Spaces of Kudrjavcev Type
I. Interpolation, Embedding, and Structure

HANS TRIEBEL
Sektion Mathematik, Friedrich-Schiller-Universität,
Schillerstrasse, DDR-69, Jena, Germany

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1. Introduction

The paper deals with a special class of function spaces with weights, which (at least for some special cases) are closely related to spaces introduced by L. D. Kudrjavcev [5, 6], and which are developed further by B. Hanouzet [1, 2, 3], Ju. S. Nikol'skii [7], and T. S. Pigolkina [12]. But the method used here is entirely different: The paper is a self-contained treatment on function spaces with weights based on decomposition procedures. Such decomposition methods seem to be a rather powerful tool for spaces of functions (and distributions) without and with weights, in particular in the interpolation theory for spaces of such a type. For Lebesgue–Besov (–Sobolev–Slobodeckij) spaces $H^s_p(\mathbb{R}^n)$, $B^s_{p,q}(\mathbb{R}^n)$, and $W^s_p(\mathbb{R}^n)$ such methods are used by S. M. Nikol'skij [8], J. Peetre [9], and the author [15, 17]. With their aid one can reduce the interpolation theory for the spaces $H^s_p(\mathbb{R}^n)$, $B^s_{p,q}(\mathbb{R}^n)$, and $W^s_p(\mathbb{R}^n)$ to the interpolation theory for spaces of type $L_p$ and $L_p$, see [9, 15, 17]. On the next level, using other decomposition methods, one can reduce the interpolation theory for special, but rather wide, classes of Lebesgue–Besov (–Sobolev–Slobodeckij) spaces with weights to the interpolation theory of the corresponding spaces $H^s_p(\mathbb{R}^n)$ and $B^s_{p,q}(\mathbb{R}^n)$ without weights, see [16]. In this paper we apply this basic idea on a new class of function spaces.

In Section 2 we define the spaces considered here, and prove some important facts. In particular, Theorem 2 shows that $w^s_{p,u}(\mathbb{R}^n)$ are spaces of Kudrjavcev type, see [5, 6] or [1, 2, 3]. Although this theorem is not needed for the further considerations (all the other proofs are based on the direct definition of these spaces), it is one of the main results of the paper, since it gives a motivation for introducing spaces of such a type. In Section 3 we shall develop an interpolation theory for the considered spaces. Section 4 contains embedding theorems for different metrics, as well as direct and inverse embedding theorems on the boundary. Finally, Section 5 is concerned with the structure of the spaces $w^s_{p,u}(\mathbb{R}^n)$. This paper is an extension of Section 3.9 in [17].
2. DEFINITIONS AND EQUIVALENT NORMS

2.1. The spaces $W_p^s(R_n)$, $H_p^s(R_n)$, and $B_{p,q}^s(R_n)$

We recall some basic facts for the Lebesgue–Besov spaces $H_p^s(R_n)$ and $B_{p,q}^s(R_n)$, and their special cases $W_p^s(R_n)$ (Sobolev–Slobodeckij spaces). $R_n$ is the real $n$-dimensional Euclidean space. $C_0^\infty(R_n)$ is the space of all complex infinitely differentiable functions, defined in $R_n$, with compact support. (Similarly $C_0^\infty(\Omega)$ is defined, where $\Omega \subset R_n$ is a domain in $R_n$). $S(R_n)$ is the Schwartz space of complex rapidly decreasing infinitely differentiable functions, $S'(R_n)$ is the space of tempered distributions. $F$ and $F^{-1}$ are the Fourier transformation and the inverse Fourier transformation in $S'(R_n)$, respectively. $L_p(R_n)$ and $L_p^{00}(R_n)$ have the usual meaning; $1 < p < \infty$.

If $1 < p < \infty$ and if $s \geq 0$, then

$$H_p^s(R_n) = \{ f \mid f \in S'(R_n), \| f \|_{H_p^s} := \| F^{-1}(1 + |\xi|^2)^{s/2} \hat{f} \|_{L_p} < \infty \}$$

are the Lebesgue spaces. (Of course, $H_p^0 = L_p$). We use $D^\alpha = \partial^{\alpha_1}_{\alpha_1} \cdots \partial^{\alpha_n}_{\alpha_n}$ for $\alpha = (\alpha_1, \ldots, \alpha_n)$ multi-index, $|\alpha| = \sum_{j=1}^n \alpha_j$,

$$(\Delta_h f)(x) = f(x + h) - f(x), \quad h \in R_n,$$

and

$$\Delta_h f = \Delta_h (\Delta_h^{-1} f) \quad \text{for } l = 2, 3, \ldots, (\Delta_h^1 = \Delta_h).$$

Let $s > 0; 1 < p < \infty$; and $1 < q < \infty$. Further, let $k$ and $l$ be two integers such that $0 \leq k < s$ and $l > s - k$. We set

$$\| f \|_{B_{p,\infty}^s} = \sum_{|\alpha|=k} \left( \int_{R_n} \left| h \right|^{-(s-k)q} \| \Delta_h^l D^\alpha f \|_{L_p} \frac{dh}{h^{n}} \right)^{1/q} \quad \text{for } l \leq q < \infty \quad (2a)$$

and

$$\| f \|_{B_{p,q}^s} = \sum_{|\alpha|=k} \sup_{h \in R_n} \left| h \right|^{-(s-k)} \| \Delta_h^l D^\alpha f \|_{L_p} \quad \text{for } q = \infty. \quad (2b)$$

Then

$$B_{p,q}^s(R_n) = \{ f \mid f \in L_p(R_n), \| f \|_{B_{p,q}^s} = \| f \|_{L_p} + \| f \|_{B_{p,q}^s} < \infty \}$$

are the usual Besov spaces. There exist many equivalent norms. In particular, for different values of $k$ and $l$ the corresponding norms are equivalent to each other (so we omitted these indices). We refer to [8], 5.6, [15], or [17], 2.5.1. We set

$$W_p^s(R_n) = \begin{cases} H_p^s(R_n) & \text{for } s = 1, 2, 3, \ldots, \\ B_{p,q}^s(R_n) & \text{for } s > 0, s \text{ is not an integer.} \end{cases}$$
Then $W_p^s(R_n)$ are the usual Sobolev–Slobodeckij spaces. In particular,
\[ \|f\|_{W_p^s} = \sum_{|\alpha| \leq s} \|D^\alpha f\|_{L_p} \quad \text{for } s = 0, 1, 2, \ldots \]  
(Sobolev spaces) and
\[ \|f\|_{W_p^s} = \|f\|_{L_p} + \sum_{|\alpha| = \lfloor s \rfloor} \left( \int_{R_n \times R_n} \frac{|D^\alpha f(x) - D^\alpha f(y)|^p}{|x - y|^{n+|\alpha|+sp}} \, dx \, dy \right)^{1/p} \]  
for $0 < s = \lfloor s \rfloor + \{s\}; \lfloor s \rfloor$ integer; $0 < \{s\} < 1$; (Slobodeckij spaces), are equivalent norms (see the above references).

For our purpose it will be useful to modify $\|f\|_{H_p^s}$. The Michlin–Hörmander multiplier theorem, see [4], yields
\[ \|F^{-1}(1 + |\xi|^s)Ff\|_{L_p} \sim \|F^{-1}(1 + |\xi|^{2s/2})Ff\|_{L_p}, \quad f \in S(R_n). \]
Setting
\[ \|f\|_{h_p^s} = \|F^{-1}|\xi|^sFf\|_{L_p}, \quad s \geq 0, \]  
and using the multiplier theorem, it follows
\[ \|f\|_{H_p^s} \sim \|f\|_{L_p} + \|f\|_{h_p^s}, \quad f \in S(R_n). \]
Since $S(R_n)$ is dense in $H_p^s(R_n)$ the last relation holds for $f \in H_p^s(R_n)$, where $\|f\|_{h_p^s}$ is to be understood in the sense of completion.

2.2. Function Spaces with Weights

In this subsection we give the definitions for the spaces considered in this paper.

Definition 1. Let
\[ K_j = \{x \mid 2^{j-1} < |x| < 2^{j+2}\}, j = 1, 2, \ldots; K_0 = \{x \mid |x| < 4\}. \]

(a) Then, $Z$ denotes the set of all systems of functions $(\xi_j(x))_{j=0}^\infty$ with the properties:

1. $\xi_j(x) \geq 0, \xi_j(x) \in \mathcal{C}_0^\infty(K_j)$, (outside of $K_j$ the function $\xi_j$ is extended by zero).

2. There exists a positive number $c$ such that
\[ c \leq \sum_{j=0}^\infty \xi_j(x). \]
(3) For each multi-index $\gamma$, there exists a positive number $c(\gamma)$ such that
\[ |D^\gamma f(x)| \leq c(\gamma)2^{-|\gamma|} \quad \text{for } x \in \mathbb{R}^n \quad \text{and} \quad j = 0, 1, 2, \ldots \] (9)

(b) $\hat{Z}$ is the subset of $Z$, where (8) is reinforced by
\[ \sum_{j=0}^{\infty} \zeta_j(x) = 1. \] (10)

Remark 1. It is not very hard to see that $\hat{Z}$ is not empty. Let $\chi_j(x); j = 1, 2, \ldots$ be the characteristic function of $\{x \mid 2^j < |x| < 2^{j+1}\}$, and let $\chi_0(x)$ be the characteristic function of $\{x \mid |x| < 2\}$. We use Sobolev's mollification method, see for instance [13, p. 381]. The mollified function of $f$ is denoted by $(f)_h$, where $h > 0$ indicates the radius of the mollification. Now we mollify $\chi_j(x)$ and $\chi_{j+1}(x)$ in a neighborhood of $\{y \mid |y| = 2^{j+1}\}$, where the radius of mollification is $2^{j-1}; j = 0, 1, 2, \ldots$ (hence, $\chi_j$ is mollified near $\{y \mid |y| = 2^j\}$ and near $\{y \mid |y| = 2^{j+1}\}$ by different radii of mollification). Let $\zeta_j$ be the function $\chi_j$ mollified in the described way. Then the well-known properties of the mollification method yield $\{\zeta_j\}_{j=0}^{\infty} \in \hat{Z}$.

