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SEPARATION PROPERTIES IN ALGEBRAIC CATEGORIES OF TOPOLOGICAL SPACES

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Full subcategories $\mathbf{C} \subseteq \mathbf{Top}$ of the category of topological spaces, which are algebraic over **Set** in the sense of Herrlich [2], have pleasant separation properties, mostly subject to additional closedness assumptions. For instance, every **C**-object is a T_1-space, if the two-element discrete space belongs to **C**. Moreover, if **C** is closed under the formation of finite powers in **Top** and even varietal [2], then every **C**-object is Hausdorff. Hence, the T_2-axiom turns out to be (nearly) superfluous in Herrlich's and Strecker's characterization of the category of compact Hausdorff spaces [1], although it is essential for the proof.

If we think of **C**-objects X as universal algebras (with possibly infinite operations), then the subalgebras of X form the closed sets of a compact topology on X, provided that the ordinal spaces $[0, \beta]$ belong to **C**. This generalizes a result in [3]. The subalgebra topology is used to prove criterions for the Hausdorffness of every space in **C**, if **C** is only algebraic.

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0. Introduction

We consider full, isomorphism-closed subcategories $\mathbf{C} \subseteq \mathbf{Top}$ of the category **Top** of topological spaces and continuous maps which are algebraic with respect to the underlying set functor $U: \mathbf{C} \to \mathbf{Set}$ in the sense of Herrlich [2]. Well-known examples for **C** are the full subcategories of all indiscrete, discrete, compact Hausdorff, compact zero-dimensional T_0-spaces, and all powers of a strongly rigid compact Hausdorff space [2]. There are a lot of further algebraic categories of topological spaces [3].

Except the indiscrete spaces, all **C**-objects in our examples have strong separation properties, they are even normal. Our aim is to show that certain separation properties are necessary for **C** to be algebraic over **Set** if we assume a few additional, but natural, more or less restrictive closedness conditions. Especially, we get that the T_2-axiom is nearly superfluous in Herrlich's and Strecker's characterization of the category $\mathbf{Comp}_2$ of compact Hausdorff spaces, although it plays an essential role in

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the proof [1]. The T_2-property is only needed for the two-element space (Theorem 2.3, Corollary 3.4).

We assume henceforth that the two-element discrete space $D_2 = \{0, 1\}$ is contained in **C**.

1. The Riesz condition (T_1)

1.1. Lemma. *Every space X in* **C** *fulfils the T_0-axiom.*

Proof. If every space in **C** is discrete, nothing has to be shown. If not, we have at least one space T in **C** containing a one-element subset $\{t_0\} \subseteq T$ which is not open. Now consider $x, y \in X \in \mathbf{C}$, $x \neq y$, and assume that every neighbourhood of x contains y and vice versa. In this case,

$$f(t) := \begin{cases} x & \text{if } t = t_0, \\ y & \text{if } t \neq t_0, \end{cases}$$

defines a continuous map $f: T \to X$, which has a two-element image in **C**. By assumption, this image has to be discrete, because bijective continuous maps are isomorphisms in **C**. Hence, $f^{-1}(x) = \{t_0\}$ is open in T and we have a contradiction. □

1.2. Proposition. *Every $X \in \mathbf{C}$ is a T_1-space.*

Proof. Take $T \in \mathbf{C}$ from the proof above. We know now that it is a T_0-space. From this we get at least one closed subset $A \subseteq T$, which is not open. Because, if every closed subset in T is open, T_0 implies T_2. Hence, one-element subsets are closed, thus open.

