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Journal of Algebra 319 (2008) 1320–1335

JOURNAL OF Algebra

www.elsevier.com/locate/jalgebra

Geometry and the zero sets of semi-invariants for homogeneous modules over canonical algebras

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Received 12 February 2007

Available online 3 December 2007

Communicated by Peter Littelmann

Abstract

We characterize the canonical algebras such that for all dimension vectors of homogeneous modules the corresponding module varieties are complete intersections (respectively, normal). We also investigate the sets of common zeros of semi-invariants of non-zero degree in important cases. In particular, we show that for sufficiently big vectors they are complete intersections and calculate the number of their irreducible components.

Keywords: Canonical algebra; Module variety; Complete intersection; Semi-invariant

Throughout the paper k denotes a fixed algebraically closed field. By $\mathbb{N}$ and $\mathbb{Z}$ we denote the sets of non-negative integers and integers, respectively. Finally, if $i, j \in \mathbb{Z}$, then $[i, j] = \{l \in \mathbb{Z} \mid i \leqslant l \leqslant j\}$.

Introduction and main result

Canonical algebras were introduced by Ringel in [28, 3.7] (for a definition see also 1.1). A canonical algebra Λ depends on a sequence $(m_1, \ldots, m_n)$, $n \geqslant 3$, of positive integers greater then 1 and on a sequence $(\lambda_3, \ldots, \lambda_n)$ of pairwise distinct non-zero elements of k. In this situation we say that Λ is a canonical algebra of type $(m_1, \ldots, m_n)$. The canonical algebras play a prominent role in the representation theory of algebras (see for example [23,30]). In particular,

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0021-8693/$ – see front matter
doi:10.1016/j.jalgebra.2007.11.008

the module categories over canonical algebras are derived equivalent to the categories of coherent sheaves over weighted projective lines (see [16]). Moreover, according to [19, Theorem 3.1] every quasi-titled algebra is derived equivalent to a hereditary algebra or to a canonical one.

Let Λ be an algebra. For each element $\mathbf{d}$ of the Grothendieck group of Λ one defines a variety $\mathrm{mod}_\Lambda(\mathbf{d})$ called the variety of Λ-modules of dimension vector $\mathbf{d}$ (see 2.1). The study of varieties of modules is an important and interesting topic in the representation theory of algebras (for some reviews of results see for example [10,17,20]). In [6] Skowroński and the author proved that if Λ is a tame canonical algebra and $\mathbf{d}$ is the dimension vector of an indecomposable Λ-module, then $\mathrm{mod}_\Lambda(\mathbf{d})$ is a complete intersection with at most 2 irreducible components. The module varieties over canonical algebras were also studied by Barot and Schröer in [3].

Let Λ be a canonical algebra of type $(m_1, \ldots, m_n)$. We call a module regular if it is periodic with respect to the action of the Auslander–Reiten translate (such modules are of special interest in the representation theory, see for example [31]). This class of modules also received special attention from a geometric point of view. Skowroński and the author showed in [7] that if $\mathbf{d}$ is the dimension vector of a regular module over a tame canonical algebra Λ, then the corresponding variety is a normal complete intersection. This result was extended in [4] by showing that the varieties $\mathrm{mod}_\Lambda(\mathbf{d})$ are normal (respectively, complete intersections) for all dimension vectors $\mathbf{d}$ of regular Λ-modules if and only if

$$\frac{1}{m_1 - 1} + \cdots + \frac{1}{m_n - 1} > 2n - 5 \quad (\geqslant 2n - 5).$$

A special type of regular modules are the homogeneous ones, which are invariant with respect to the action of the Auslander–Reiten translate. The first result of the paper is the following.

Theorem 1. *Let Λ be a canonical algebra of type $(m_1, \ldots, m_n)$.*

(1) *The varieties $\mathrm{mod}_\Lambda(\mathbf{d})$ are complete intersections for all dimension vectors $\mathbf{d}$ of homogeneous Λ-modules if and only if*

$$\frac{1}{m_1} + \cdots + \frac{1}{m_n} \geqslant n - 4.$$

(2) *The varieties $\mathrm{mod}_\Lambda(\mathbf{d})$ are normal for all dimension vectors $\mathbf{d}$ of homogeneous Λ-modules if and only if*

$$\frac{1}{m_1} + \cdots + \frac{1}{m_n} > n - 4.$$

If $\frac{1}{m_1} + \cdots + \frac{1}{m_n} \geqslant n - 2$, then the algebra Λ is tame, hence in this case the assertion follows from the quoted result [7]. Thus we may assume that

$$\frac{1}{m_1} + \cdots + \frac{1}{m_n} < n - 2.$$

In this case the algebra Λ is wild and $\mathbf{d}$ is the dimension vector of a homogeneous module if and only if $\mathbf{d} = p\mathbf{h}$ for some $p \in \mathbb{N}$, where $\mathbf{h}$ is the dimension vector with all the coordinates equal to 1.

We take a closer look into the boundary situation.

Theorem 2. *Let Λ be a canonical algebra of type $(m_1, \ldots, m_n)$ with*

$$\frac{1}{m_1} + \cdots + \frac{1}{m_n} = n - 4,$$

and let m be the least common multiple of $m_1, \ldots, m_n$.

(1) *If m divides p, then $\mathrm{mod}_\Lambda(p\mathbf{h})$ is a complete intersection with exactly two irreducible components.*
(2) *If m does not divide p, then $\mathrm{mod}_\Lambda(p\mathbf{h})$ is a normal complete intersection.*

If Λ is an algebra then for each dimension vector $\mathbf{d}$ a product $\mathrm{GL}(\mathbf{d})$ of general linear groups acts on the variety $\mathrm{mod}_\Lambda(\mathbf{d})$ (see 2.1). This action induces an action on the ring $k[\mathrm{mod}_\Lambda(\mathbf{d})]$ of regular functions on $\mathrm{mod}_\Lambda(\mathbf{d})$ (see 3.1). It is known that for a triangular algebra (no cycles in the Gabriel quiver), hence in particular for canonical algebras, only the constant functions are invariant with respect to this action, however the ring $\mathrm{SI}[\mathrm{mod}_\Lambda(\mathbf{d})]$ of semi-invariants has a richer structure (see for example [18,27,33]). In particular, rings of semi-invariants arising for regular modules over canonical algebras were studied [14,15,32].

In connection with rings of semi-invariants one may also ask, for a dimension vector $\mathbf{d}$ over an algebra Λ, about properties of the set $\mathcal{Z}(\mathbf{d})$ of the common zeros of the semi-invariants of non-zero weight. This line of research (in the context of module varieties) was initiated by Chang and Weyman [11] and continued by Riedtmann and Zwara [24–26]. A motivation for this research is that $\mathcal{Z}(\mathbf{d})$ reflects properties of $k[\mathrm{mod}_\Lambda(\mathbf{d})]$ as a module over $\mathrm{SI}[\mathrm{mod}_\Lambda(\mathbf{d})]$.

The last result of the paper concerns this topic.

Theorem 3. *Let Λ be a canonical algebra of type $(m_1, \ldots, m_n)$. If*

$$\frac{1}{m_1} + \cdots + \frac{1}{m_n} < n - 4,$$

then there exists N such that, for $p \geqslant N$, $\mathcal{Z}(p\mathbf{h})$ is a set theoretic complete intersection with

$$(p-n)m_1 \cdots m_n + \sum_{l \in [1,n-1]} \sum_{i_1 < \cdots < i_l \in [1,n]} m_{i_1} \cdots m_{i_l} + 1$$

irreducible components and $k[\mathrm{mod}_\Lambda(\mathbf{d})]$ is a free $\mathrm{SI}[\mathrm{mod}_\Lambda(\mathbf{d})]$-module.