**Definition 2.** Let $\{\zeta_j\}_{j=0}^{\infty} \in Z$. Let $1 < p < \infty; 1 < q < \infty$ and $-\infty < \mu < \infty$.

(a) Then, for $s \geq 0$,
\[ h^s_{p,\mu}(\mathbb{R}^n) = \left\{ f \mid f \in H^{s,\text{loc}}_{p,\mu}(\mathbb{R}^n), \|f\|_{H^{s,\text{loc}}_{p,\mu}(\mathbb{R}^n)} = \left( \sum_{j=0}^{\infty} 2^{js} \|f\|_{L^p_{\zeta_j}(\mathbb{R}^n)}^2 \right)^{1/2} < \infty \right\} \] (11)

and, for $s > 0$,
\[ b^s_{p,\mu}(\mathbb{R}^n) = \left\{ f \mid f \in B^{s,\text{loc}}_{p,\mu}(\mathbb{R}^n), \|f\|_{B^{s,\text{loc}}_{p,\mu}(\mathbb{R}^n)} = \left( \sum_{j=0}^{\infty} 2^{js} \|f\|_{L^p_{\zeta_j}(\mathbb{R}^n)}^2 \right)^{1/2} < \infty \right\} . \] (12)

(b) Let $R_n^+ = \{x \mid x \in \mathbb{R}^n, x_n > 0\}$. Then, $h^s_{p,\mu}(R_n^+)$ is the restriction of $h^s_{p,\mu}(\mathbb{R}^n)$ on $R_n^+$,
\[ \|f\|_{h^s_{p,\mu}(R_n^+)} = \inf_{f(x) = g(x)} \|g\|_{h^s_{p,\mu}(\mathbb{R}^n)} \] (13)

and $b^s_{p,\mu}(R_n^+)$ is the restriction of $b^s_{p,\mu}(\mathbb{R}^n)$ on $R_n^+$,
\[ \|f\|_{b^s_{p,\mu}(R_n^+)} = \inf_{f(x) = g(x)} \|g\|_{b^s_{p,\mu}(\mathbb{R}^n)}. \] (14)
(c) $\hat{h}^s_{p,u}(R^n_+)$ and $\hat{b}^s_{p,q,u}(R^n_+)$ denote the completion of $C_0^\infty(R^n_+)$ in $\hat{h}^s_{p,u}(R^n_+)$ and $\hat{b}^s_{p,q,u}(R^n_+)$, respectively.

(d) $\omega^s_{p,u}(R^n) = \frac{|\hat{h}^s_{p,u}(R^n)|}{|\hat{b}^s_{p,u}(R^n)|}$ for $s = 0, 1, 2, \ldots$, for $0 < s \neq \text{integer}$. (15)

Similarly for $\omega^s_{p,u}(R^n_+)$ and $\omega^s_{p,u}(R^n_+)$.

**Remark 2.** It will be shown, that $\hat{h}^s_{p,u}(R^n)$ and $\hat{b}^s_{p,q,u}(R^n)$ are Banach spaces. Then (13) and (14) is the usual procedure of constructing factor spaces. Clearly, all these norms depend on the choice of the system $\{\zeta_j\}_{j=0}^\infty \in Z$. But it will be shown that for different systems $\{\zeta_j\}_{j=0}^\infty$ the corresponding norms are equivalent to each other. This is the reason why we omitted additional indices indicating the choice of $\{\zeta_j\}_{j=0}^\infty$.

2.3. Properties of the Spaces $\hat{h}^s_{p,u}$ and $\hat{b}^s_{p,q,u}$

We prove some simple, but important, properties for the defined spaces, which are the basis for the further considerations.

**Lemma 1.** Let $c > 0$. Let $g(x) = f(cx)$. Then holds

\[
\|g\|_{\hat{h}^s_p} = c^{s-\frac{n}{p}} \|f\|_{\hat{h}^s_p}, \quad f \in H^s_p(R^n), \tag{16}
\]

\[
\|g\|_{\hat{b}^s_{p,q}} = c^{s-\frac{n}{q}} \|f\|_{\hat{b}^s_{p,q}}, \quad f \in B^s_{p,q}(R^n). \tag{17}
\]

**Proof.** (17) is an easy consequence of (2). To prove (16) we assume $f \in S(R^n)$ and use

\[ (Fg)(x) = c^{-n}(Ff)(x/c) \quad \text{(similarly for } F^{-1}). \]

Then (16) is a consequence of (6). Now one obtains (16) for arbitrary $f \in H^s_p(R^n)$ by completion.

**Remark 3.** (16) and (17) show that $\|f\|_{\hat{h}^s_p}$ and $\|f\|_{\hat{b}^s_{p,q}}$ are the "homogeneous" parts of the usual norms in $H^s_p(R^n)$ and $B^s_{p,q}(R^n)$. That these parts are sufficient for our purpose will be shown in the next lemma.

**Lemma 2.** Let $\Omega$ be a compact subset in $R^n$. Then

\[
\|f\|_{H^s_p} \sim \|f\|_{\hat{h}^s_p}, \quad f \in H^s_p(R^n), \quad \text{supp } f \subset \Omega, \tag{18}
\]

\[
\|f\|_{B^s_{p,q}} \sim \|f\|_{\hat{b}^s_{p,q}}, \quad f \in B^s_{p,q}(R^n), \quad \text{supp } f \subset \Omega. \tag{19}
\]

("\sim" means, that the right-hand side of (18), respectively (19), can be estimated from below and from above by the left-hand side with the aid of positive numbers, independently of $f$).
Proof. We have to prove
\[ \|f\|_{L_p} \leq c \|f\|_{H^{s}} \], respectively \[ \|f\|_{L_p} \leq c \|f\|_{b^s_{p,q}} \], \( s > 0 \). (20)
where \( c \) is a suitable positive number. Assume that there does not exist such a number \( c \). Then we can construct a sequence \( \{f_k\}_{k=1}^{\infty} \) such that
\[ \|f_k\|_{L_p} = 1, \ \supp f_k \subseteq \Omega, \ |f_k|_{H^{s}} \leq (1/k) \|f_k\|_{L_p} = 1/k \] (21)
(similarly for the space \( b^s_{p,q} \)). Since \( \{f_k\}_{k=1}^{\infty} \) is bounded in \( H^s_{p} (\mathbb{R}^n) \), so it is also bounded in \( H^s_{p} (\omega) \), where \( \omega \) is an open bounded domain. \( \Omega \subseteq \omega \). But the embedding from \( H^s_{p} (\omega) \) into \( L^p_{p} (\omega) \) is compact. Now it is easy to see that \( \{f_k\}_{k=1}^{\infty} \) is a precompact set in \( L^p_{p} (\mathbb{R}^n) \). Without loss of generality we assume, that \( \{f_k\}_{k=1}^{\infty} \) is a Cauchy sequence in \( L^p_{p} (\mathbb{R}^n) \). By (21) it follows that \( \{f_k\}_{k=1}^{\infty} \) is also a Cauchy sequence in \( H^s_{p} (\mathbb{R}^n) \). Let \( f_k \to f \in H^s_{p} (\mathbb{R}^n) \). Then
\[ \|f\|_{L_p} = 1, \ \supp f \subseteq \Omega, \ |f|_{H^{s}} = 0 \] (respectively \( |f|_{b^s_{p,q}} = 0 \)). (22)
Since \( |f|_{H^{s}} \) is determined only by a limit process we consider the mollified function \( (f)_{\delta} = \omega_{\delta} * f \in C_0^\infty (\mathbb{R}^n) \) in the above-mentioned sense, where * is the convolution, and \( \omega_{\delta} \) is the kernel of the mollifier. Let \( g \in S(\mathbb{R}^n) \). Using
\[ F(\omega_{\delta} * g) = c F \omega_{\delta} \cdot F g \]
\[ F^{-1}(|\xi|^s F(g)_{\delta}) = F^{-1}(|\xi|^s F \omega_{\delta} F g) = \omega_{\delta} * F^{-1}(|\xi|^s F g) \]
it follows
\[ |(f)_{\delta}|_{H^{s}} \leq c |g|_{H^{s}} \] (23)
(here we used \( |(k)_{\delta}|_{L_p} \leq |k|_{L_p} \)). Completion shows that (23) is valid for all functions belonging to \( H^s_{p} (\mathbb{R}^n) \), in particular for \( f \). Consequently,
\[ F^{-1}(|\xi|^s F(f)_{\delta}) = 0. \]
Since \( (f)_{\delta} \in C_0^\infty (\mathbb{R}^n) \), one obtains \( (f)_{\delta} = 0 \). It follows \( f = 0 \). This is a contradiction to (22). A similar conclusion can be made for \( b^s_{p,q} \).

