. $\square$

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