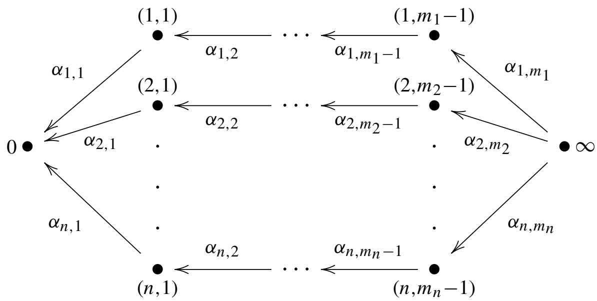
Explicit bounds for N can be found in Propositions 4.1, 4.2, and 4.5.

The paper is organized as follows. In Section 1 we present definition and necessary facts about canonical algebras, and in Section 2 we prove Theorems 1 and 2. Next, in Section 3, we collect useful facts about semi-invariants, which in Section 4 are used in the proof of Theorem 3.

1. Preliminaries on canonical algebras

In this section we present facts about canonical algebras necessary in our proofs.

1.1. Let $\mathbf{m} = (m_1, \ldots, m_n)$, $n \geqslant 3$, be a sequence of integers greater than 1 and let $\boldsymbol{\lambda} = (\lambda_3, \ldots, \lambda_n)$ be a sequence of pairwise distinct non-zero elements of k. We define $\Lambda(\mathbf{m}, \boldsymbol{\lambda})$ as the path algebra of the bound quiver $(\Delta(\mathbf{m}), R(\mathbf{m}, \boldsymbol{\lambda}))$, where $\Delta(\mathbf{m})$ is the quiver



and

$$R(\mathbf{m}, \boldsymbol{\lambda}) = \left\{ \alpha_{1,1} \cdots \alpha_{1,m_1} + \lambda_i \alpha_{2,1} \cdots \alpha_{2,m_2} - \alpha_{i,1} \cdots \alpha_{i,m_i} \;\middle|\; i \in [3, n] \right\}.$$

The algebras of the above form are called canonical. In particular, we call $\Lambda(\mathbf{m}, \boldsymbol{\lambda})$ a canonical algebra of type $\mathbf{m}$. If $\mathbf{m}$ and $\boldsymbol{\lambda}$ are fixed, then we usually write Λ and (Δ, R), instead of $\Lambda(\mathbf{m}, \boldsymbol{\lambda})$ and $(\Delta(\mathbf{m}), R(\mathbf{m}, \boldsymbol{\lambda}))$, respectively. In this case we further denote by Δ_0 the set of vertices of Δ. Until the end of the section we assume that $\Lambda = \Lambda(\mathbf{m}, \boldsymbol{\lambda})$ is a fixed canonical algebra and (Δ, R) is the corresponding bound quiver. The following invariant

$$\delta = \frac{1}{2}\left(n - 2 - \frac{1}{m_1} - \cdots - \frac{1}{m_n}\right)$$

controls the representation type of Λ (see [28]). Namely, Λ is tame if and only if $\delta \leqslant 0$. Moreover, Λ is domestic if and only if $\delta < 0$.

1.2. By a representation of the bound quiver (Δ, R) we mean a collection $M = (M_x, M_{i,j})_{x \in \Delta_0,\ i \in [1,n],\ j \in [1,m_i]}$ of finite dimensional vector spaces M_x, $x \in \Delta_0$, and linear maps $M_{i,j} : M_{(i,j)} \to M_{(i,j-1)}$, $i \in [1, n]$, $j \in [1, m_i]$, such that

$$M_{1,1} \cdots M_{1,m_1} + \lambda_i M_{2,1} \cdots M_{2,m_2} - M_{i,1} \cdots M_{i,m_i} = 0, \quad i \in [3, n],$$

where $M_{(i,0)} = M_0$ and $M_{(i,m_i)} = M_\infty$ for $i \in [1, n]$. The category of representations of (Δ, R) is equivalent to the category of Λ-modules, and we identify Λ-modules and representations of (Δ, R). For a representation M we define its dimension vector $\mathbf{dim}\, M \in \mathbb{N}^{\Delta_0}$ by $(\mathbf{dim}\, M)_x = \dim_k M_x$, $x \in \Delta_0$.

1.3. The Ringel bilinear form $\langle -, - \rangle : \mathbb{Z}^{\Delta_0} \times \mathbb{Z}^{\Delta_0} \to \mathbb{Z}$ is defined by

$$\begin{aligned} \langle \mathbf{d}', \mathbf{d}'' \rangle = d'_0 d''_0 &+ \sum_{i \in [1,n],\ j \in [1,m_i - 1]} d'_{i,j} d''_{i,j} + d'_\infty d''_\infty \\ &- \sum_{i \in [1,n],\ j \in [1,m_i]} d'_{i,j} d''_{i,j-1} + (n-2) d'_\infty d''_0, \end{aligned}$$

where we use the convention that $d_{i,0} = d_0$ and $d_{i,m_i} = d_\infty$ for $\mathbf{d} \in \mathbb{Z}^{\Delta_0}$ and $i \in [1, n]$, and $d_{i,j} = d_{(i,j)}$ for $i \in [1, n]$ and $j \in [1, m_i - 1]$, plays an important role in describing the representation theory of Λ. It is known (see [8, 2.2]), that if M and N are Λ-modules, then

$$\langle \mathbf{dim}\, M, \mathbf{dim}\, N\rangle = [M, N] - [M, N]^1 + [M, N]^2,$$

where, following Bongartz [9], $[M, N] = \dim_k \operatorname{Hom}_\Lambda(M, N)$, $[M, N]^1 = \dim_k \operatorname{Ext}^1_\Lambda(M, N)$, and $[M, N]^2 = \dim_k \operatorname{Ext}^2_\Lambda(M, N)$.

1.4. Let $\mathbf{h}$ be the dimension vector with all the coordinates equal to 1, and

$$\mathbf{e}_{i,0} = \mathbf{h} - (\mathbf{e}_{i,1} + \cdots + \mathbf{e}_{i,m_i-1})$$

for $i \in [1, n]$, where $(\mathbf{e}_x)_{x\in\mathbb{Z}^{\Delta_0}}$ is the standard basis of $\mathbb{Z}^{\Delta_0}$ and $\mathbf{e}_{i,j} = \mathbf{e}_{(i,j)}$ for $i \in [1, n]$ and $j \in [1, m_i - 1]$. One easily checks that

$$\begin{aligned}
\langle \mathbf{d}, \mathbf{h}\rangle &= d_0 - d_\infty = -\langle \mathbf{h}, \mathbf{d}\rangle,\\
\langle \mathbf{d}, \mathbf{e}_{i,j}\rangle &= d_{i,j} - d_{i,j+1}, \quad i \in [1, n],\ j \in [0, m_i - 1],\\
\langle \mathbf{e}_{i,j}, \mathbf{d}\rangle &= d_{i,j} - d_{i,j-1}, \quad i \in [1, n],\ j \in [1, m_i - 1],
\end{aligned}$$

and

$$\langle \mathbf{e}_{i,0}, \mathbf{d}\rangle = d_{i,m_i} - d_{i,m_i-1}, \quad i \in [1, n],$$

for $\mathbf{d} \in \mathbb{Z}^{\Delta_0}$.

1.5. Let $\mathcal{P}$ ($\mathcal{R}$, $\mathcal{Q}$, respectively) be the subcategory of all Λ-modules which are direct sums of indecomposable Λ-modules X such that

$$\langle \mathbf{dim}\, X, \mathbf{h}\rangle > 0 \quad \bigl(\langle \mathbf{dim} X, \mathbf{h}\rangle = 0,\ \langle \mathbf{dim}\, X, \mathbf{h}\rangle < 0,\ \text{respectively}\bigr).$$

We have the following properties of the above decomposition of the category of Λ-modules (see [28, 3.7]).