**Lemma 3.** Let \( g \) be a complex infinitely differentiable function, defined in \( \mathbb{R}^n \), such that holds for all multi-indices \( \gamma \)
\[ |D^\gamma g(x)| \leq b(\gamma) \ |x|^{-|\gamma|}, \quad x \in \mathbb{R}^n, \] (24)
where \( b(\gamma) \) are appropriate positive numbers, independently of \( x \). If \( f \in h^s_{p,u}(\mathbb{R}^n) \), resp. \( f \in b^s_{p,q,u}(\mathbb{R}^n) \), then \( fg \in h^s_{p,u}(\mathbb{R}^n) \), respectively \( fg \in b^s_{p,q,u}(\mathbb{R}^n) \). There exists
a positive number $d$, depending only on $b(\gamma)$, where $0 \leq |\gamma| \leq [s] + 1$ (by fixed $s$, $p$, $q$, $\mu$, and $n$) such that

$$
\|fg\|_{h_{p,u}^s(R_n)} \leq d \|f\|_{h_{p,u}^s(R_n)},
$$

respectively

$$
\|fg\|_{b_{p,q,u}^s(R_n)} \leq d \|f\|_{b_{p,q,u}^s(R_n)}.
$$

Proof. Step 1. Let $\{\zeta_j\}_{j=0}^\infty \in Z$. Let $f \in h_{p,u}^s(R_n)$ (respectively $f \in b_{p,q,u}^s(R_n)$). Lemma 1 yields

$$
\|(f\zeta_j)(2^j x)\|_{h_{p,u}^s} \leq 2^{j(s-n/p)} \|(f\zeta_j)(x)\|_{h_{p,u}^s}
$$

(similarly for $b_{p,q,u}^s$). Of course, in the last relation we can replace $f$ by $fg$, where $g$ has the above meaning. The transformation $y = 2^{-j}x$ maps $K_j$ (Definition 1) on the standard domain $\Omega = \{y \mid \frac{3}{2} < |y| < 4\}$, $j = 1, 2, \ldots$. It holds that

$$
|D_{2^{-j}} g(2^j x)| \leq 2^{j\gamma} b(\gamma) \quad \text{for} \quad x \in \Omega.
$$

Now we apply Lemma 1 and Lemma 2, where $\Omega$ is either the above standard domain or $K_0$. It follows

$$
2^{j(s-n/p)} \|(fg\zeta_j)(x)\|_{h_{p,u}^s} = \|(fg\zeta_j)(2^j x)\|_{h_{p,u}^s} \sim \|(fg\zeta_j)(2^j x)\|_{h_{p,u}^s}
\leq c \|(f\zeta_j)(2^j x)\|_{h_{p,u}^s} \sim c \|(f\zeta_j)(2^j x)\|_{h_{p,u}^s},
$$

The estimate in the last line is based on (27), the number $c$ depends only on $b(\gamma)$, where $0 \leq |\gamma| \leq [s] + 1$. (For Sobolev spaces this is clear by direct computation, for Lebesgue–Besov spaces it follows by interpolation.) A similar conclusion can be made for the case $b_{p,q,u}^s$.

Step 2. Let $f \in b_{p,u}^s(R_n)$, respectively $f \in b_{p,q,u}^s(R_n)$. Then (25) is an immediate consequence of (28) and a corresponding formula for $b_{p,q,u}^s$.

Theorem 1. Let $-\infty < \mu < \infty$; $1 < p < \infty$; $1 \leq q \leq \infty$; and $s > 0$ (for the $h$-spaces $s = 0$ is also admissible).

(a) $h_{p,u}^s(R_n)$ and $b_{p,q,u}^s(R_n)$ are Banach spaces. If $q < \infty$, then $C_0^\infty(R_n)$ is dense in $h_{p,u}^s(R_n)$ and dense in $b_{p,q,u}^s(R_n)$.

(b) If $\{\zeta_j\}_{j=0}^\infty \in Z$ and $\{\xi_j\}_{j=0}^\infty \in Z$ are two systems in the sense of definition 1, then the corresponding norms (11) for the space $h_{p,u}^s(R_n)$ with respect to the two systems are equivalent to each other; and the corresponding norms (12) for the space $b_{p,q,u}^s(R_n)$ are also equivalent to each other.
(c) For \( f \in h^s_{p,u}(R^n) \) holds

\[
\|f\|_{h^s_{p,u}(R^n)} \sim \left( \sum_{j=0}^{\infty} 2^{js} \|f\|_{h^s_{p,u}(R^n)} \right)^{1/p}.
\]

For \( f \in b^s_{p,q,u}(R^n) \) it holds (29) after replacing there \( h^s_{p,u}(R^n) \) by \( b^s_{p,q,u}(R^n) \),

Proof. Step 1. Using (8) and Lemma 2 it follows that \( \|\cdot\|_{h^s_{p,u}(R^n)} \) and

\( \|\cdot\|_{b^s_{p,q,u}(R^n)} \) are norms. If \( \{f_k\}_{k=1}^{\infty} \) is a Cauchy sequence, then there exists

\( f \in H^s_{p,loc}(R^n) \), respectively \( f \in B^s_{p,q}(R^n) \), such that

\( f_k \to f \) in \( h^s_{p,u}(R^n) \), resp. \( b^s_{p,q,u}(R^n) \).

Hence, \( h^s_{p,u}(R^n) \) and \( b^s_{p,q,u}(R^n) \) are Banach spaces.

Step 2. Let \( \{\xi_j\}_{j=0}^{\infty} \in \mathbb{Z} \) and let \( \{\tilde{\xi}_j\}_{j=0}^{\infty} \in \mathbb{Z} \). For fixed \( j \), where \( j = 0, 1, 2,..., \) and \( f \in h^s_{p,u}(R^n) \), we obtain

\[
\|f\xi_j\|_{h^s_p} \sim \left( \sum_{k=-J}^{j} 2^{ks} \|f\|_{h^s_{p,u}(R^n)} \right)^{1/p},
\]

(24) is satisfied, and \( b(\gamma) \) is independent of \( j \). It follows that

\[
|c| \leq \sum_{k=-J}^{j} 2^{ks} \|\xi_j\|_{h^s_p},
\]

where \( c \) is independent of \( j \). This proves (b) for the \( h \)-spaces. A similar conclusion can be made for the \( b \)-spaces.

Step 3. We prove the last part of (a). Let \( f \in h^s_{p,u}(R^n) \). Further let \( \{\xi_j\}_{j=0}^{\infty} \in \mathbb{Z} \). Then

\[
\|f - f \sum_{j=0}^{N} \xi_j\|_{h^s_{p,u}(R^n)} = \left\|f \sum_{j=0}^{\infty} \xi_j\right\|_{h^s_{p,u}(R^n)}.
\]
Repeating the argumentation of the last step, we obtain

$$\left\| \zeta_k f \sum_{j=N+1}^{\infty} \zeta_j \right\|_{h_p^s} \leq c \left\| \zeta_k f \right\|_{h_p^s}$$

for \( k = N - 1, N, N + 1, \ldots \),

where \( c \) is independent of \( k \) (and \( N \)). Hence,

$$\left\| f - f \sum_{j=0}^{N} \zeta_j \right\|_{h_p^s(R_n)} \leq c \left( \sum_{h=N-1}^{\infty} 2^{h} \left\| \zeta_h f \right\|_{h_p^s} \right)^{1/p} < \epsilon \quad \text{for } N \geq N_0(\epsilon).$$

But \( \sum_{j=0}^{N} \zeta_j \) can be approximated by functions belonging to \( C_0^\omega(R_n) \) in the desired way (for instance with the aid of the above mentioned mollification method). Similarly, one concludes for \( b_\nu^{s}(R_n) \), provided that \( q \leq \omega \).

(Here we used that \( C_0^\omega(R_n) \) is a dense subset in \( H_d^\omega(R_n) \), as well as in \( B^s_{\nu,0}(R_n) \), \( q < \omega \). This is untrue for \( B^s_{\nu,0}(R_n) \)).

**Step 4.** We prove (c). The both last equivalences follow immediately from Lemma 1 and from Lemma 2, where \( \Omega \) in Lemma 2 is either \( K_0 \) or \( \{ y : 1/3 < |y| < 4 \} \). (See the proof of Lemma 3.) But the first equivalence is only the transformation of the last equivalence. Of course, the assertions are also true for the \( b \)-spaces.

**Remark 4.** Now it is clear that \( h_\nu^{s}(R_n) \) and \( b_\nu^{s}(R_n) \) are also Banach spaces, since they are factor spaces of Banach spaces.

**2.4. Equivalent Norms in the Spaces \( w_\nu^{s}(R_n) \)**

For abbreviation we write

$$p(x) = (1 + |x|^2)^{1/2}.$$

The following theorem will show that the spaces \( w_\nu^{s}(R_n) \) are closely related to spaces considered by L. D. Kudrjavcev [5, 6] and B. Hanouzet [1, 2, 3]. But for convenience we start with a lemma.

**Lemma 4.** Let \( 1 < \rho < \infty \) and let \( 0 < \eta < 1 \). Further let \( \{\zeta_j\}_{j=0}^{\infty} \in Z \). Then there exists a positive number \( c \), independent of \( j \) and \( y \), where \( y \in K_j \) (Definition 1), such that

$$\int_{K_j} \frac{\left| p^{\nu/p}(x) D^\nu \zeta_j(x) - p^{\nu/p}(y) D^\nu \zeta_j(y) \right|^p}{|x - y|^{n+\eta p}} \, dx \leq c 2^{2^{j} |\alpha| p^{v-j} p}.$$

(30)

\( \alpha \) is an arbitrary multi-index. If \( |\alpha| - 0 \), then one can replace \( D^\nu \zeta_j \) by 1.
\textbf{Proof.} Let \( y \in K_j \) be fixed. Then one obtains
\[
\left\{ \begin{array}{l}
\int_{K_j} \frac{|\rho^{\mu/p}(x) D^s \zeta_j(x) - \rho^{\mu/p}(y) D^s \zeta_j(y)|^p}{|x - y|^{n+sp}}
+ \int_{|x - y| \leq 2^{-j-2}} \frac{|\nabla (\rho^{\mu/p} D^s \zeta_j)(z)|^p}{|x - y|^{n-\mu + sp}}
\leq \int_{|x - y| \geq 2^{-j-2}} \frac{dx}{|x - y|^{n+sp}}.
\end{array} \right.
\]

Here \( z = z(x) = y + \sigma(x)(x - y), 0 \leq \sigma(x) \leq 1 \). It holds that
\[
|\nabla (\rho^{\mu/p} D^s \zeta_j)(z)|^p \leq (2^{-j/2})^{2(\mu/p)2^{-j}} = c' 2^{-j/2} \mu + \mu.
\]

Equation (30) is an easy consequence of the last estimate.