Now consider again $x, y \in X \in \mathbf{C}$, $x \neq y$, and assume that every neighbourhood of x contains y. In this case, we have a continuous map $g: T \to X$ defined by

$$g(t) = \begin{cases} x & \text{if } t \in A, \\ y & \text{if } t \notin A. \end{cases}$$

Again, the two-element image of g in **C** has to be discrete and $g^{-1}(x) = A$ has to be open, but it is not. □

2. The Hausdorff-property (T_2)

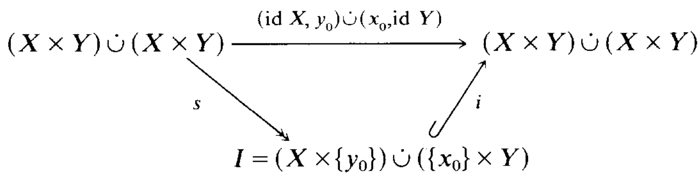
The following observation is basic for our considerations (see [3, 1.2]):

2.1. Proposition. **C** *is closed under the formation of finite coproducts (sums) in* **Top**.

Proof. By assumption, there is a **C**-topology on the cartesian product $X \times \{0, 1\}$ for every $X \in \mathbf{C}$, which is finer than or equal to the usual product topology. But the two injections of X, $x \mapsto (x, 0)$, $x \mapsto (x, 1)$, are still continuous. This means that the **C**-topology on $X \times \{0, 1\}$ is coarser than or equal to the sum-topology on $X \times \{0, 1\} = X \dot\cup X$ which coincides with the product-topology in **Top**. Therefore, we have

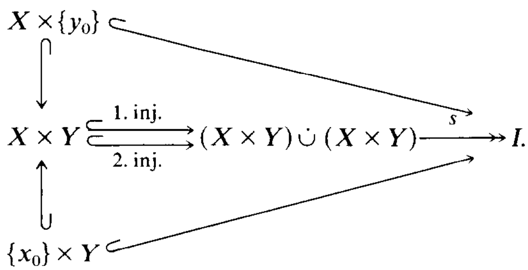
$$X \dot\cup X \in \mathbf{C}.$$

Now let X, $Y \in \mathbf{C}$ be non-empty, $x_0 \in X$, $y_0 \in Y$. By assumption on **C**, there is a **C**-topology on $X \times Y$ such that the natural projections have the usual universal property in **C**. Therefore, the following map is continuous with respect to this topology and it has a unique surjective-injective-factorization in **C**:



(with $(\mathrm{id}\, X, y_0)(x, y) := (x, y_0)$ and $(x_0, \mathrm{id}\, Y)(x, y) := (x_0, y)$).

The coarsest topology on I for which the map i becomes continuous is the coproduct-topology, i.e. the finest topology for which the injections $X \times \{y_0\}$, $\{x_0\} \times Y \hookrightarrow I$ become continuous. But these injections have to be continuous, because they are compositions of continuous maps in **C**:



Therefore, I carries the coproduct-topology, and we have

$$X \dot\cup Y \cong I \in \mathbf{C}. \quad \square$$

2.2. Corollary. $U: \mathbf{C} \to \mathbf{Set}$ *preserves and reflects finite unions.*

Proof. Finite unions in **C** are regular images of finite coproducts. By Proposition 2.1, U preserves and reflects finite coproducts and, by assumption, regular epimorphisms. $\square$

2.3. Theorem. *If* $U: \mathbf{C} \to \mathbf{Set}$ *is even varietal* [2] *and if* **C** *is closed under the formation of finite powers in* **Top**, *then every* $X \in \mathbf{C}$ *is Hausdorff.*

Proof. We have to show that the diagonal in $X \times X$ is always closed. The diagonal is the image of the diagonal map $\Delta_X: X \hookrightarrow X \times X$, which is a **C**-morphism. Therefore we prove that the set-theoretical image of every monomorphism in **C** is a closed subset of its codomain:

Let X, Y be spaces in **C** with $X \subseteq Y$ such that the natural inclusion $i: X \hookrightarrow Y$ is continuous. Up to isomorphisms, all monomorphisms in **C** are of this kind. Now consider the **C**-union of the diagonal $\Delta_Y: Y \hookrightarrow Y \times Y$ and the inclusion $i \times i: X \times X \hookrightarrow Y \times Y$, which is preserved by $U: \mathbf{C} \to \mathbf{Set}$ (Corollary 2.2). In **Set** it is just the equivalence relation R on Y which belongs to the decomposition of Y into X and one-element subsets. The corresponding natural projection $r: Y \to Y/X$ can be 'lifted' along U, because U is varietal. Hence there is a (unique) **C**-topology on Y/X such that r becomes continuous. Using Proposition 1.2 we get that $X = r^{-1}(\{X\})$ is a closed subset of Y. □