First, $[N, M] = 0$ and $[M, N]^1 = 0$, if either $N \in \mathcal{R} \vee \mathcal{Q}$ and $M \in \mathcal{P}$, or $N \in \mathcal{Q}$ and $M \in \mathcal{P} \vee \mathcal{R}$. Here, for two subcategories $\mathcal{X}$ and $\mathcal{Y}$ of the category of Λ-modules, we denote by $\mathcal{X} \vee \mathcal{Y}$ the additive closure of their union. Moreover, one knows that $\operatorname{pd}_\Lambda M \leqslant 1$ for $M \in \mathcal{P} \vee \mathcal{R}$ and $\operatorname{id}_\Lambda N \leqslant 1$ for $N \in \mathcal{R} \vee \mathcal{Q}$. Secondly, $\mathcal{R}$ decomposes into a $\mathbb{P}^1(k)$-family $\coprod_{\lambda\in\mathbb{P}^1(k)} \mathcal{R}_\lambda$ of uniserial categories. In particular, $[M, N] = 0$ and $[M, N]^1 = 0$ if $M \in \mathcal{R}_\lambda$ and $N \in \mathcal{R}_\mu$ for $\lambda \neq \mu$. We put $\mathcal{R}' = \coprod_{i\in[1,n]} \mathcal{R}_{\lambda_i}$ and $\mathcal{R}'' = \coprod_{\lambda\in\mathbb{P}^1(k)\setminus\{\lambda_1,\ldots,\lambda_n\}} \mathcal{R}_\lambda$, where $\lambda_1 = 0$ and $\lambda_2 = \infty$. If $\lambda \in \mathbb{P}^1(k) \setminus \{\lambda_1, \ldots, \lambda_n\}$, then there is a unique simple object R_λ in $\mathcal{R}_\lambda$ and its dimension vector is $\mathbf{h}$. On the other hand, if $\lambda = \lambda_i$ for $i \in [1, n]$, then there are m_i simple objects $R_\lambda^{(0)}, \ldots, R_\lambda^{(m_i-1)}$ in $\mathcal{R}_{\lambda_i}$ and their dimension vectors are $\mathbf{e}_{i,j}$, $j \in [0, m_i - 1]$, respectively.

1.6. Let $\mathbf{P}$, $\mathbf{R}$ and $\mathbf{Q}$ denote the sets of the dimension vectors of modules from $\mathcal{P}$, $\mathcal{R}$, $\mathcal{Q}$, respectively. According to [4, 2.6], $\mathbf{d} \in \mathbf{P}$ if and only if either $\mathbf{d} = 0$ or $d_0 > d_\infty \geqslant 0$ and $d_{i,j} \geqslant d_{i,j+1}$ for all $i \in [1,n]$ and $j \in [0, m_i - 1]$. Dually, $\mathbf{d} \in \mathbf{Q}$ if and only if either $\mathbf{d} = 0$ or $0 \leqslant d_0 < d_\infty$ and $d_{i,j} \leqslant d_{i,j+1}$ for all $i \in [1,n]$ and $j \in [0, m_i - 1]$.

For $l_1 \in [0, m_1 - 1], \dots, l_n \in [0, m_n - 1]$ we set

$$\mathbf{e}(l_1, \dots, l_n) = \mathbf{e}_0 + \sum_{i \in [1,n]} \sum_{j \in [1, l_i]} \mathbf{e}_{i,j}.$$

We will need the following fact.

Lemma. *If $\mathbf{d} \in \mathbf{P}$ is such that $\langle \mathbf{d}, \mathbf{h} \rangle = 1$, then*

$$\mathbf{d} = r\mathbf{h} + \mathbf{e}(l_1, \dots, l_n)$$

for some $r \in \mathbb{N}$ and $l_i \in [0, m_i - 1]$, $i \in [1,n]$. In particular, $\langle \mathbf{d}, \mathbf{d} \rangle = 1$.

Proof. The first part follows immediately from the above description of $\mathbf{P}$ and the equality $\langle \mathbf{d}, \mathbf{h} \rangle = d_0 - d_\infty$. The second part follows by direct calculations. □

1.7. The following inequality will be extremely useful in our proofs.

Lemma. *Let $\mathbf{d} \in \mathbb{Z}^{\Delta_0}$. Then*

$$\langle \mathbf{d}, \mathbf{d} \rangle \geqslant -\delta (d_0 - d_\infty)^2.$$

Moreover, the equality holds if and only if

$$d_{i,j} = \frac{1}{m_i} \big((m_i - j) d_0 + j d_\infty \big)$$

for $i \in [1,n]$ and $j \in [1, m_i - 1]$.

Proof. Let $\mathbf{d}' = \mathbf{d} - d_\infty \mathbf{h}$. Then $\langle \mathbf{d}, \mathbf{d} \rangle = \langle \mathbf{d}', \mathbf{d}' \rangle$ and $d_0' = d_0 - d_\infty$, hence the claim follows from the following equality

$$\langle \mathbf{d}', \mathbf{d}' \rangle = -\delta d_0'^2 + \frac{1}{2} \sum_{i \in [1,n]} \sum_{j \in [1, m_i - 1]} \frac{1}{(m_i - j)(m_i - j + 1)} \big((m_i - j + 1) d'_{i,j} - (m_i - j) d'_{i,j-1} \big)^2,$$

which was suggested to me by Professor Riedtmann. □

2. Preliminaries on module varieties and proofs of Theorems 1 and 2

Throughout this section Λ is a fixed canonical algebra of type $\mathbf{m}$ and Δ is its quiver. We first define in 2.1 varieties of modules, then formulate in 2.2 numerical criteria characterizing geometric properties of these varieties, and finally we apply these criteria in 2.3 and 2.4 in order to prove Theorems 1 and 2, respectively.

2.1. For $\mathbf{d} \in \mathbb{N}^{\Delta_0}$ let $\mathbb{A}(\mathbf{d}) = \prod_{i\in[1,n], j\in[1,m_i]} \mathbb{M}(d_{i,j-1}, d_{i,j})$. By $\mathrm{mod}_\Lambda(\mathbf{d})$ we denote the subset of $\mathbb{A}(\mathbf{d})$ formed by all tuples $(M_{i,j})$ such that

$$M_{1,1} \cdots M_{1,m_1} + \lambda_i M_{2,1} \cdots M_{2,m_2} - M_{i,1} \cdots M_{i,m_i} = 0, \quad i \in [3, n].$$

We identify the points M of $\mathrm{mod}_\Lambda(\mathbf{d})$ with Λ-modules of dimension vector $\mathbf{d}$ by taking $M_x = k^{d_x}$ for $x \in \Delta_0$. The product $\mathrm{GL}(\mathbf{d}) = \prod_{x\in\Delta_0} \mathrm{GL}(d_x)$ of general linear groups acts on $\mathrm{mod}_\Lambda(\mathbf{d})$ by conjugation

$$(g \cdot M)_{i,j} = g_{(i,j-1)} M_{i,j} g_{(i,j)}^{-1}, \quad i \in [1, n],\ j \in [1, m_i],$$

for $g \in \mathrm{GL}(\mathbf{d})$ and $M \in \mathrm{mod}_\Lambda(\mathbf{d})$, where $g_{(i,0)} = g_0$ and $g_{(i,m_i)} = g_\infty$ for $i \in [1, n]$. The orbits with respect to this action correspond bijectively to the isomorphism classes of Λ-modules of dimension vector $\mathbf{d}$. For $M \in \mathrm{mod}_\Lambda(\mathbf{d})$ we denote by $\mathcal{O}(M)$ the $\mathrm{GL}(\mathbf{d})$-orbit of M. We put

$$a(\mathbf{d}) = \dim \mathbb{A}(\mathbf{d}) - (n-2)d_0 d_\infty.$$

Note that $a(\mathbf{d}) = \dim \mathrm{GL}(\mathbf{d}) - \langle \mathbf{d}, \mathbf{d}\rangle$.