\textbf{Theorem 2.} Let \( 1 < p < \infty \) and \( -\infty < \mu < \infty \). If \( s = 0, 1, 2, \ldots \), then
\[
\mathcal{W}^p_{\mu,s}(R_n) = \left\{ f \mid f \in W^p_{\mu,s}(R_n), \| f \|_{\mathcal{W}^p_{\mu,s}(R_n)} = \left( \int_{R_n} \sum_{|a| \leq s} \rho^{\mu - (s - |a|)p}(x) |D^a f(x)|^p dx \right)^{1/p} < \infty \right\}.
\]

If \( 0 < s = [s] + \{s\} \), \([s]\) integer, \(0 < \{s\} < 1\), then
\[
\mathcal{W}^p_{\mu,s}(R_n)
= \left\{ f \mid f \in W^p_{\mu,s}(R_n), \| f \|_{\mathcal{W}^p_{\mu,s}(R_n)} = \| f \|_{\mathcal{W}^p_{\mu,[s] + \{s\}}(R_n)}
+ \left( \int_{R_n \times R_n} \sum_{|a| = [s]} \frac{|\rho^{\mu/p}(x) D^a f(x) - \rho^{\mu/p}(y) D^a f(y)|^p}{|x - y|^{n + (s - \{s\})p}} \, dx \, dy \right)^{1/p} < \infty \right\}.
\]

In both cases \( \| f \|_{\mathcal{W}^p_{\mu,s}(R_n)} \) is an equivalent norm in \( \mathcal{W}^p_{\mu,s}(R_n) \).

\textbf{Proof. Step 1.} Let \( f \in \mathcal{W}^p_{\mu,s}(R_n) \). Further let \( \{ \zeta_j \}_{j=0}^\infty \in \hat{Z} \), where we assume that there exists a positive number \( C \) such that
\[
\inf_{x \in \supp \zeta_j, y \in K_j} |x - y| \geq C 2^j; \quad j = 0, 1, 2, \ldots
\]
(For instance, the system constructed in Remark 1, has this property, where \( C = 1/4 \).) If \( s \) is not an integer, then it follows that

\[
\| f \|_{W_p^{s},\mu(R_n)}^{p} \leq c \sum_{j=0}^{\infty} \sum_{|\alpha| \leq [s]} 2^{j \mu - js \mu + j |\alpha| |s|} \int_{R_n} |D^\alpha (f_{k_j}(x))|^p \, dx \\
+ c \sum_{j=0}^{\infty} \sum_{|\alpha| = [s]} \int_{K_j \times K_j} \frac{|N^{\mu/p}(x) D^\alpha (f_{k_j}(x)) - N^{\mu/p}(y) D^\alpha (f_{k_j}(y))|^p}{|x - y|^{n+|\alpha| |p|}} \, dx \, dy \\
+ c \sum_{j=0}^{\infty} \sum_{|\alpha| = [s]} \int_{(R_n - K_j) \times K_j} \frac{N^{\mu}(y) |D^\alpha (f_{k_j}(y))|^p}{|x - y|^{n+|\alpha| |p|}} \, dy.
\]  

(34)

(Here we used that for fixed \( x \in R_n \) at most three terms \( (f_{k_j}(x)) \) are not zero.) (33) yields that the last term on the right-hand side of (34) can be estimated by the first term. Applying (30), where \( \alpha = 0 \) and \( \eta = \{s\} \), and where \( D^\alpha z_j \) is replaced by 1, it follows that

\[
\| f \|_{w_p^{s},\mu(R_n)}^{p} \leq c \sum_{j=0}^{\infty} \sum_{|\alpha| \leq [s]} 2^{j \mu - js \mu + j |\alpha| |s|} \int_{R_n} |D^\alpha (f_{k_j}(x))|^p \, dx \\
+ c \sum_{j=0}^{\infty} \sum_{|\alpha| = [s]} 2^{j \mu} \int_{R_n \times R_n} \frac{|D^\alpha (f_{k_j}(x)) - D^\alpha (f_{k_j}(y))|^2}{|x - y|^{n+|\alpha| |p|}} \, dx \, dy.
\]  

(35)

Considering the transformation \( \tilde{x} = \varepsilon x \), where \( \varepsilon > 0 \), one obtains for \( 0 < |\alpha| < [s] \) and \( g \in W_p^{s}(R_n) \)

\[
e^{-|\varepsilon| |\alpha| |s|} \int_{R_n} |D^\alpha g(x)|^p \, dx \\
\leq c \sum_{|\beta| = [s]} \int_{R_n \times R_n} \frac{|D^\beta g(x) - D^\beta g(y)|^p}{|x - y|^{n+|\beta| |p|}} \, dx \, dy + c \varepsilon^{-|\beta| |s|} \| g \|_{L_p},
\]

(36)

where \( c \) is independent of \( \varepsilon \). Choosing \( g = f_{k_j} \) and \( \varepsilon = 2^j \), and applying (29) (with the \( b \)-spaces instead of the \( h \)-spaces) then (35) yields

\[
\| f \|_{w_p^{s},\mu(R_n)}^{p} \leq c \| f \|_{b_p^{s},\mu}^{p}.
\]  

(37)

A similar conclusion can be made if \( s \) is an integer. Then one has to replace \( b_p^{s},\mu \) in (37) by \( h_p^{s} \).
Step 2. Assume $f \in W^s_{p,1,0c}(R_n)$ and $\|f\|_{W^{s}_{p,1,0c}(R_n)} < \infty$. Let $(\xi_j)_{j=0}^\infty \in \hat{Z}$ be the same system as in the first step. If $s$ is an integer, then it follows from the last equivalence in (29) and the transformation which leads to the first equivalence in (29)

$$\|f\|_{L_p}^p \sim \sum_{j=n}^{\infty} \left(2^{ju} \sum_{|\alpha| = e} \|D^\alpha(f\xi_j)(x)\|_{L_p}^p + 2^{ju-j\alpha} \|f\xi_j\|_{L_p}^p \right).$$

(38)

The properties of the system $(\xi_j)_{j=0}^\infty$ prove

$$\|f\|_{L_p}^p \leq c \|f\|_{W^{s}_{p,1,0c}(R_n)}^p.$$  

(39)

Let $s$ be not an integer. Then

$$\|f\|_{L_p}^p \leq c \sum_{j=0}^{\infty} \sum_{|\beta| + |\gamma| = [s]} 2^{ju} \int_{R_n \times R_n} \frac{|D^\beta f(x) D^\gamma \xi_j(x) - D^\gamma f(y) D^\beta \xi_j(y)|^p}{|x - y|^{n+|\beta|+|\gamma|}^p} \, dx \, dy.$$  

(40)

On the terms with $|\gamma| < [s]$ we apply the counterpart to (36), namely

$$\int_{R_n \times R_n} \frac{|g(x) - g(y)|^p}{|x - y|^{n+|\gamma|}^p} \, dx \, dy \leq c \epsilon^{(1-(s))p} \sum_{|\beta| = 1} \|D^\beta g\|_{L_p}^p + c \epsilon^{-|\gamma|p} \|g\|_{L_p}^p.$$  

We set $g = D^\gamma f \cdot D^\beta \xi_j$ and $\epsilon = 2^{j}$. For estimating the terms with $|\gamma| = [s]$ we use (30) (for $\mu = 0$, $|\alpha| = 0$, and $\eta = [s]$) and the method of the last step, in particular (34). It follows that

$$\|f\|_{L_p}^p \leq c \sum_{j=0}^{\infty} \sum_{|\gamma| < [s]} 2^{ju-j\alpha} \|D^\gamma f(x)|^p \, dx \int_{K_j} + c \sum_{j=0}^{\infty} \sum_{|\gamma| = [s]} \int_{K_j \times K_j} 2^{ju} \frac{|D^\gamma f(x) - D^\gamma f(y)|^p}{|x - y|^{n+|\gamma|}^p} \, dx \, dy.$$
Using again (30) ($|\alpha| = 0, \eta = \{s\}$, and $D^\alpha \zeta_j$ replaced by 1), it follows

$$\|f\|_{b_p,p,\mu(R_n)}^{b_p}$$

$$\ll c \|f\|_{b_p,p,\mu-\{s\},\mu(R_n)}^{b_p} + c \sum_{j=0}^{\infty} \sum_{\gamma \in \delta_j} \int_{K_j \times K_j} \frac{|\rho^{\mu/p}(x) D^\alpha f(x) - \rho^{\mu/p}(y) D^\alpha f(y)|^p}{|x - y|^{n+\delta}} \, dx \, dy$$

$$\ll c' \|f\|_{b_p,p,\mu(R_n)}^{b_p} .$$

This proves the theorem.