2.4. Remarks. (1) Obviously, the inclusion $i: X \hookrightarrow Y$ in the proof above is the **C**-equalizer of r and the constant map $Y \ni y \mapsto X \in Y/X$. Hence, every monomorphism in **C** is regular and, therefore, every epimorphism too, because **C** is (regular-epi, mono)-factorizable. If the inclusion $\mathbf{C} \hookrightarrow \mathbf{Top}$ preserves finite limits, for instance, if **C** is a reflective subcategory, then every monomorphism in **C** is an embedding.

(2) Conversely, if **C** is not necessarily varietal but has the property that every **C**-monomorphism is regular, then every **C**-object X is Hausdorff, provided that the inclusion $\mathbf{C} \hookrightarrow \mathbf{Top}$ preserves finite limits.

To prove this, consider the diagonal $\Delta_X: X \hookrightarrow X \times X$ and the embedding of a single point $\{(x, y)\} \hookrightarrow X \times X$, $x \neq y$. The latter is a **C**-morphism too, because it is the **C**-image of a constant map. The induced map $X \dot{\cup} \{(x, y)\} \hookrightarrow X \times X$ is a **C**-morphism (Proposition 2.1) hence, by assumption, an embedding. Thus there is a neighbourhood of (x, y) in $X \times X$ which is disjoint from the diagonal. It follows that the diagonal is closed in $X \times X$.

Note that the regularity assumption is only needed for the monomorphism $X \dot{\cup} \{(x, y)\} \hookrightarrow X \times X$. It is always regular if every pair of constant **C**-morphisms (with a single point as domain) has a coequalizer which is preserved by $U: \mathbf{C} \to \mathbf{Set}$.

To be varietal is an essential assumption for **C** in the theorem above. It is not clear whether it remains valid if **C** is only algebraic or not, although there is some evidence for a result in this direction:

2.5. Proposition. *Let the ordinal space* $[0, \omega]$ *be contained in* **C**. *Then every* **C**-*subobject of any* **C**-*product of* **C**-*objects which satisfy the first axiom of countability is Hausdorff.*

Proof. Obviously, it is enough to show that any **C**-object X which satisfies the first axiom of countability is Hausdorff.

Let $\{U_i \mid i \in \mathbb{N}\}$, $\{V_i \mid i \in \mathbb{N}\}$ be local bases at $x, y \in X$, $x \neq y$, such that $U_i \supseteq U_{i+1}$ and $V_i \supseteq V_{i+1}$ for all $i \in \mathbb{N}$. Now assume that there is an element $x_i \in U_i \cap V_i$ for every $i \in \mathbb{N}$, with $x_i \neq x, y$ and $x_i \neq x_j$ for $i \neq j$, using the T_1-axiom (Proposition 1.2).

By Proposition 2.1, the sum $S := [0, \omega] \dot{\cup} [0, \omega]$ is contained in **C** and we get a **C**-morphism

$$f: S \to X$$

defined by

$$f(i) = \begin{cases} x_i & \text{for } i \neq \omega \text{ in both summands,} \\ x & \text{for } i = \omega \text{ in the 1st summand,} \\ y & \text{for } i = \omega \text{ in the 2nd summand.} \end{cases}$$

The **C**-image $f(S)$ of f carries a topology which is coarser than or equal to the final topology with respect to f. In any case, we get a bijective continuous map $g: [0, \omega] \to f(S)$, hence an isomorphism, defined as follows:

$$g(i) = \begin{cases} x_{i-1} & \text{for } 0 < i < \omega, \\ x & \text{for } i = \omega, \\ y & \text{for } i = 0. \end{cases}$$