2.2. For a subcategory $\mathcal{X}$ of the category of Λ-modules and a dimension vector $\mathbf{d}$ we denote by $\mathcal{X}(\mathbf{d})$ the set of all $M \in \mathrm{mod}_\Lambda(\mathbf{d})$ such that $M \in \mathcal{X}$. One knows that if $\mathbf{d} \in \mathbb{N}^{\Delta_0}$ then $\mathcal{P}(\mathbf{d})$ and $(\mathcal{R} \vee \mathcal{Q})(\mathbf{d})$ are open subsets of $\mathrm{mod}_\Lambda(\mathbf{d})$ (see [4, Lemmas 3.7 and 3.8]). Together with the properties of the categories $\mathcal{P}$, $\mathcal{R}$ and $\mathcal{Q}$ listed in 1.5, it implies that we can apply the results of [4, Section 4] with $\mathcal{X} = \mathcal{P}$ and $\mathcal{Y} = \mathcal{R} \vee \mathcal{Q}$.

Observe that if $\mathbf{d} \in \mathbf{P}$ and $p \geqslant d_0$, then $p\mathbf{h} - \mathbf{d} \in \mathbf{Q}$. Moreover,

$$\langle p\mathbf{h} - \mathbf{d}, \mathbf{d}\rangle = -p(d_0 - d_\infty) - \langle \mathbf{d}, \mathbf{d}\rangle.$$

Thus, the following proposition is a consequence of [4, Propositions 4.3, 4.5 and 4.9].

Proposition 2.2.1. *Let $p \geqslant 1$.*

(1) *The variety $\mathrm{mod}_\Lambda(p\mathbf{h})$ is a complete intersection if and only if $\langle \mathbf{d}, \mathbf{d}\rangle \geqslant -p(d_0 - d_\infty)$ for all $\mathbf{d} \in \mathbf{P}$ such that $d_0 \leqslant p$.*
(2) *The variety $\mathrm{mod}_\Lambda(p\mathbf{h})$ is normal if and only if $\langle \mathbf{d}, \mathbf{d}\rangle > -p(d_0 - d_\infty)$ for all $\mathbf{d} \in \mathbf{P}$, $\mathbf{d} \neq 0$, such that $d_0 \leqslant p$.*

We will also need the following consequence of the proof of [4, Proposition 4.5].

Proposition 2.2.2. *Let $p \geqslant 1$ and assume that $\langle \mathbf{d}, \mathbf{d}\rangle \geqslant -p(d_0 - d_\infty)$ for all $\mathbf{d} \in \mathbf{P}$ such that $d_0 \leqslant p$. Then the irreducible components of $\mathrm{mod}_\Lambda(p\mathbf{h})$ are in bijection with the dimensions vectors $\mathbf{d} \in \mathbf{P}$ such that $d_0 \leqslant p$ and $\langle \mathbf{d}, \mathbf{d}\rangle = -p(d_0 - d_\infty)$.*

2.3. We now prove Theorem 1. Recall that it is enough to prove the theorem for $\mathbf{d} = p\mathbf{h}$ with $p \geqslant 1$ (see the discussion after Theorem 1). Assume that

$$\frac{1}{m_1} + \cdots + \frac{1}{m_n} \geqslant n - 4.$$

Then $\delta \leqslant 1$ and according to Lemma 1.7

$$\langle \mathbf{d}, \mathbf{d} \rangle \geqslant -\delta (d_0 - d_\infty)^2 \geqslant -d_0(d_0 - d_\infty) \geqslant -p(d_0 - d_\infty)$$

for each $p \geqslant 1$ and $\mathbf{d} \in \mathbf{P}$ such that $d_0 \leqslant p$. According to Proposition 2.2.1(1), this implies that $\operatorname{mod}_\Lambda(p\mathbf{h})$ is a complete intersection. Analogously, we prove that $\operatorname{mod}_\Lambda(p\mathbf{h})$ is normal if

$$\frac{1}{m_1} + \cdots + \frac{1}{m_n} > n - 4,$$

since in this case $\delta < 1$ and the second inequality in the above string is strict for $\mathbf{d} \neq 0$.

It remains to prove that if

$$\frac{1}{m_1} + \cdots + \frac{1}{m_n} < n - 4 \quad (\leqslant n - 4),$$

then there exists p such that $\operatorname{mod}_\Lambda(p\mathbf{h})$ is not a complete intersection (respectively, normal), or in other words, there exists $\mathbf{d} \in \mathbf{P}$ ($\mathbf{d} \neq 0$) such that

$$\langle \mathbf{d}, \mathbf{d} \rangle < -p(d_0 - d_\infty) \quad \bigl(\leqslant p(d_0 - d_\infty)\bigr).$$

A construction of such p and $\mathbf{d}$ is suggested by Lemma 1.7. Namely, let $p = m_1 \cdots m_n$ and $\mathbf{d}$ be given by the formulas

$$d_0 = p, \qquad d_\infty = 0, \qquad d_{i,j} = \frac{m_i - j}{m_i} p, \quad i \in [1, n],\ j \in [1, m_i - 1].$$

Then $\mathbf{d} \in \mathbf{P}$, $\mathbf{d} \neq 0$, and $\langle \mathbf{d}, \mathbf{d} \rangle = -\delta p(d_0 - d_\infty)$, what finishes the proof.

2.4. We now prove Theorem 2. Assume that

$$\frac{1}{m_1} + \cdots + \frac{1}{m_n} = n - 4$$

and $p > 0$. We already know that $\operatorname{mod}_\Lambda(p\mathbf{h})$ is a complete intersection. Moreover, according to [4, Proposition 4.9] it is normal if and only it is irreducible. Thus our task it to classify the irreducible components of $\operatorname{mod}_\Lambda(p\mathbf{h})$. According to Proposition 2.2.2 this is equivalent to classifying the dimension vectors $\mathbf{d} \in \mathbf{P}$ such that $d_0 \leqslant p$ and $\langle \mathbf{d}, \mathbf{d} \rangle = -p(d_0 - d_\infty)$. Obviously, one such vector is the zero vector. Hence assume that $\mathbf{d} \neq 0$. It follows from Lemma 1.7 that $\langle \mathbf{d}, \mathbf{d} \rangle \geqslant -(d_0 - d_\infty)^2$ (recall that $\delta = 1$ in our case). Thus the condition $\langle \mathbf{d}, \mathbf{d} \rangle = -p(d_0 - d_\infty)$ implies that $d_0 = p$ and $d_\infty = 0$. Moreover, we know again from Lemma 1.7 that $\langle \mathbf{d}, \mathbf{d} \rangle = -(d_0 - d_\infty)^2$ if and only if $d_{i,j} = \frac{m_i - i}{m_i} p$, $i \in [1, n]$, $j \in [1, m_i - 1]$. Note that $\mathbf{d}$ defined by the above formulas belongs to $\mathbf{P}$ if and only if it belongs to $\mathbb{N}^{\Delta_0}$, i.e., if and only if m_i divides p for $i \in [1, n]$. This observation concludes the proof.