3. INTERPOLATION THEOREMS

3.1. Preliminaries

It is assumed that the reader is acquainted with the basic notations of abstract interpolation theory in Banach spaces. A detailed description may be found in [17]. But for convenience we recall some notations. Further we prove two lemmas, out of the main line of interpolation theory.

A pair $\{A_0, A_1\}$ of two Banach spaces is said to be an interpolation couple, if both spaces are continuously embedded in a common linear Hausdorff space $A$. Then

$$A_0 + A_1 = \{a \mid a \in A, \exists a_0 \in A_0, \exists a_1 \in A_1, a = a_0 + a_1\}$$

is meaningful. Equipped with the norm

$$K(t, a) = K(t, a; A_0, A_1) = \inf_{a_0 + a_1} (\|a_0\|_{A_0} + t \|a_1\|_{A_1})$$

it becomes a Banach space. Here $0 < t < \infty$. Of course, all these norms for different values of $t$ are equivalent to each other. If $0 < \Theta < 1$ and $1 \leq q \leq \infty$ then

$$\begin{align*}
(A_0, A_1)_{\Theta,q} &= \left\{ a \mid a \in A_0 + A_1, \|a\|_{(A_0, A_1)_{\Theta,q}} = \left( \int_0^\infty [t^{-q}K(t, a)]^q \frac{dt}{t} \right)^{1/q} < \infty \right\} \\
&= \left( \int_0^\infty [t^{-q}K(t, a)]^q \frac{dt}{t} \right)^{1/q}
\end{align*}$$

is the real interpolation functor. (For $q = \infty$ one has to replace $\left( \int_0^\infty [t^{-q}K(t, a)]^q \frac{dt}{t} \right)^{1/q}$ by $\sup | \cdot |$). Let $S = \{x \mid 0 < \Re x < 1\}$ be a strip in the complex plane,
let \( S \) be its closure. Then \( F(A_0, A_1) \) is the set of all vector-valued functions \( f \) defined in \( S \) with the following properties:

1. \( f(x) \) is \((A_0 + A_1)\)-continuous and \((A_0 + A_1)\)-bounded in \( \bar{S} \).
2. \( f(x) \) is \((A_0 + A_1)\)-analytic in \( S \).
3. \( f(it) \) is \( A_0 \)-continuous and \( A_0 \)-bounded for \( t \in \mathbb{R} \), \( f(1 + it) \) is \( A_1 \)-continuous and \( A_1 \)-bounded for \( t \in \mathbb{R} \).

\( F(A_0, A_1) \), equipped with the norm

\[
\| f \|_{F(A_0, A_1)} = \max \left[ \sup_{t \in \mathbb{R}} \| f(it) \|_{A_0}, \sup_{t \in \mathbb{R}} \| f(1 + it) \|_{A_1} \right]
\]
becomes a Banach space. If \( 0 < \Theta < 1 \), then

\[
[A_0, A_1]_{\Theta} = \{ a \mid a \in A_0 + A_1, \exists f \in F(A_0, A_1), f(\Theta) = a \},
\]

\[
\| a \|_{[A_0, A_1]_{\Theta}} = \inf_{f \in F(A_0, A_1)} \| f \|_{[A_0, A_1]_{\Theta}}
\]
is the complex interpolation functor.

If \( A \) is a Banach space with the norm \( \| a \|_A \), then \( cA \) denotes (temporarily) the same Banach space, equipped with the norm \( c \| a \|_A \). Here \( c > 0 \).

**Lemma 5.** Let \( \{A_0, A_1\} \) be an interpolation couple, and let \( c_0 \) and \( c_1 \) be two positive numbers. Then holds

\[
\| a \|_{[c_0 A_0, c_1 A_1]_{\Theta}} = c_0^{1-\Theta} c_1^{\Theta} \| a \|_{[A_0, A_1]_{\Theta}}, \quad a \in [A_0, A_1]_{\Theta},
\]

\[
\| a \|_{(c_0 A_0, c_1 A_1)_{\Theta}} = c_0^{1-\Theta} c_1^{\Theta} \| a \|_{(A_0, A_1)_{\Theta}}, \quad a \in (A_0, A_1)_{\Theta}.
\]

**Proof.** (44) is an easy consequence of (41). We prove (43).

\[
g(x) = c_0^{\frac{1}{2\beta}} c_1^{\frac{1}{2\beta}} f(x)
\]

is an isomorphic mapping from \( F(c_0 A_0, c_1 A_1) \) onto \( F(A_0, A_1) \). But now, using (42), we obtain (43).

We prove a second lemma about interpolation. Let \( A \) and \( B \) be two Banach spaces. Then the linear and bounded operator \( R \in L(A, B) \) is said to be a retraction, if there exists an operator \( S \in L(B, A) \) such that

\[
RS = E \quad \text{(identity in } B)\]

\( S \) is called a (corresponding) coretraction. Clearly, \( R \) is a mapping from \( A \) onto \( B \), since \( R(b) = b \) for all \( b \in B \). The operator \( S \) has a trivial null-space.

Further,

\[
(SR)^{\beta} = SRSR = SR.
\]
Hence, $SR$ is a projection in $A$. Therefore, the range of $SR$ is a complemented subspace of $A$, the projection space. $S$ is an isomorphic mapping from $B$ onto this projection space, and $R$, restricted on this projection space, is the corresponding inverse operator.

**Lemma 6.** Let $\{A_0, A_1\}$ and $\{B_0, B_1\}$ be two interpolation couples. Let $R \in L(A_0 + A_1, B_0 + B_1)$ and $S \in L(B_0 + B_1, A_0 + A_1)$, such that the restriction of $R$ on $A_j$; $j = 0, 1$; is a retraction from $A_j$ onto $B_j$; and the restriction of $S$ on $B_j$ is a corresponding coretraction. Then the restriction of $R$ on $[A_0, A_1]_\Theta$; $0 < \Theta < 1$; is a retraction from $[A_0, A_1]_\Theta$ onto $[B_0, B_1]_\Theta$, and the restriction of $S$ on $[B_0, B_1]_\Theta$ is a corresponding coretraction. The same statement holds for the spaces $(A_0, A_1)_{\Theta_{1,q}}$, respectively $(B_0, B_1)_{\Theta_{1,q}}$; $0 < \Theta < 1$; $1 \leq q \leq \infty$.

**Proof.** The hypotheses show that (45) holds for $A = A_0 + A_1$ and $B = B_0 + B_1$. On the other hand, it follows from the interpolation property $R \in L([A_0, A_1]_\Theta, [B_0, B_1]_\Theta)$, $S \in L([B_0, B_1]_\Theta, [A_0, A_1]_\Theta)$ (46) and a similar assertion for the real interpolation functor. (We use the symbols $R$ and $S$ also for their restrictions.) But the two last facts show that (45) holds also for the operators from (46). This proves the lemma.

**Remark 5.** The lemma is simple. But it seems to be one of the most powerful tools in interpolation theory, and, in particular, in its applications to spaces of functions and distributions. The idea is due to J. Peetre [10] (who used this method more implicitly also in other papers). This method is extensively used in [17]. Perhaps, the most valuable fact is that $S$ is an isomorphic mapping from $[B_0, B_1]_\Theta$, respectively $(B_0, B_1)_{\Theta_{1,q}}$, onto a complemented subspace of $[A_0, A_1]_\Theta$, respectively $(A_0, A_1)_{\Theta_{1,q}}$.

### 3.2. Interpolation of the h-Space and the b-Spaces over $R_n$

With the aid of the last two lemmas it is not very hard to develop an interpolation theory for the $h$-spaces and the $b$-spaces.

**Theorem 3.** Let $-\infty < \mu_0 < \infty$; $-\infty < \mu_1 < \infty$; $1 < p_0 < \infty$; $1 < p_1 < \infty$; $1 \leq q_0 \leq \infty$; $1 \leq q_1 \leq \infty$; and $s_0 > 0$; $s_1 > 0$ (in the case of the $h$-spaces $s_0 = 0$, respectively $s_1 = 0$, is also admissible). Further let $0 < \Theta < 1$. Let

$$s = (1 - \Theta) s_0 + \Theta s_1; \quad \frac{1}{p} = \frac{1 - \Theta}{p_0} + \frac{\Theta}{p_1}; \quad \frac{1}{q} = \frac{1 - \Theta}{q_0} + \frac{\Theta}{q_1};$$

(47)

$$\frac{\mu}{p} = (1 - \Theta) \frac{\mu_0}{p_0} + \Theta \frac{\mu_1}{p_1}.$$  

(48)
If additionally $p_0 = p_1 = p$ and $s \neq s_1$, then holds
\[ b^s_{p_0, p_1, u_0, u_1}(R_n) = (h_{p_0, u_0}^s(R_n), h_{p_1, u_1}^s(R_n))_{\Theta, p} = (h_{p_0, u_0}^s(R_n), b^s_{p_1, u_1}(R_n))_{\Theta, p} = (b^s_{p_0, p_1, u_0, u_1}(R_n), h_{p_1, u_1}^s(R_n))_{\Theta, p}. \]

(b) It holds that
\[ (b^s_{p_0, p_0, u_0, u_0}(R_n), b^s_{p_0, p_0, u_0, u_0}(R_n))_{\Theta, p} = b^s_{p_0, p_0, u_0}(R_n). \]

(c) If additionally $s_0 \neq s_1$, then
\[ (b^s_{p_0, p_0, u_0, u_0}(R_n), h_{p_1, u_1}^s(R_n))_{\Theta, p} = (h_{p_0, u_0, u_0}^s(R_n), h_{p_1, u_1}^s(R_n))_{\Theta, p} = b^s_{p_0, p_0, u_0}(R_n). \]