Especially, $\{y\}$ is open in $f(S)$, thus $f^{-1}(y) = \{\omega\}$ has to be open in $[0, \omega]$, which is a contradiction! □

3. The subobject topology

The set-theoretical images of all **C**-monomorphisms $A \hookrightarrow X$, X fixed, are closed under intersections and finite unions (Corollary 2.2). Therefore they can be considered as the closed sets of a topology on X, the *subobject-topology*. Using the same technique as in [3, 1.4] we get:

3.1. Theorem. *Let the ordinal spaces* $[0, \beta]$, β *a limit ordinal, be contained in* **C**. *Then every* $X \in$ **C** *is compact in its subobject-topology.*

Proof. Just as in the proof of [3, 1.4], it can be calculated that every decreasing family $(A_\alpha)_{\alpha \in [0, \beta[}$ of non-empty images of **C**-subobjects of X, has a non-empty intersection because every **C**-morphism remains continuous with respect to the subobject-topology. Moreover, it is immediate that every ordinal space $[0, \alpha]$ and $[\alpha + 1, \beta]$ is contained in **C**, $0 < \alpha < \beta$, and that the U-universal maps $\eta: I \to UFI$ are dense with respect to the subobject-topology. □

3.2. Corollary (see [3, 1.4]). *If* **C** *contains the ordinal spaces* $[0, \beta]$ *for all limit ordinals* β *and is* weakly closed hereditary, *i.e., every closed subset* A *of a* **C**-*object* X *carries a* **C**-*topology such that the inclusion* $A \hookrightarrow X$ *is continuous, then every space in* **C** *is compact.*

Proof. By assumption, the subobject-topology on $X \in \mathbf{C}$ is finer than or equal to the original one. Hence, Theorem 3.1 applies. □

3.3. Remark. As we have seen in the proof of Theorem 2.3, every **C**-subobject has a closed image, if **C** is even varietal over **Set**. Hence, the subobject-topology is coarser than or equal to the original **C**-topology in this case. Both coincide if **C** is weakly closed hereditary.

Combining Theorem 2.3 and Corollary 3.2 we get:

3.4. Corollary (Herrlich–Strecker [1]). *If* **C** *is closed-hereditary and productive in* **Top** *and if* $U: \mathbf{C} \to \mathbf{Set}$ *is varietal then* $\mathbf{C} = \mathbf{Comp}_2$ (*and vice versa*).

Proof. See, for instance, the proof of [3, 1.6]. □

4. Compactness and normality

As we have seen above, every space in **C** is compact Hausdorff, hence normal, if **C** is closed hereditary and productive in **Top**, and if $U: C \to \mathbf{Set}$ is even varietal. The algebraic case is much more difficult, unless we assume a rather restrictive closedness condition:

4.1. Proposition. *Let* **C** *be closed hereditary and productive in* **Top**. *Then the following are equivalent*:

(1) *Every space in* **C** *is* (*compact*) *Hausdorff.*

(2) *Compact* **C***-topologies are maximal compact.*

(3) *Compact refinements of* **C***-topologies are* **C***-topologies.*

Proof. By the general assumption, every closed subspace of the powers D_2^I, hence every compact zero-dimensional T_0-space, especially every ordinal space $[0, \beta]$ is contained in **C**. Thus Corollary 3.2 applies, and every $X \in \mathbf{C}$ is compact. Therefore (1)⇒(2); (2)⇒(3) is trivial.

(3)⇒(1). By Theorem 3.1, every space $(X, \mathscr{X}) \in \mathbf{C}$ is compact in its subobject-topology $\mathscr{S}$. By the general assumption, $\mathscr{S}$ is a refinement of $\mathscr{X}$, which means that $(X, \mathscr{S}) \in \mathbf{C}$, and

$$\mathrm{id}\, X: (X, \mathscr{S}) \to (X, \mathscr{X})$$

is a bijective **C**-morphism, hence an isomorphism.