3. Preliminaries on semi-invariants

Throughout this section $\Lambda = \Lambda(\mathbf{m}, \boldsymbol{\lambda})$ is a fixed canonical algebra, and $\Delta = \Delta(\mathbf{m})$. Moreover, we put

$$|\mathbf{m}| = m_1 + \cdots + m_n.$$

Our main aim in this section is to prove Proposition 3.9, which reduces the proof of Theorem 3 to a certain inequality. In order to achieve this aim we first recall basic facts about semi-invariants in 3.1 and 3.2. The main result of this first part is Corollary 3.3 giving a new formulation of Theorem 3, which we subsequently improve in 3.4–3.9.

3.1. Let $\mathbf{d} \in \mathbb{N}^{\Delta_0}$. The action of $\mathrm{GL}(\mathbf{d})$ on $\mathrm{mod}_\Lambda(\mathbf{d})$ induces an action of $\mathrm{GL}(\mathbf{d})$ on the coordinate ring $k[\mathrm{mod}_\Lambda(\mathbf{d})]$ of $\mathrm{mod}_\Lambda(\mathbf{d})$ in the usual way, i.e.

$$(g \cdot f)(M) = f\bigl(g^{-1} \cdot M\bigr)$$

for $g \in \mathrm{GL}(\mathbf{d})$, $f \in k[\mathrm{mod}_\Lambda(\mathbf{d})]$, and $M \in \mathrm{mod}_\Lambda(\mathbf{d})$. The product $\mathrm{SL}(\mathbf{d}) = \prod_{x \in \Delta_0} \mathrm{SL}(d_x)$ of special linear groups is a closed subgroup of $\mathrm{GL}(\mathbf{d})$. The ring $\mathrm{SI}[\mathrm{mod}_\Lambda(\mathbf{d})]$ of invariants with respect to the induced action of $\mathrm{SL}(\mathbf{d})$ on $k[\mathrm{mod}_\Lambda(\mathbf{d})]$ is called the ring of semi-invariants.

By a weight we mean a group homomorphism $\sigma : \mathbb{Z}^{\Delta_0} \to \mathbb{Z}$. We identify the weights with the elements of the group $\mathbb{Z}^{\Delta_0}$ in the usual way. If σ is a weight, then we define the weight space

$$\mathrm{SI}\bigl[\mathrm{mod}_\Lambda(\mathbf{d})\bigr]_\sigma = \Bigl\{ f \in k\bigl[\mathrm{mod}_\Lambda(\mathbf{d})\bigr] \Bigm| g \cdot f = \Bigl(\prod_{x \in \Delta_0} \det{}^{\sigma(x)}(g)\Bigr) f \Bigr\}$$

(observe that $\mathrm{SI}[\mathrm{mod}_\Lambda(\mathbf{d})]_\sigma \subset \mathrm{SI}[\mathrm{mod}_\Lambda(\mathbf{d})]$). It is known that

$$\mathrm{SI}\bigl[\mathrm{mod}_\Lambda(\mathbf{d})\bigr] = \bigoplus_{\sigma \in \mathbb{Z}^{\Delta_0}} \mathrm{SI}\bigl[\mathrm{mod}_\Lambda(\mathbf{d})\bigr]_\sigma$$

provided $\mathbf{d}$ is sincere, i.e. $d_x \neq 0$ for $x \in \Delta_0$. Moreover $\mathrm{SI}[\mathrm{mod}_\Lambda(\mathbf{d})]_0 = k$. In this situation the set $\mathcal{Z}(\mathbf{d})$ of common zeros of homogeneous semi-invariants with non-zero weights is called the zero set of semi-invariants.

3.2. We recall now a construction of semi-invariants described in [15] (being a generalization of a construction of Schofield [29]—compare also [12,13]). Let M be a Λ-module of projective dimension at most 1, and let

$$0 \to P_1 \xrightarrow{\varphi} P_0 \to M \to 0$$

be its minimal projective resolution. If $\mathbf{d} \in \mathbb{N}^{\Delta_0}$ satisfies $\langle \mathbf{dim}\, M, \mathbf{d} \rangle = 0$, then the map $d_{\mathbf{d}}^M : \mathrm{mod}_\Lambda(\mathbf{d}) \to k$ given by $d_{\mathbf{d}}^M(N) = \det \mathrm{Hom}_\Lambda(\varphi, N)$ is well defined (up to scalars) and is a homogeneous semi-invariant of weight $-\langle \mathbf{dim}\, M, - \rangle$. Moreover, $d_{\mathbf{d}}^M(N) = 0$ if and only if $[M, N] \neq 0$.

3.3. For $\mathbf{d} \in \mathbb{N}^{\Delta_0}$ let $\mathrm{Reg}_\Lambda(\mathbf{d})$ denote the closure of $\mathcal{R}(\mathbf{d})$. If $\mathcal{R}(\mathbf{d})$ is non-empty, i.e. $\mathbf{d} \in \mathbf{R}$, then $\mathrm{Reg}_\Lambda(\mathbf{d})$ is an irreducible component of $\mathrm{mod}_\Lambda(\mathbf{d})$. The action of $\mathrm{GL}(\mathbf{d})$ on $k[\mathrm{mod}_\Lambda(\mathbf{d})]$ restricts to an action on $k[\mathrm{Reg}_\Lambda(\mathbf{d})]$. In particular, by $\mathrm{SI}[\mathrm{Reg}_\Lambda(\mathbf{d})]$ we denote the ring of $\mathrm{SI}(\mathbf{d})$-invariant regular functions on $\mathrm{Reg}_\Lambda(\mathbf{d})$. The rings $\mathrm{SI}[\mathrm{Reg}_\Lambda(\mathbf{d})]$ for $\mathbf{d} \in \mathbf{R}$ have been studied in [32] (in case of characteristic 0) and in [15] (in case of arbitrary characteristic). We now list their properties which are important for our investigations.

Proposition. *Let $p \geqslant 1$. Then $\mathrm{SI}[\mathrm{Reg}_\Lambda(p\mathbf{h})]$ is generated by $d^M_{p\mathbf{h}}$, $M \in \mathcal{R}$. Moreover, if $p \geqslant n-1$ then $\mathrm{SI}[\mathrm{Reg}_\Lambda(p\mathbf{h})]$ is a polynomial algebra in $|\mathbf{m}| + p + 1 - n$ variables.*

The following consequence of the above proposition will be crucial for us.

Corollary. *Let $p \geqslant 1$. If $\mathrm{mod}_\Lambda(p\mathbf{h})$ is irreducible, then*

$$\mathcal{Z}(p\mathbf{h}) = \bigl\{N \in \mathrm{mod}_\Lambda(p\mathbf{h}) \bigm| [M,N] \neq 0 \textit{ for all } M \in \mathcal{R}\bigr\}.$$

Moreover, if $p \geqslant n-1$ and

$$\dim \mathcal{Z}(p\mathbf{h}) = a(\mathbf{d}) - |\mathbf{m}| - p - 1 + n,$$

then $\mathcal{Z}(p\mathbf{h})$ is a set theoretic complete intersection and $k[\mathrm{mod}_\Lambda(p\mathbf{h})]$ is a free $\mathrm{SI}[\mathrm{mod}_\Lambda(p\mathbf{h})]$-module.

Proof. Recall from [4] that if $\mathrm{mod}_\Lambda(p\mathbf{h})$ is irreducible then $\mathrm{mod}_\Lambda(p\mathbf{h})$ is a complete intersection of dimension $a(\mathbf{d})$, thus the above corollary is a direct consequence of the above proposition and [5, Section 4]. □

We note that always $\mathcal{Z}(p\mathbf{h}) \geqslant a(\mathbf{d}) - |\mathbf{m}| - p - 1 + n$, hence the hard part of the proof is show that $\mathcal{Z}(p\mathbf{h}) \leqslant a(\mathbf{d}) - |\mathbf{m}| - p - 1 + n$.