(d) If additionally $s_0 = s_1 = s$, then
\[ (h_{p_0, u_0}^s(R_n), h_{p_1, u_1}^s(R_n))_{\Theta, p} = h_{p_1, u_1}^s(R_n). \]

(e) It holds
\[ [h_{p_0, u_0}^s(R_n), h_{p_1, u_1}^s(R_n)]_{\Theta} = h_{p, u}^s(R_n), \]
\[ [b^s_{p_0, p_0, u_0, u_0}(R_n), h_{p_1, u_1}^s(R_n)]_{\Theta} = b^s_{p_0, p_0, u_0}(R_n). \]

Proof. Step 1. We prove (53). Using (29), one obtains that the operator $S$, defined by
\[ Sf = \{f(\xi_j)_{j=0}^\infty\}, \]
is a linear and bounded mapping from $h_{p, u}^s(R_n)$ into $l_p(H_{p, u}^s(R_n))$, where $H_{p, u}^s(R_n)$ is the space $H_{p, u}^s(R_n)$ equipped with the norm
\[ \|g\|_{H_{p, u}^s(R_n)} = 2^{j(\mu/p - j + j(n/p))} \|g(2^jx)\|_{H_{p, u}^s(R_n)}. \]
Here $l_p(A_J)$ is the $l_p$-space with $A_J$-valued components; $j = 0, 1, \ldots$; normed in the usual way. For the system $\{\xi_j\}_{j=0}^\infty \in \hat{Z}$ it is supposed that (33) is satisfied. By the method described in Remark 1 we can construct a second system $\{\varphi_j\}_{j=0}^\infty \in Z$, such that
\[ \varphi_j(x) = 1 \quad \text{for} \quad x \in \text{supp } \xi_j. \]
Using again (29), it is not very hard to see, that $R$, defined by

$$R(g) = \sum_{j=0}^{\infty} \varphi_j(x) g_j(x),$$

is a linear and bounded mapping from $l_p(H^{p,1}(R_n))$ into $h^s_{p,u}(R_n)$. One obtains with the help of (55)

$$RS = E.$$ 

Hence, $R$ is a retraction, and $S$ is a corresponding coretraction in the sense of (45). Now we use the interpolation theory for vector-valued $l_p$-spaces, see [17, 1.18.1], or [16, p. 75]. If $\{A_j, B_j\}$ are interpolation couples, then holds

$$[l_{p_0}(A_j), l_{p_1}(B_j)]_\Theta = l_p([A_j, B_j]_\Theta),$$

where $p_0, p_1, p$, and $\Theta$ satisfy the above conditions. Applying the above considerations on the spaces $h^{p_0, u_0}_{p_0}(R_n)$ and $h^{s_{i_1}, u_1}_{p_1}(R_n)$, and using Lemma 6 (as well as the remarks before and after this lemma), then it follows that $S$ is also an isomorphic mapping from

$$[h^{p_0, u_0}_{p_0}(R_n), h^{s_{i_1}, u_1}_{p_1}(R_n)]_\Theta$$

onto a complemented subspace of

$$[l_{p_0}(H^{p_0, u_0}_{p_0}(R_n)), l_{p_1}(H^{p_1, v_1}_{p_1}(R_n))]_\Theta = l_p([H^{p_0, u_0}_{p_0}(R_n), H^{s_{i_1}, u_1}_{p_1}(R_n)]_\Theta).$$

To interpolate $H^{p_0, u_0}_{p_0}(R_n)$ and $H^{s_{i_1}, v_1}_{p_1}(R_n)$ we use again the last equivalence in (29). Clearly, $g(x) \rightarrow g(2x)$ is an isometric mapping from $H^{p, v}_{p_k}(R_n)$ onto $H^{p, v}_{p_k}(R_n)$; $k = 0, 1$; equipped with the norm

$$2^{j(u_k/p_k)-j(v_k/v)(n/p_k)} \| \cdot \|_{H^{p, v}_{p_k}}.$$ 

The interpolation property and Lemma 5 yield that $g(x) \rightarrow g(2x)$ is also an isometric mapping from $[H^{p_0, u_0}_{p_0}(R_n), H^{s_{i_1}, v_1}_{p_1}(R_n)]_\Theta$ onto $[H^{p_0, u_0}_{p_0}(R_n), H^{s_{i_1}, v_1}_{p_1}(R_n)]_\Theta$ equipped with the norm

$$2^{j(u/p)-j(v/n)} \| \cdot \|_{[H^{p_0, u_0}(R_n), H^{s_{i_1}, v_1}(R_n)]_\Theta}.$$ 

Here we used (47), and (48). But

$$[H^{p_0}_{p_0}(R_n), H^{s_{i_1}}_{p_1}(R_n)]_\Theta = H^{s_{i}}_{p}(R_n),$$

(58)
see [17, Formula (2.4.2/11)], or [15, p. 631]. Hence,

$$[H^b_{p,q}(R^n), H^b_{p_1,q}(R^n)]_{\Theta} = H^b_{p,q}(R^n).$$

Putting this result in (57) and using the isomorphic properties of $S$ and (29), it follows that

$$\|f\|_{[H^b_{p,q}^s, u_0, u_1(R^n)]_{\Theta}} \sim \left( \sum_{j=0}^{\infty} 2^{-j} \|f^{(j)}\|_{L^p_{p_1}} \right)^{1/p} = \|f\|_{H^b_{p,q}^s(R^n)}.
$$

This proves (53).

**Step 2.** All the other assertions of the theorem can be proved in exactly the same way. Essentially, the proof of (53) was reduced to (56) and (58). We must replace these both statements by corresponding relations. The needed interpolation theorems for the vector-valued $l_p$-spaces may be found in [17, 1.18.1] (see also [16, p. 75]). The interpolation theory for the Lebesgue–Besov (Sobolev–Slobodeckij) spaces $H^s_p(R^n)$ and $B^s_{p,q}(R^n)$ is developed in [17, 2.4] (see also [15]).

**Remark 6.** The method described in the first step explains one of the main features of interpolation theory for function spaces: Firstly, one has to prove interpolation theorems for vector-valued $l_p$-spaces and $L_p$-spaces in a more or less direct way. Secondly, one reduces the interpolation theory for the Lebesgue–Besov (Sobolev–Slobodeckij) spaces $H^s_p(R^n)$ and $B^s_{p,q}(R^n)$ to the theory of the vector-valued $l_p$-spaces and $L_p$-spaces. This is done systematically in [17] and [15]. Thirdly, one reduces the interpolation theory for function spaces with weights to the theory for Lebesgue–Besov spaces $H^s_p(R^n)$ and $B^s_{p,q}(R^n)$. Step 1 of the above proof is an example of this procedure. Another, rather wide, class of functions spaces with weights is treated on this basis in [17] and [16]. In particular, the above theorem is the counterpart to theorem 3.4.2 in [17].

**Remark 7.** The most interesting spaces of the above type are the spaces $w^s_{p,q}(R^n)$, since they can be described explicitly by Theorem 2. So it will be interesting to specialize the above theorem on these spaces. Then one obtains a generalization of Theorem 3.9.2 in [17]. If the hypotheses of the last theorem are satisfied, then

$$\left( w^s_{p_0,q_0}(R^n), w^s_{p_1,q_1}(R^n) \right)_{\Theta,p} = \begin{cases} \|h^s_{p_0,0}(R^n) \| & \text{if } s = s_0 = s_1 \neq \text{integer}, \\ \|w^s_{p,0}(R^n) \| & \text{if } s = s_0 = s_1 = \text{integer}. \end{cases}$$

(59)
This is a consequence of (50), (51), and (52). Clearly, only in the case $s_0 \neq s_1$, $s$ integer, the right-hand side of (59) is not $w_{p,\mu}(R_n)$. This is a generalization of formula (3.9.2/1) in [17]. A little more complicated is the complex interpolation. If $s_0$ and $s_1$ are not integers, then (see (54))

$$[w_{p_0,\mu_0}^{s_0}(R_n), w_{p_1,\mu_1}^{s_1}(R_n)]_\Theta = b_{p,\mu}^s(R_n).$$

If $s_0$ and $s_1$ are integers, then it follows from (53)

$$[w_{p_0,\mu_0}^{s_0}(R_n), w_{p_1,\mu_1}^{s_1}(R_n)]_\Theta = h_{p,\mu}^s(R_n).$$

But if $s_0$ is an integer and $s_1$ is not an integer, then the interpolation of the $w$-spaces requires the complex interpolation of $H_{p_0}^{s_0}(R_n)$ and $B_{p_1,\mu_1}^{s_1}(R_n)$. This cannot be done in the framework of the Lebesgue–Besov spaces, see [15], formula (14), where the interpolation space is described with the aid of new spaces.

3.3. Interpolation of the $h$-Spaces and the $b$-Spaces over $R_n^+$

It is not very hard to carry over Theorem 3 on the $h$-spaces and the $b$-spaces (and so also to the $w$-spaces) over $R_n^+$.

**Lemma 7.** The restriction operator $R$ from $h_{p,\mu}^s(R_n)$ onto $h_{p,\mu}^s(R_n^+)$, where $s > 0$; $-\infty < \mu < \infty$; $1 < p < \infty$; and from $b_{p,\mu}^s(R_n)$ onto $b_{p,\mu}^s(R_n^+)$, where $s > 0$; $-\infty < \mu < \infty$; $1 < p < \infty$; $1 < q < \infty$; is a retraction. If $N$ is a given positive number, then there exists a common corresponding coretraction $S$ (extension operator) for all the $h$-spaces and all the $b$-spaces, provided that $s < N$. (More precisely: There exists an extension operator $S$ from $L_p^{\infty}(R_n^+)$ into $L_p^{\infty}(R_n)$, whose restriction on $h_{p,\mu}^s(R_n^+)$; $0 \leq s < N$; $1 < p < \infty$; $-\infty < \mu < \infty$; is an extension operator into $h_{p,\mu}^s(R_n)$, and whose restriction on $b_{p,\mu}^s(R_n^+)$; $0 < s < N$; $1 < p < \infty$; $1 < q < \infty$; $-\infty < \mu < \infty$; is an extension operator into $b_{p,\mu}^s(R_n)$).

