Lyapunov Functions for Second-Order Differential Inclusions: A Viability Approach

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In this paper the existence of Lyapunov functions for second-order differential inclusions is analyzed by using the methodology of the Viability Theory. A necessary assumption on the initial states and sufficient conditions for the existence of local and global Lyapunov functions are obtained. An application is also provided.

1. INTRODUCTION

Lyapunov functions play a central role in the qualitative theory of differential equations. In particular, they allow us to deduce many properties of the asymptotic behavior of the solutions of a differential equation.

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In recent years this topic has been studied in the framework of differential inclusions by means of the methods provided by the Viability Theory (see [3]). In this way, it is interesting to note the connection between Lyapunov functions and viscosity solutions of Hamilton–Jacobi equations (see [8] or [9]) or the relation of that kind of function with the viability kernel of some problems (see [3] or [4]).

The aim of this paper is to investigate the existence of Lyapunov functions for second-order differential inclusions by using viability results for higher-order differential inclusions (see [5, 6, 11, 12]). We consider a second-order initial value problem given by a differential inclusion

\[ x''(t) \in F(t, x(t), x'(t)) \]  

with initial conditions

\[ x(0) = x_0, \quad x'(0) = u_0. \]  

We also consider a scalar differential equation

\[ \beta''(t) = -g(t, \beta(t), \beta'(t)), \]  

\[ g: [0, +\infty[ \times \mathbb{R}^2 \to \mathbb{R} \] being a continuous function with linear growth, i.e., satisfying the inequality \(|g(t, \tau)| \leq c(1 + \|	au\|)\) for all \((t, \tau) \in [0, +\infty[ \times \mathbb{R}^2\) \((c > 0)\). Our purpose is to look for functions \(V: X \to \mathbb{R}\), \(X\) being a finite-dimensional vector space such that

\[ V(x(t)) \leq \beta(t) \]  

holds, for at least a solution \(x(\cdot)\) of (1)–(2) and a solution \(\beta(\cdot)\) of (3), satisfying

\[ \beta(0) = V(x_0), \quad \beta'(0) = D_1V(x_0)(u_0), \]  

where \(D_1V\) is the contingent epiderivative of \(V\) (see Subsection 2.2).

The paper is organized as follows. In Section 2 some preliminaries of set-valued and nonsmooth analysis are presented. After that, the second-order epiderivative of a function is introduced and studied. Section 3 is properly devoted to Lyapunov functions for second-order differential inclusions. First a necessary condition is given (Theorem 3.1) and then a sufficient condition for the local existence of Lyapunov functions is obtained (Theorem 3.2). The main result in the paper (Theorem 3.3) provides conditions ensuring the existence of Lyapunov functions in a global sense. In Section 4 an application is considered.
2. PRELIMINARIES

First of all we will recall some notions of set-valued and nonsmooth analysis; for a detailed discussion of these concepts we refer the reader to [2, 3, 7, or 13]. Nevertheless, some contents in Subsection 2.2 appear here for the first time. Throughout the paper $X, Y$ are finite-dimensional vector spaces and $2^X$ denotes the family of all subsets of $X$.

2.1. Set-Valued Analysis

The domain of a set-valued map $F: X \to 2^Y$, denoted by dom $F$, is the set of points $x \in X$ such that $F(x) \neq \emptyset$, and it is said to be nontrivial if dom$(F) \neq \emptyset$. The graph of $F$ is the set $\mathcal{G}(F) = \{(x, y) \in X \times Y : y \in F(x)\}$. A set-valued map $F$ is said to be upper semicontinuous (u.s.c. for short) on $\Omega \subseteq X$ if $F^{-1}(C) = \{x \in X : F(x) \cap C \neq \emptyset\}$ is closed in $\Omega$ for all closed sets $C \subseteq Y$. In the case $F$ is defined from $\mathbb{R} \times X$ to $2^Y$, it is said to be almost u.s.c. on $I \times \Omega$, $I$ being a compact interval, if for every $\varepsilon > 0$ there exists a closed $I_\varepsilon \subseteq I$ with $\mu(I \setminus I_\varepsilon) \leq \varepsilon$, such that $F$ is u.s.c. on $I_\varepsilon \times \Omega$, with $I_\varepsilon \times \Omega \subseteq \text{dom}(F)$; here $\mu$ denotes the Lebesgue measure. If $I$ is not compact, $F$ is called almost u.s.c. on $I \times \Omega$ if it satisfies this property on $J \times \Omega$ for each compact $J \subseteq I$.