3.4. We will need the following well-known fact.

Lemma. *If $p \geqslant 1$, then $\mathcal{R}''(p\mathbf{h})$ is an open subset of $\mathcal{R}(p\mathbf{h})$. In particular,* $\dim \mathcal{R}''(p\mathbf{h}) = a(p\mathbf{h})$.

Proof. Let $M \in \mathcal{R}$. Then $M \in \mathcal{R}''$ if and only if $\mathrm{Hom}_\Lambda(R^{(j)}_{\lambda_i}, M) = 0$ for all $i \in [1,n]$ and $j \in [0, m_i - 1]$, which implies the first part of the lemma. The second part is an immediate consequence of the well-known fact that $\mathcal{R}(p\mathbf{h})$ is an irreducible set of dimension $a(p\mathbf{h})$ (see for example remarks after [4, Lemma 3.7]). □

3.5. Fix $\mathbf{d}' \in \mathbf{P}$, $\mathbf{d}'' \in \mathbf{Q}$ and $X \in \mathcal{R}'$ such that $\mathbf{d}' + \mathbf{d}'' + \mathbf{dim}\, X = q\mathbf{h}$ for some $q \geqslant 1$. For each $p \geqslant q$ we consider the set $\mathcal{C}_p(\mathbf{d}', \mathbf{d}'', X)$ consisting of all $M \in \mathrm{mod}_\Lambda(p\mathbf{h})$ which are isomorphic to modules of the form $M' \oplus M'' \oplus X \oplus Y$, where $M' \in \mathcal{P}(\mathbf{d}')$, $M'' \in \mathcal{Q}(\mathbf{d}'')$ and $Y \in \mathcal{R}''((p-q)\mathbf{h})$. We will need the following properties of the set $\mathcal{C}_p(\mathbf{d}', \mathbf{d}'', X)$.

Lemma. *Let $\mathbf{d}'$, $\mathbf{d}''$, X, q and p be as above, and $\mathcal{C} = \mathcal{C}_p(\mathbf{d}', \mathbf{d}'', X)$. Then $\mathcal{C}$ is an irreducible constructible set of dimension*

$$a(p\mathbf{h}) - \bigl((2p-q)\langle \mathbf{d}', \mathbf{h}\rangle + \langle \mathbf{d}', \mathbf{d}'\rangle + \langle \mathbf{d}', \mathbf{dim}\, X\rangle + [X,X]\bigr).$$

Proof. The claim follows from [4, Corollary 3.4]. Indeed

$$\mathcal{C} = \mathcal{P}(\mathbf{d}') \oplus \mathcal{O}(X) \oplus \mathcal{R}''\big((p-q)\mathbf{h}\big) \oplus \mathcal{Q}(\mathbf{d}'')$$

in the notation of [4, 3.4]. Moreover, according to [4, Lemma 3.8]

$$\dim \mathcal{P}(\mathbf{d}') = a(\mathbf{d}') \quad \text{and} \quad \dim \mathcal{Q}(\mathbf{d}'') = a(\mathbf{d}'').$$

Further, by a well-known formula for the dimension of $\mathcal{O}(X)$ (see for example [21, 2.2])

$$\dim \mathcal{O}(X) = \dim \mathrm{GL}(\mathbf{d}) - [X, X] = a(\mathbf{d}) + \langle \mathbf{d}, \mathbf{d} \rangle - [X, X],$$

where $\mathbf{d} = \mathbf{dim}\, X$. In addition, according to Lemma 3.4

$$\dim \mathcal{R}''\big((p-q)\mathbf{h}\big) = a\big((p-q)\mathbf{h}\big).$$

Finally, for any $M' \in \mathcal{P}(\mathbf{d}')$, $M'' \in \mathcal{Q}(\mathbf{d}'')$ and $M \in \mathcal{R}''((p-q)\mathbf{h})$

$$\begin{aligned}
&[M', X] = \langle \mathbf{d}', \mathbf{d} \rangle, && [X, M'] = 0,\\
&[M', M] = \big\langle \mathbf{d}', (p-q)\mathbf{h} \big\rangle, && [M, M'] = 0,\\
&[M', M''] = \langle \mathbf{d}', \mathbf{d}'' \rangle, && [M', M''] = 0,\\
&[X, M] = 0 = \big\langle \mathbf{d}, (p-q)\mathbf{h} \big\rangle, && [M, X] = 0,\\
&[X, M''] = \langle \mathbf{d}, \mathbf{d}'' \rangle, && [M'', X] = 0,\\
&[M, M''] = \big\langle (p-q)\mathbf{h}, \mathbf{d}'' \big\rangle, && [M'', M] = 0,
\end{aligned}$$

hence we are in position to apply [4, Corollary 3.4]. Since $\langle \mathbf{h}, \mathbf{d}' \rangle = -\langle \mathbf{d}', \mathbf{h} \rangle$ and $\langle \mathbf{d}'', \mathbf{h} \rangle = \langle q\mathbf{h} - \mathbf{d} - \mathbf{d}', \mathbf{h} \rangle = -\langle \mathbf{d}', \mathbf{h} \rangle$, the formula follows by direct calculations. $\square$

3.6. Another important property is the following.

Lemma. *Let $\mathbf{d}'$, $\mathbf{d}''$, X, q and p be as above, and $\mathcal{C} = \mathcal{C}_p(\mathbf{d}', \mathbf{d}'', X)$. If $\operatorname{mod}_\Lambda(p\mathbf{h})$ is irreducible, then $\mathcal{C} \cap \mathcal{Z}(p\mathbf{h}) \neq \emptyset$ if and only if $\mathbf{d}' \neq 0$ and for each $i \in [1, n]$ and $j \in [0, m_i - 1]$ either $\langle \mathbf{d}', \mathbf{e}_{i,j} \rangle \neq 0$ or $[X, R_{\lambda_i}^{(j)}] \neq 0$. In particular, $\mathcal{C} \cap \mathcal{Z}(p\mathbf{h}) \neq \emptyset$ if and only if $\mathcal{C} \subset \mathcal{Z}(p\mathbf{h})$.*

Proof. Take $N \in \mathcal{C}$. Recall that, according to Corollary 3.3, under the assumptions of the lemma $N \in \mathcal{Z}(p\mathbf{h})$ if and only if $[N, R] \neq 0$ for all $R \in \mathcal{R}$. This is equivalent to saying that $[N, R_\lambda] \neq 0$ for all $\lambda \in \mathbb{P}^1(k) \setminus \{\lambda_1, \ldots, \lambda_n\}$ and $[N, R_{\lambda_i}^{(j)}] \neq 0$ for all $i \in [1, n]$ and $j \in [0, m_i - 1]$. Write $N \simeq M' \oplus X \oplus M \oplus M''$ for $M' \in \mathcal{P}$, $M \in \mathcal{R}''$ and $M \in \mathcal{Q}$. The former condition is equivalent to $M' \neq 0$ (i.e., $\mathbf{d}' \neq 0$) since $[X \oplus M'', R] = 0$ for all $R \in \mathcal{R}''$ and $[M, R_\lambda] = 0$ for all but a finite number of $\lambda \in \mathbb{P}^1(k) \setminus \{\lambda_1, \ldots, \lambda_n\}$. Similarly, the latter condition leads to the second condition of the lemma. $\square$

3.7. For $p \geqslant 1$ let $\mathfrak{Z}_p$ be the set of all triples $(\mathbf{d}', \mathbf{d}'', [X])$, where $\mathbf{d}' \in \mathbf{P}$, $\mathbf{d}'' \in \mathbf{Q}$ and $X \in \mathcal{R}'$ are such that $\mathbf{d}' + \mathbf{d}'' + \mathbf{dim}\, X = q\mathbf{h}$ for some $q \leqslant p$, $\mathbf{d}' \neq 0$, and for each $i \in [1, n]$ and $j \in [0, m_i - 1]$ either $\langle \mathbf{d}', \mathbf{e}_{i,j} \rangle \neq 0$ or $[X, R_{\lambda_i}^{(j)}] \neq 0$. Here $[X]$ denotes the isomorphism class of X. Observe that $\mathfrak{Z}_p$ is a finite set, hence as a consequence of two preceding lemmas and Corollary 3.3 we get the following.