**Proof.** It is sufficient to construct an appropriate extension operator. We use the known fact that there exists a common extension operator $\tilde{S}$ from $H_p^\infty(R_n^+)$; $0 \leq s < N$; $1 < p < \infty$; into $H_p^\infty(R_n)$, and from $B_{p,\mu}^s(R_n^+)$; $0 < s < N$; $1 < p < \infty$; $1 < q < \infty$; into $B_{p,\mu}^s(R_n)$, see [17, Lemma 2.9.3]. Let $(\lambda_j)_{j=0}^{\infty} \in \mathcal{Z}$, where (33) is satisfied. Let $(\varphi_j)_{j=0}^{\infty} \in \mathbb{Z}$, such that (55) holds. Then

$$\hat{(Sf)}(x) = \sum_{j=0}^{\infty} \varphi_j(x) \hat{S}[(f(y))(2^jy)] (2^{-j}x)$$

has the desired properties: $\hat{S} [(f(y))(2^jy)] (2^jx)$ means that we apply $\hat{S}$ on the function $g(y)$, where $g(y) = (f(y))(2^jy)$, and this function, defined in $R_n$, is to
take in the point $2^{-i}x$. If $x \in \mathbb{R}^+$ then $(Sf)(x) = f(x)$. Using the third equivalence in (29) it follows that

$$
\|Sf\|_{h^{\infty}_{p,u}(\mathbb{R}^+)} \leq c \sum_{k=0}^{\infty} \sum_{r=0}^{\infty} 2^{ku-ksp+k} \|\varphi_k(2^kx) (Sf)(2^kx)\|_{h^{\infty}_{p,u}}
$$

(61)

$(\zeta_1 = 0$ for $1 < 0)$. If $g \in h^{s}_{p,u}(\mathbb{R}^+)$ is an arbitrary function whose restriction on $\mathbb{R}^+$ coincides with $f$, then it follows that

$$
\|Sf\|_{h^{\infty}_{p,u}(\mathbb{R}^+)} \leq c' \sum_{j=0}^{\infty} 2^{j+jsp+is} \|\varphi_k(2^kx)\|_{h^{\infty}_{p,u}} \sim \|g\|_{h^{s}_{p,u}(\mathbb{R}^+)}
$$

Taking the infimum with respect to $g$ on the right-hand side, then it follows the desired property. In the same manner one concludes for the $b$-spaces.

**Theorem 4.** Under the hypotheses of Theorem 3 all the statements of Theorem 3 are true, after replacing there $\mathbb{R}^+$ by $\mathbb{R}^++$.

**Proof.** We prove the counterpart to (53). Lemma 7 and Lemma 6 yield

$$
\|Sf\|_{h^{\infty}_{p,u}(\mathbb{R}^+)} \leq c' \sum_{j=0}^{\infty} 2^{j+jsp+is} \|\varphi_k(2^kx)\|_{h^{\infty}_{p,u}} \sim \|g\|_{h^{s}_{p,u}(\mathbb{R}^+)}
$$

This proves (53), where $\mathbb{R}^+$ is replaced by $\mathbb{R}^++$. In exactly the same manner one proves all the other statements.

**Remark 8.** Of course, the special cases of Remark 7 can be carried over, too.

**4. Embedding Theorems**

In this section we consider two types of embedding theorems: embedding from $\mathbb{R}^+$ into itself, and embedding from $\mathbb{R}^+$ into $\mathbb{R}^+-$, the boundary of $\mathbb{R}^+$. Of course, the second type seems to be more important, since embedding theorems on the boundary are the basis for considering boundary value problems for degenerate elliptic differential operators.

**4.1. Embedding Theorems for Different Metrics**

Let $g^{s}_{p,u}(\mathbb{R}^+)$ be either $h^{s}_{p,u}(\mathbb{R}^+)$ or $b^{s}_{p,u}(\mathbb{R}^+)$, where the index $q$ is omitted for the moment, since it does not play any role in the following theorem. (For the $b$-space we assume $s > 0$, for the $h$-space $s \geq 0$).
Theorem 5. Let \( \sigma > s \geq 0 \). Let \( 1 < r \leq p < \infty \). Further let \( \mu \in R_1 \). If \( \sigma - n/r > s - n/p \), and \( (v/r) - \sigma + (n/r) = (\mu/p) - s + (n/p) \), then holds

\[
g^\sigma_{r,v}(R_n) \subset g^\sigma_{p,u}(R_n).
\]

The last relation is also valid after replacing \( R_n \) by \( R_n^+ \).

Proof. Let (for instance) \( f \in h^\sigma_{r,v}(R_n) \). Using (29) and the well-known embedding theorems for the \( H \)-spaces, see for instance 2.8.1 in [17], it follows that

\[
\|f\|_{h^\sigma_{p,u}(R_n)} \leq c \left( \sum_{j=0}^{\infty} 2^{ju-j\sigma p-jn} \|f_j\|_{H^\sigma_r} \right)^{1/p} \leq c \left( \sum_{j=0}^{\infty} 2^{ju-j\sigma p-jn} \|f_j\|_{H^\sigma_r} \right)^{1/r} = c \left( \sum_{j=0}^{\infty} 2^{ju-j\sigma p-jn} \|f_j\|_{H^\sigma_r} \right)^{1/r} \sim \|f\|_{h^\sigma_{p,u}(R_n)}.
\]

Similarly one proves the assertion if \( g^\sigma_{p,u} \) and/or \( g^\sigma_{r,v} \) is an \( b \)-space. One obtains the desired statements for the spaces over \( R_n^+ \) on the basis of Lemma 7.

Remark 9. Using other embedding theorems for the Lebesgue–Besov spaces [17, 2.8.1] one can reinforce the last theorem. For instance, in some cases one can admit the limit case \( \sigma - n/r = s - n/p \). Further, one can formulate conditions which ensure that functions belonging to \( b \)-spaces or \( h \)-spaces are (at least locally) elements of Hölder spaces.

4.2. Direct and Inverse Embedding Theorems (Embedding on the Boundary)

Let \( R_n \in x = (x', x_n) \), where \( x' = (x_1, \ldots, x_{n-1}) \). If \( \lambda \) is a positive number we write

\[
\lambda = [\lambda]^- + [\lambda]^+; [\lambda]^+ \text{ integer; } 0 < [\lambda]^+ \leq 1.
\]

Let \( 1 < p < \infty \) and \( s > 1/p \). Then the operator \( R \) is defined in this section by

\[
Rf = \left\{ f(x', 0), \frac{\partial f(x', 0)}{\partial x_n}, \ldots, \frac{\partial[s-(1/p)]^{-}\partial f(x', 0)}{\partial x_n[s-(1/p)]^{-}}(x', 0) \right\}.
\]

Theorem 6. (a) If \( 1 < p < \infty \); \( 1/p < s < \infty \); and \(-\infty < \mu < \infty \); then \( R \) is a retraction from \( h^\sigma_{p,u}(R_n) \), respectively \( h^\sigma_{p,u}(R_n^+) \), onto

\[
\prod_{k=0}^{[s-(1/p)\mu]} b^s-(1/p)(k)(R_{n-k}).
\]
It holds
\[ h_{p,u}^s(R_+^n) = \{ f \mid f \in h_{p,u}^s(R_+^n), Rf = 0 \}. \] (65)

(b) If \( 1 < p < \infty; 1/p < s < \infty; 1 \leq q \leq \infty; \) and \( -\infty < \mu < \infty; \) then \( R \) is a retraction from \( b_{p,q,u}^s(R_n), \) respectively \( b_{p,q,u}^s(R_+^n), \) onto
\[ \prod_{k=0}^{\lfloor s-(1/p) \rfloor - 1} b_{p,q,u}^{s-(1/p)-k}(R_{n-1}). \]

If additionally \( 1 < q < \infty, \) then
\[ b_{p,q,u}^s(R_+^n) = \{ f \mid f \in b_{p,q,u}^s(R_+^n), Rf = 0 \}. \] (66)

(c) Let \( 1 < q, p > \infty; 0 < s \leq 1/p; \) and \( -\infty < \mu < \infty. \) Then holds
\[ h_{p,u}^s(R_+^n) = h_{p,u}^s(R_+^n), \quad b_{p,q,u}^s(R_+^n) = b_{p,q,u}^s(R_+^n). \] (67)

(For the \( h \)-spaces \( s = 0 \) is admissible.)