Let $\{S_\sigma\}_{\sigma \in \Sigma}$ be a family of sets; the upper limit in the Painlevé–Kuratowski sense is the set defined by

$$\limsup_{\sigma \in \Sigma} S_\sigma = \left\{x \in X : \liminf_{\sigma \in \Sigma} d(x, S_\sigma) = 0\right\},$$

where $d$ denotes the usual distance in $X$. The contingent derivative of a set-valued map $F: X \to 2^Y$ at $(x_0, y_0) \in \mathcal{G}(F)$ is another set-valued map, denoted by $DF(x_0, y_0)$, given by means of its graph as

$$\mathcal{G}(DF(x_0, y_0)) = \limsup_{h \to 0^+} \frac{\mathcal{G}(F) - (x_0, y_0)}{h}.$$ 

Therefore $DF(x_0, y_0)[z] = \{u \in Y : (z, u) \in T_{\mathcal{G}(F)}(x_0, y_0)\}$ for all $z \in X$, $T_{\mathcal{G}(F)}(x_0, y_0)$ being the contingent or Bouligand cone to $\mathcal{G}(F)$ at $(x_0, y_0)$.

Given a nonempty set $C \subseteq X$ and $(x_0, u_0) \in \mathcal{G}(T_C)$, i.e., $u_0 \in T_C(x_0)$, the Ben-Tal second-order tangent set of $C$ at that point is defined by

$$A^{(2)}_C(x_0, u_0) = \limsup_{h \to 0^+} \frac{C - x_0 - h u_0}{h^2/2}$$

and the second-order interior tangent set of $C$ at $(x_0, u_0)$ is the set introduced in [5] as follows: $\omega \in A^{(2)}_C(x_0, u_0)$ if and only if there are $\varepsilon, \eta > 0$ satisfying

$$x_0 + h u_0 + \frac{h^2}{2}(\omega + \varepsilon B_X) \subseteq C, \quad 0 \leq h \leq \eta,$$

$B_X$ being the closed unit ball in $X$. 

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2.2. Contingent Epiderivatives

Let \( \phi : X \to \mathbb{R} \cup \{+\infty\} \) be an extended function. The domain of \( \phi \), \( \text{dom } \phi \), is the set \( \text{dom } \phi = \{ x \in X : \phi(x) \in \mathbb{R} \} \). For each \( x_0 \in \text{dom } \phi \), the contingent epiderivative of \( \phi \) at \( x_0 \) in the direction \( u \in X \) is defined by

\[
D_1 \phi(x_0)(u) = \liminf_{h \to 0^+, \ u' \to u} \frac{\phi(x_0 + h u') - \phi(x_0)}{h}.
\]

It is said that \( \phi \) is contingently epidifferentiable at \( x_0 \) when \( D_1 \phi(x_0)(u) > -\infty \) for any \( u \in X \) or, equivalently, if \( D_1 \phi(x_0)(0) = 0 \) (see Proposition 6.1.3 in [2]). From Proposition 6.1.4 in [2] we have that

\[
\text{epi } D_1 \phi(x_0) = T_{\text{epi } \phi}(x_0, \phi(x_0)), \quad \forall \ x_0 \in \text{dom } \phi,
\]

where epi refers to the epigraph (epi \( f = \{(x, \lambda) : f(x) \leq \lambda \} \)). Thus \( D_1 \phi(x_0) \) is lower semicontinuous (l.s.c. for short) and positively homogeneous whenever \( \phi \) is contingently epidifferentiable at \( x_0 \).

When \( \liminf \) can be replaced by \( \lim \), following the terminology in [13], the value of the limit

\[
D_1 \phi(x_0)(u) = \lim_{h \to 0^+, \ u' \to u} \frac{\phi(x_0 + h u') - \phi(x_0)}{h} \in [-\infty, +\infty]
\]

is called the semiderivative of \( \phi \) at \( x_0 \) in the direction \( u \). The function \( \phi \) is said to be semidifferentiable at \( x_0 \) for \( u \in X \) if that limit exists in \( \mathbb{R} \) and semidifferentiable at \( x_0 \) if this holds for every \( u \in X \).

**Lemma 2.1.** Let \( \phi : X \to \mathbb{R} \cup \{+\infty\} \) be an extended function. If the contingent epiderivative of \( \phi \) at \( x_0 \in \text{dom } \phi \) in the direction \( u_0 \in X \) is a real number, then \( u_0 \in T_{\text{dom } \phi}(x_0) \). Furthermore, if \( \phi \) is semidifferentiable at \( x_0 \) for \( u_0 \), then

\[
A^{(2)}_{\text{dom } \phi}(x_0, u_0) = X.
\]

**Proof.** The first statement of the lemma follows from the very definition of the Bouligand tangent cone. To prove the second one, let us take \( \omega \in X \). Since \( \phi \) is semidifferentiable at \( x_0 \) for \( u_0 \), the limit

\[
\lim_{h \to 0^+, \ \omega' \to \omega} \frac{\phi(x_0 + h u_0 + h^2/2 \omega') - \phi(x_0)}{h}
\]

is equal to \( D_1 \phi(x_0)(u_0) \in \mathbb{R} \). Hence \( x_0 + h u_0 + (h^2/2) \omega' \) must be in \( \text{dom } \phi \) for \( h \) small enough and \( \omega' \) close enough to \( \omega \), which implies \( \omega \in A^{(2)}_{\text{dom } \phi}(x_0, u_0) \).