Proposition. *Let $p \geqslant 1$ and assume that* $\operatorname{mod}_\Lambda(p\mathbf{h})$ *is irreducible. If*

$$(2p - q)\langle \mathbf{d}', \mathbf{h} \rangle + \langle \mathbf{d}', \mathbf{d}' \rangle + \langle \mathbf{d}', \mathbf{dim}\, X \rangle + [X, X] \geqslant |\mathbf{m}| + p + 1 - n \qquad (*)$$

for all $(\mathbf{d}', \mathbf{d}'', [X]) \in \mathfrak{Z}_p$, where $q\mathbf{h} = \mathbf{d}' + \mathbf{d}'' + \mathbf{dim}\, X$, then $\mathcal{Z}(p\mathbf{h})$ is a set theoretic complete intersection and $k[\operatorname{mod}_\Lambda(p\mathbf{h})]$ *is free as a module over* $\operatorname{SI}[\operatorname{mod}_\Lambda(p\mathbf{h})]$. *Moreover, if this is the case then the map*

$$\mathfrak{Z}_p \ni \big(\mathbf{d}', \mathbf{d}'', [X]\big) \mapsto \overline{\mathcal{C}_p(\mathbf{d}', \mathbf{d}'', X)} \subset \mathcal{Z}(p\mathbf{h})$$

induces a bijection between those members of $\mathfrak{Z}_p$ with equality in $(*)$ and the irreducible components of $\mathcal{Z}(p\mathbf{h})$.

Proof. The only missing part is the well-known fact that irreducible components of complete intersections have the same dimension [22, 3.12]. □

3.8. The following inequality will give us a more accessible version of the previous fact.

Lemma. *Let $p \geqslant 1$. If $(\mathbf{d}', \mathbf{d}'', [X]) \in \mathfrak{Z}_p$, then*

$$[X, X] \geqslant |\mathbf{m}| - n\langle \mathbf{d}', \mathbf{h} \rangle.$$

Proof. Since the categories $\mathcal{R}_{\lambda_i}$, $i \in [1, n]$, are uniserial and pairwise orthogonal it follows for indecomposable $R \in \mathcal{R}'$, that if $[R, R_{\lambda_{i_0}}^{(j_0)}] \neq 0$ for some $i_0 \in [1, n]$ and $j_0 \in [0, m_i - 1]$, then $[R, R_{\lambda_i}^{(j)}] = 0$ for all $i \in [1, n]$ and $j \in [0, m_i - 1]$ such that $(i, j) \neq (i_0, j_0)$. For $i \in [1, n]$ let s_i denote the number of the indecomposable direct summands of X which belong to $\mathcal{R}_{\lambda_i}$. Since

$$\langle \mathbf{d}', \mathbf{h} \rangle = \big\langle \mathbf{d}', \mathbf{e}_{i,0} \big\rangle + \cdots + \big\langle \mathbf{d}', \mathbf{e}_{i,m_i-1} \big\rangle,$$

for each $i \in [1, n]$, $\langle \mathbf{d}', \mathbf{e}_{i,j} \rangle \geqslant 0$ for all $i \in [1, n]$ and $j \in [0, m_i - 1]$, it follows from the definition of $\mathfrak{Z}_p$ that $\langle \mathbf{d}', \mathbf{h} \rangle \geqslant m_i - s_i$. Using that $[X, X] \geqslant s_1 + \cdots + s_n$, we obtain our claim. □

3.9. We now reformulate Proposition 3.7.

Proposition. *Let $p \geqslant 1$ and assume that* $\operatorname{mod}_\Lambda(p\mathbf{h})$ *is irreducible. If*

$$(p - q)\langle \mathbf{d}', \mathbf{h} \rangle + (p - n)\big(\langle \mathbf{d}', \mathbf{h} \rangle - 1\big) + \big(\langle \mathbf{d}', \mathbf{d}' \rangle - 1\big) \geqslant 0$$

for all $(\mathbf{d}', \mathbf{d}'', [X]) \in \mathfrak{Z}_p$, where $q\mathbf{h} = \mathbf{d}' + \mathbf{d}'' + \mathbf{dim}\, X$, then $\mathcal{Z}(p\mathbf{h})$ is a set theoretic complete intersection and $k[\operatorname{mod}_\Lambda(p\mathbf{h})]$ *is free as a module over* $\operatorname{SI}[\operatorname{mod}_\Lambda(p\mathbf{h})]$. *Moreover, if this is the* *case and the above inequality is strict for all* $(\mathbf{d}', \mathbf{d}'', [X]) \in \mathfrak{Z}_p$ *such that* $\langle \mathbf{d}', \mathbf{h} \rangle > 1$*, then the irreducible components are indexed by the triples* $(\mathbf{d}', \mathbf{d}'', [X]) \in \mathfrak{Z}_p$ *such that*

$$\begin{aligned} &\langle \mathbf{d}', \mathbf{dim}\, X \rangle = 0, && [X, X] = |\mathbf{m}| - n \langle \mathbf{d}', \mathbf{h} \rangle, \\ &\langle \mathbf{d}', \mathbf{h} \rangle = 1, && \mathbf{d}' + \mathbf{d}'' + \mathbf{dim}\, X = p\mathbf{h}. \end{aligned} \tag{$+$}$$

Proof. The first part follows from Proposition 3.7 and Lemma 3.8 together with the obvious inequality $\langle \mathbf{d}', \mathbf{dim}\, X \rangle \geqslant 0$. The second part is obtained in a similar way: one has to use in addition Lemma 1.6. □

For future reference we introduce the following notation:

$$\operatorname{diff}\bigl(\mathbf{d}', \mathbf{d}'', [X]\bigr) = (p - q) \langle \mathbf{d}', \mathbf{h} \rangle + (p - n)\bigl(\langle \mathbf{d}', \mathbf{h} \rangle - 1\bigr) + \bigl(\langle \mathbf{d}', \mathbf{d}' \rangle - 1\bigr)$$

for $(\mathbf{d}', \mathbf{d}'', [X]) \in \mathfrak{Z}_p$, with $\mathbf{d}' + \mathbf{d}'' + \mathbf{dim}\, X = q\mathbf{h}$.