Consequently, the **C**-topology of every $X \in \mathbf{C}$ coincides with its subobject-topology. Especially, this holds for $X \times X$. But the diagonal in $X \times X$ is closed with respect to the subobject-topology, hence in the product-topology, too. □

4.2. Remark. It is well known [4], that there are (maximal) compact spaces for which every compact subset is closed, although they are not Hausdorff. The full subcategory $\mathbf{K} \subseteq \mathbf{Top}$ of all topological spaces in which compact and closed subsets coincide is closed hereditary and closed under the formation of compact subobjects in **Top**. Every surjective continuous map in **K** is a quotient map, every injective an embedding, every bijective an isomorphism. But **K** is very far from being productive in **Top**, because the diagonal is always compact in $X \times X$ for $X \in \mathbf{K}$, although not closed with respect to the product-topology, in general.

In the following we try to weaken the closedness conditions in Proposition 4.1 above.

4.3. Lemma. *Let $Y \in \mathbf{Top}$ be a T_4-space, $X \in \mathbf{Top}$ a T_1-space, and $s: Y \twoheadrightarrow X$ closed, surjective, and continuous, then X is Hausdorff.*

4.4. Lemma. *Let $\mathbf{C}$ be closed under the formation of limits in* **Top**. *Then* **C** *contains all compact zero-dimensional T_0-spaces, especially the ordinal spaces* $[0, \beta]$.

Proof. Let $A \subseteq D_2^I$ be closed. Then A is compact, and for any point $x \in D_2^I \backslash A$ there is a clopen neighbourhood U of A with $x \notin U$. Therefore, A is the equalizer of the following family:

$$\{f: D_2^I \to D_2 \mid f \text{ continuous and } f(A) = \{0\}\}.$$

Hence, every closed subspace of powers of D_2 belongs to **C**, thus every compact zero-dimensional T_0-space. □

By assumption, $U: \mathbf{C} \to \mathbf{Set}$ has a left adjoint $F: \mathbf{Set} \to \mathbf{C}$. The images FI of F are called the *free* **C**-objects.

4.5. Lemma. *Let* **C** *be contained in* $\mathbf{Comp}_2$. *Then every free* **C**-*object is zero-dimensional.*

Proof. In this case, the embedding $\mathbf{C} \hookrightarrow \mathbf{Comp}_2$ is algebraic in the sense of Herrlich [2], especially, **C** is closed under the formation of limits in $\mathbf{Comp}_2$, hence in **Top**. By Lemma 4.4, **C** contains all compact zero-dimensional T_0-spaces, among them the Stone–Čech compactifications βI of discrete spaces I. But βI is universal with respect to the underlying set functor of $\mathbf{Comp}_2$, hence with respect to its restriction $U: \mathbf{C} \to \mathbf{Set}$. Thus we have $FI = \beta I$. □

4.6. Lemma. *Let A be a closed subset of a compact zero-dimensional T_0-space X. Then the quotient space X/A is compact zero-dimensional and T_0.*

This tells us that the algebraic but not varietal full subcategory of compact zero-dimensional T_0-spaces is closed under the formation of certain quotients in **Top**!

Proof. Let $\hat{x}$, $\hat{y} \in X/A$ be distinct points. Without loss of generality, we may assume that $\hat{x} = \{x\}$ is a single point and $\hat{y}$ at least compact (a single point or equal to A) in X. Therefore, there is a clopen neighbourhood U of $\hat{y}$ with $x \notin U$, hence a continuous map

$$f: X \to D_2$$

with $f(U) = 1$ and $f(x) = 0$. This map induces a continuous map $\hat{f}: X/A \to D_2$ with $\hat{f}(\hat{x}) = 0$ and $\hat{f}(\hat{y}) = 1$. This shows that X/A is a **Top**-subobject of a certain power of D_2. Moreover, it is compact, hence a closed subspace. □

4.7. Theorem. *Let* **C** *be weakly closed hereditary and assume that the inclusion* $\mathbf{C} \hookrightarrow \mathbf{Top}$ *preserves limits. Then the following are equivalent*:

(1) *Every space* $X \in \mathbf{C}$ *is (compact) Hausdorff.*

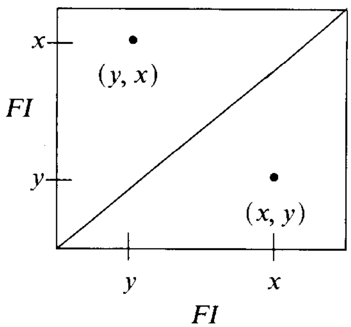
(2) (i) *Every compact Hausdorff refinement of a* **C**-*topology is a* **C**-*topology.*

(ii) **C** *is closed under the formation of quotients* $FI/\{x, y\}$, $\{x, y\} \subseteq FI$, *in* **Top**.

Proof. By assumption and by Corollary 3.2, Lemma 4.3, every space in **C** is compact.

(1)⇒(2). Condition (i) is trivial, and (ii) follows from Lemmas 4.4, 4.5 and 4.6.

(2)⇒(1). By assumption (ii) there is a **C**-topology on $FI/\{x, y\}$ such that the natural projection $FI \to FI/\{x, y\}$ becomes continuous. By our general assumption, its kernel pair R in **Top** belongs to **C**:



$$R = \{(t, t) \mid t \in FI\} \cup \{(x, y), (y, x)\}.$$

Now there is an obvious bijective, continuous map

$$FI \,\dot{\cup}\, D_2 \to R,$$

which has to be an isomorphism (Proposition 2.1). Hence the diagonal is closed in R, thus in $FI \times FI$, because $x, y \in FI$ are arbitrary. Therefore, every free **C**-object is compact Hausdorff, hence normal, and carries its subobject-topology.

Now consider an arbitrary space $X \in \mathbf{C}$ and a continuous surjection $s: FI \to X$. This map remains continuous, if we consider the possibly finer subobject-topology on X. Moreover, it becomes closed. Since the subobject-topology is T_1, Lemma 4.6 applies, and we get that X is compact (Corollary 3.2) Hausdorff in this topology. Using (i), we get that the original **C**-topology of X coincides with the subobject-topology. □

4.8. Remarks. (1) For '(2)⇒(1)' it is enough to assume that the inclusion preserves finite limits and that the ordinal spaces $[0, \beta]$, β a limit ordinal, are contained in **C**.

(2) Condition (ii) in Theorem 4.7 (2) can be replaced by a weaker version:

(ii') For every pair of distinct points x, y in a free object FI there is a **C**-topology on $FI/\{x, y\}$ which is coarser than or equal to the quotient topology (i.e. **C** is *weakly* closed under the formation of such quotients in **Top**).

The proof of Theorem 4.7 simplifies a lot if we replace (ii) by the rather strong assumption

(ii″) Every free **C**-object is Hausdorff.

4.9. Corollary. *Let* **C** *be weakly closed hereditary such that the inclusion* $\mathbf{C} \hookrightarrow \mathbf{Top}$ *preserves finite limits. Moreover, let* $\mathbf{Comp}_2$ *be contained in* **C**. *Then the following are equivalent*:

(1) $\mathbf{C} = \mathbf{Comp}_2$.

(2) **C** *is closed under the formation of quotients* $FI/\{x, y\}$, $\{x, y\} \subseteq FI$, *in* **Top**.

Unfortunately, it is not clear whether there is a proper algebraic extension **C** of $\mathbf{Comp}_2$ in **Top** or not. On the one hand, such an extension cannot be cogenerated by a space which satisfies the first axiom of countability (Proposition 2.5). Moreover, its free objects must be rather strange (Corollary 4.9(2)), and non-regular monomorphisms have to exist (Remark 2.4(2)), provided that the inclusion $\mathbf{C} \hookrightarrow \mathbf{Top}$ preserves finite limits. On the other hand, there are a lot of ridiculous topological spaces [4].

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