Proof. Step 1. Temporarily we denote the system \( Z \) and the system \( Z \) from Definition 1 by \( Z_n, \) respectively \( Z_n, \) to indicate the dimension. Let \( \{ \zeta_j \}_{j=0}^\infty \in \hat{Z}_n. \) Then holds \( \{ \zeta_j(x', 0) \}_{j=0}^\infty \in \hat{Z}_n. \) We assume additionally that \( \zeta_j(2^i x) \) is independent of \( x_n \) provided that \( |x_n| \leq \delta, \) where \( \delta > 0 \) is a sufficiently small number. It is not hard to see, that there exist systems of such a type. Let \( f \in h_{p,u}^s(R_n), \) where \( s > 1/p. \) We use the known fact that \( R \) is a retraction from \( H^s_p(R_n) \) onto \( \prod_{k=0}^{\lfloor s-(1/p) \rfloor - 1} B^{s-(1/p)-k}(R_n), \) the corresponding coretraction is denoted by \( \mathcal{S}; \) see [17, Theorem 2.9.3]. One obtains with the aid of (29)

\[ \sum_{k=0}^{\lfloor s-(1/p) \rfloor - 1} \left\| \frac{\partial^k f}{\partial x_n^k}(x, 0) \right\|_{H_{p,v}^{s-(1/p)-k}(R_{n-1})}^p \]
\[ \sim \sum_{k=0}^{\lfloor s-(1/p) \rfloor - 1} \sum_{j=0}^{\lfloor s-(1/p) \rfloor} 2^{j+1/2} \left\| \frac{\partial^k f}{\partial y_n^k}(x') \right\|_{H_{p,v}^{s-(1/p)-k}(R_{n-1})}^p \]
\[ \quad \sum_{k=0}^{\lfloor s-(1/p) \rfloor - 1} \sum_{j=0}^{\lfloor s-(1/p) \rfloor} 2^{j+1/2} \left\| \frac{\partial^k f}{\partial y_n^k}(x') \right\|_{H_{p,v}^{s-(1/p)-k}(R_{n-1})}^p \]
\[ \leq C \sum_{j=0}^{\infty} 2^j \left\| f_j \right\|_{H_{p,v}^{s}(R_n)} \sim \left\| f \right\|_{H_{p,v}^{s}(R_n)} \] (68)
This proves that $R$ is a linear and bounded operator from $h^s_{p,u}(R^n)$ into $\prod_{k=0}^{[s-(1/p)]-} b^s_{p,p,\mu}(R_{n-1})$. To prove that $R$ is a retraction we assume
\[ g_k(x') \in b^{s-(1/p)-k}_{p,p,\mu}(R_{n-1}); \quad k = 0, \ldots, [s - (1/p)]^{-}. \]

Let \( \{\varphi_j(x)\}_{j=0}^{\infty} \in Z_n \) with the property (55). Again we suppose that \( \varphi_j(x) \) is independent of \( x_n \) near the plane \( \{y \mid y_n = 0\} \). If $\hat{S}$ is the extension operator from $H^s_p(R^{n+}_n)$ into $H^s_p(R_n)$, then we act
\[ \mathcal{G}\{g_0, \ldots, g_{[s-(1/p)]-}\}(x) = \sum_{j=0}^{\infty} \varphi_j(x) (\hat{S}\varphi_j) \{\ldots, 2^{jk}g_k(2^jx'), \ldots\} (2^{-j}x). \]

Using the technique developed in (61), then one obtains in analogy to the estimates in (68)
\[ \| \mathcal{G}\{g_0, \ldots, g_{[s-(1/p)]-}\}(x)\|_{[s-(1/p)]^{-}} \leq c \sum_{k=0}^{[s-(1/p)]^{-}} \| g_k(x')\|_{b^{s-(1/p)-k}_{p,p,\mu}}(R_{n-1}), \quad (70) \]

Further, it holds $R\mathcal{G} = E$. Hence, $\mathcal{G}$ is a coretraction, corresponding to $R$. In the same way one proves that $R$ is a retraction from $b^s_{p,q,\mu}(R_n)$ onto
\[ \prod_{k=0}^{[s-(1/p)]-} b^{s-(1/p)-k}_{p,q,\mu}(R_{n-1}). \]

**Step 2.** Let $f \in h^s_{p,u}(R^{n+}_n)$. Let $s > 1/p$. Using the extension operator $S$ from Lemma 7, it follows that $R$ is also a linear and bounded operator from $h^s_{p,u}(R^{n+}_n)$ into
\[ \prod_{k=0}^{[s-(1/p)]-} b^{s-(1/p)-k}_{p,q,\mu}(R_{n-1}). \]

The restriction of $\mathcal{G}$ on $R^{n+}_n$ is a corresponding coretraction. In the same way one concludes for the $b$-spaces.

**Step 3.** Now we prove (65), (66), (67). Clearly, the left-hand sides are contained in the right-hand sides. Let $f$ be a function of the right-hand side of (65), (66), (67), respectively. Then $R(f\xi_j) = 0$ (since $\xi_j$ does not depend on $x_n$ near the hyperplane $\{y \mid y_n = 0\}$). But in all these cases $f\xi_j$ can be approximated by functions belonging to $C^\infty_0(R^{n+}_n)$ in $H^s_p(R^{n+}_n)$, respectively $B^s_{p,q}(R^{n+}_n)$. This follows from [17, Theorem 2.9.3]. If $Sf$ is the extended function in the sense of Lemma 7, then $\xi_j Sf$ can be approximated by functions belonging to $C^\infty_0(R_n)$, vanishing near $\{y \mid y_n = 0\}$. (See the
extension method in [17, Lemma 1, subsection 2.9.1]. But now the desired statements follow from the method of the third step of the proof to Theorem 1.

**Remark 10.** The theorem is the generalization of Theorem 3.9.3 in [17]. The theorem contains the "direct" embedding ($R$ is a linear and bounded operator) as well as the "inverse" embedding ($R$ is a retraction, that means that there exists a right-inverse operator $\mathcal{E}$ such that $R\mathcal{E} = E$).

5. Structure Theorems

This section is the continuation of the structure theory for the Lebesgue-Besov spaces $H^{s}_{p}(\mathbb{R})$ and $B^{s}_{p,q}(\mathbb{R})$ developed in [17] 2.11 (see also [14]), and for Lebesgue-Besov spaces with weights developed in [17, 3.7 and 3.9.4] (some results are also proved in [16]).

**Theorem 7.** Let $1 < p < \infty$ and $-\infty < \mu < \infty$.

(a) The spaces $h^{s}_{p,\mu}(\mathbb{R})$, $h^{s}_{p,\mu}(\mathbb{R}^{+})$ and $h^{s}_{p,\mu}(\mathbb{R}^{+})$ are isomorphic to $L_{p}(0,1)$.

(b) The spaces $b^{s}_{p,\mu}(\mathbb{R})$, $b^{s}_{p,\mu}(\mathbb{R}^{+})$, and $b^{s}_{p,\mu}(\mathbb{R}^{+})$ are isomorphic to $L_{p}$.

**Proof.** Step 1. We use the method developed in the first step of theorem 3. One obtains that

$$\tilde{S}f = (2^{j(n/p) - j + j(n/p)}(f_{j+1}^{x})(2^{j}x))_{j=0}^{\infty}$$

is an isomorphic mapping from $h_{p,\mu}(\mathbb{R})$ and $b_{p,\mu}(\mathbb{R})$ onto a complemented subspace of $L_{p}(H^{s}_{p}(\mathbb{R}))$ and $L_{p}(B^{s}_{p,q}(\mathbb{R}))$, respectively. Since $H^{s}_{p}(\mathbb{R})$ is isomorphic to $L_{p}(0,1)$ and $B^{s}_{p,q}(\mathbb{R})$ is isomorphic to $L_{p}$ (see the above references) it follows that $h_{p,\mu}(\mathbb{R})$ is isomorphic to a complemented subspace of $L_{p}(0,1)$ and hence also isomorphic to a complemented subspace of $L_{p}(0,1)$. Similarly, $b_{p,\mu}(\mathbb{R})$ is isomorphic to a complemented subspace of $L_{p}$. But each infinitely dimensional complemented subspace of $L_{p}$ is isomorphic to $L_{p}$, see [11]. Hence, $b_{p,\mu}(\mathbb{R})$ is isomorphic to $L_{p}$. To prove that $h_{p,\mu}(\mathbb{R})$ is isomorphic to $L_{p}(0,1)$ we need an additional consideration. If $\omega \subset R$, is a ball, then the above-mentioned extension method yields that $H^{s}_{p}(\omega)$ is isomorphic to a complemented subspace of $H^{s}_{p}(\mathbb{R})$, where the functions of this subspace vanishes outside of a compact set. Then $H^{s}_{p}(\omega)$ is also isomorphic to a complemented subspace of $H^{s}_{p,\mu}(\mathbb{R})$. But $H^{s}_{p}(\omega)$ is isomorphic to $L_{p}(0,1)$, see [17, 4.9.3]. Consequently, we have the following situation: $h^{s}_{p,\mu}(\mathbb{R})$ is isomorphic to a complemented subspace of $L_{p}(0,1)$, and on the other hand there exists a complemented subspace of $h^{s}_{p,\mu}(\mathbb{R})$ which is isomorphic to $L_{p}(0,1)$. Since $L_{p}(L_{p}(0,1))$ is isomorphic to $L_{p}(0,1)$,
we have the situation treated by A. Pelczyński [11]. It follows that $h_{p,u}^s(R_n)$ is isomorphic to $L_p(0, 1)$.

**Step 2.** If $R$ has the meaning of (64) and if $\mathcal{E}$ is a corresponding coretraction in the sense of Theorem 6 for the spaces defined over $R_n^+$, then $E \rightarrow \mathcal{E} R$ is a projection from $h_{p,u}^s(R_n^+)$ onto $h_{p,u}^s(R_n^+)$, and from $b_{p,u}^s(R_n^+)$ onto $b_{p,u}^s(R_n^+)$, respectively. Using Lemma 7 it follows that $h_{p,u}^s(R_n^+)$ and $h_{p,u}^s(R_n^+)$ are complemented subspaces of $h_{p,u}^s(R_n)$. Similarly for the $b$-spaces. Now we can conclude in the same manner as in the first step. This proves the theorem.

**References**