3.10. We calculate now the number of triples described in the above proposition.

Lemma. *If* $p \geqslant n$ *then the number of triples* $(\mathbf{d}', \mathbf{d}'', [X]) \in \mathfrak{Z}_p$ *satisfying* $(+)$ *is*

$$(p - n) m_1 \cdots m_n + \sum_{l \in [1, n-1]} \sum_{i_1 < \cdots < i_l \in [1, n]} m_{i_1} \cdots m_{i_l} + 1.$$

Proof. It follows from Lemma 1.6 that the condition $\langle \mathbf{d}', \mathbf{h} \rangle = 1$ implies that

$$\mathbf{d}' = r\mathbf{h} + \mathbf{e}_0 + \mathbf{e}(l_1, \ldots, l_n)$$

for some $r \in \mathbb{N}$ and $l_i \in [0, m_i - 1]$, $i \in [1, n]$. Note that if $i \in [1, n]$ and $j \in [0, m_i - 1]$, then $\langle \mathbf{d}', \mathbf{e}_{i,j} \rangle > 0$ if and only if $j = l_i$. Consequently, for each $i \in [1, n]$ and $j \in [0, m_i - 1]$, $j \neq l_i$, there exists an indecomposable direct summand $X_{i,j}$ of X such that $[X_{i,j}, R_{\lambda_i}^{(j)}] \neq 0$. It follows that $X_{i,j} = R_{\lambda_i}^{(j)}$ since otherwise either $[X_{i,l}, X_{i,j}] \neq 0$ for $l \neq j$ and consequently $[X, X] > |\mathbf{m}| - n$, or $\langle \mathbf{d}', \mathbf{dim}\, X_{i,j} \rangle \neq 0$. It is possible to find $\mathbf{d}'' \in \mathbf{Q}$ such that $\mathbf{d}' + \mathbf{d}'' + \mathbf{dim}\, X = p\mathbf{h}$ if and only if $r + |\{i \in [1, n] \mid l_i > 0\}| \leqslant p - 1$, which implies the formula in the lemma. □

4. Proof of Theorem 3

Throughout this section we assume that Λ is a fixed canonical algebra of type $\mathbf{m}$. Our aim in this section is to show how Proposition 3.9 and Lemma 3.10 imply Theorem 3.

4.1. We start with the domestic case.

Proposition. *Let* $\delta < 0$. *If* $p \geqslant n$ *then* $\mathcal{Z}(p\mathbf{h})$ *is a set theoretic complete intersection and* $k[\operatorname{mod}_\Lambda(\mathbf{d})]$ *is a free* $\operatorname{SI}[\operatorname{mod}_\Lambda(\mathbf{d})]$*-module. For* $p > n$ *the number of the irreducible components of* $\mathcal{Z}(p\mathbf{h})$ *is*

$$(p - n) m_1 \cdots m_n + \sum_{l \in [1, n-1]} \sum_{i_1 < \cdots < i_l \in [1, n]} m_{i_1} \cdots m_{i_l} + 1.$$

Proof. The claim follows from Proposition 3.9, Lemma 3.10 and Theorem 1. It is enough to observe that, according to Lemma 1.7, $\langle \mathbf{d}', \mathbf{d}' \rangle \geqslant 1$ for $\mathbf{d}' \in \mathbf{P}$, hence obviously $\operatorname{diff}(\mathbf{d}', \mathbf{d}'', [X]) \geqslant 0$ for all $(\mathbf{d}', \mathbf{d}'', [X]) \in \mathfrak{Z}_p$ if $p \geqslant n$. Moreover, this inequality is strict if $\langle \mathbf{d}', \mathbf{h} \rangle > 1$ and $p > n$. $\square$

4.2. We consider now the tubular case.

Proposition. *Let $\delta = 0$. If $p \geqslant n + 1$ then $\mathcal{Z}(p\mathbf{h})$ is a set theoretic complete intersection and $k[\operatorname{mod}_\Lambda(\mathbf{d})]$ is a free $\mathrm{SI}[\operatorname{mod}_\Lambda(\mathbf{d})]$-module. The number of irreducible components of $\mathcal{Z}(p\mathbf{h})$ is*

$$(p-n)m_1 \cdots m_n + \sum_{l \in [1,n-1]} \sum_{i_1 < \cdots < i_l \in [1,n]} m_{i_1} \cdots m_{i_l} + 1$$

for $p > n + 1$.

Proof. Fix $(\mathbf{d}', \mathbf{d}'', [X]) \in \mathfrak{Z}_p$. Observe that if $\langle \mathbf{d}', \mathbf{h} \rangle = 1$ then $\langle \mathbf{d}', \mathbf{d}' \rangle = 1$, according to Lemma 1.6, and

$$\operatorname{diff}\bigl(\mathbf{d}', \mathbf{d}'', [X]\bigr) = p - q \geqslant 0.$$

On the other hand, if $\langle \mathbf{d}', \mathbf{h} \rangle > 1$ then it follows from Lemma 1.7 that $\langle \mathbf{d}', \mathbf{d}' \rangle \geqslant 0$, hence

$$\operatorname{diff}\bigl(\mathbf{d}', \mathbf{d}'', [X]\bigr) \geqslant (p-n)\bigl(\langle \mathbf{d}', \mathbf{h} \rangle - 1\bigr) - 1 \geqslant 0$$

provided $p \geqslant n + 1$. Moreover, this inequality is strict if $p > n + 1$. Now the claim follows again from Proposition 3.9, Lemma 3.10 and Theorem 1. $\square$

4.3. It remains to consider the case $0 < \delta < 1$. We start with the following observation.

Lemma. *If $0 < \delta < 1$ then $4\delta + n + 1 < \frac{1}{1-\delta}(n+1)$.*

Proof. Recall that $n \geqslant 3$. Consequently,

$$\frac{1}{1-\delta}(n+1) = \frac{n+1}{1-\delta}\delta + (n+1) > 4\delta + n + 1$$

and the claim follows. $\square$

4.4. For a fixed $p \geqslant 1$ consider the real-valued function f given by

$$f(t) = -\delta t^2 + t(p-n) + (n-p-1).$$

Lemma. *Let $p \geqslant 1$ and f be as above. If $p \geqslant \frac{1}{1-\delta}(n+1)$ then $f(t) > 0$ for all $t \in [2, p]$.*

Proof. It is enough to show that $f(2) > 0$ and $f(p) > 0$. The first inequality follows from the previous lemma, since

$$f(2) = p - (4\delta + n + 1).$$

On the other hand,

$$f(p)=p\big((1-\delta)p-(n+1)\big)+(n-1)\geqslant n-1>0,$$

which finishes the proof. □

4.5. Now we can finish our proof.

Proposition. *Let* $0<\delta<1$. *If* $p\geqslant\frac{1}{1-\delta}(n+1)$ *then* $\mathcal{Z}(p\mathbf{h})$ *is a set theoretic complete intersection with*

$$(p-n)m_1\cdots m_n+\sum_{l\in[1,n-1]}\sum_{i_1<\cdots<i_l\in[1,n]}m_{i_1}\cdots m_{i_l}+1$$

irreducible components and $k[\mathrm{mod}_\Lambda(\mathbf{d})]$ *is a free* $\mathrm{SI}[\mathrm{mod}_\Lambda(\mathbf{d})]$*-module.*

Proof. Fix $(\mathbf{d}',\mathbf{d}'',[X])\in\mathfrak{Z}_p$. Observe again that if $\langle\mathbf{d}',\mathbf{h}\rangle=1$ then $\langle\mathbf{d}',\mathbf{d}'\rangle=1$, according to Lemma 1.6, and

$$\mathrm{diff}\big(\mathbf{d}',\mathbf{d}'',[X]\big)=p-q\geqslant 0.$$

On the other hand, if $t=\langle\mathbf{d}',\mathbf{h}\rangle>1$ then it follows from Lemma 1.7 that $\langle\mathbf{d}',\mathbf{d}'\rangle\geqslant-\delta t^2$, hence we obtain from the previous lemma that

$$\mathrm{diff}\big(\mathbf{d}',\mathbf{d}'',[X]\big)\geqslant f(t)>0$$

provided $p\geqslant\frac{1}{1-\delta}(n+1)$. The claim follows again from Proposition 3.9, Lemma 3.10 and Theorem 1. □

Acknowledgments

For basic background on the representation theory of algebras we refer to [1,2]. Basic algebraic geometry used in the article can be found for example in [22]. The author gratefully acknowledges the support from the Polish Scientific Grant KBN No. 1 P03A 018 27 and the Schweizerischer Nationalfonds. The research leading to the results presented in this paper was initiated while the author held a one year post-doc position at the University of Bern.

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