**Lemma 2.2.** If \( \phi : X \to \mathbb{R} \cup \{+\infty\} \) is semidifferentiable at \( x_0 \in \text{dom } \phi \), then it is continuous at that point.
Let \( x_n \) be a sequence in \( \text{dom} \phi \) such that \( x_n \to x_0 \). Let us take the sequences \( h_n = \|x_n - x_0\| \) and \( u_n = h_n^{-1}(x_n - x_0) \). Obviously, we can assume that \( h_n \to 0^+ \) and \( u_n \to u \), which allow us to write

\[
\frac{\phi(x_n) - \phi(x_0)}{h_n} = \frac{\phi(x_0 + h_n u_n) - \phi(x_0)}{h_n} \to D_1 \phi(x_0)(u).
\]

Hence

\[
|\phi(x_n) - \phi(x_0)| = \left( \frac{\phi(x_n) - \phi(x_0)}{h_n} \right) h_n \to 0,
\]

and the proof is done.

We shall now define a second-order epiderivative, which is slightly different from the second-order contingent epiderivative introduced in [2]. Such an epiderivative arises in a natural way when one studies the existence of Lyapunov functions for second-order differential inclusions, as we will show in the next section.

**Definition 2.1.** Let \( \phi : X \to \mathbb{R} \cup \{+\infty\} \). Given \( x_0 \in \text{dom} \phi \) and \( u_0 \in \text{dom} D_1 \phi(x_0) \), the second-order epiderivative of \( \phi \) at \((x_0, u_0)\) in the direction \( \omega \in X \) is defined by

\[
D^{(2)}_1 \phi(x_0, u_0)(\omega) = \liminf_{h \to 0^+, \omega' \to \omega} \frac{\phi(x_0 + hu_0 + \frac{h^2}{2} \omega') - \phi(x_0) - h D_1 \phi(x_0)(u_0)}{h^2/2}.
\]

From the very definition of \( D^{(2)}_1 \phi(x_0, u_0) \), its epigraph coincides with the Ben-Tal second-order tangent set of \( \text{epi} \phi \) at \((x_0, \phi(x_0))\), \((u_0, D_1 \phi(x_0)(u_0))\); i.e.,

\[
\text{epi} D^{(2)}_1 \phi(x_0, u_0) = A^{(2)}_{\text{epi} \phi}((x_0, \phi(x_0)), (u_0, D_1 \phi(x_0)(u_0))). \tag{7}
\]

At the remaining points in the graph of \( T_{\text{epi} \phi} \), the Ben-Tal tangent set of \( \text{epi} \phi \) becomes trivial, i.e., equal to the whole space, if \( \phi \) is assumed to be u.s.c. and semidifferentiable at \( x_0 \) in the direction \( u_0 \). To show it, let \(((x_0, \lambda), (u_0, \beta))\) be in \( \partial(T_{\text{epi} \phi}) \):

- If \( \phi(x_0) < \lambda \), then by using a typical upper semicontinuity argument (see for instance the proof of Proposition 6.1.4 in [2]) and Lemma 2.1, we obtain that \( A^{(2)}_{\text{epi} \phi}((x_0, \lambda), (u_0, \beta)) \) is equal to \( X \times \mathbb{R} \).

- If \( \phi(x_0) = \lambda \) and \( D_1 \phi(x_0)(u_0) < \beta \), then by the semidifferentiability assumed on \( \phi \), we have that

\[
\lim_{h \to 0^+} \frac{\phi(x_0 + hu_0 + \frac{h^2}{2} \omega) - \phi(x_0)}{h} = D_1 \phi(x_0)(u_0) < \beta
\]
for all $\omega \in X$. Hence for any $\eta \in \mathbb{R}$ and $h > 0$ small enough

$$\phi(x_0 + h u_0 + \frac{h^2}{2} \omega) \leq \phi(x_0) + h \beta + \frac{h^2}{2} \eta,$$

which implies $(\omega, \eta) \in A^{(2)}_{\text{epi} \phi}((x_0, \phi(x_0)), (u_0, \beta))$.

Similar relationships are satisfied by second-order interior tangent sets. Indeed, if $(\omega, \eta)$ belongs to $A^{(2)}_{\text{epi} \phi}((x_0, \phi(x_0)), (u_0, D_x \phi(x_0)(u_0)))$ then there is $\varepsilon > 0$ with

$$\frac{\phi(x_0 + h u_0 + \frac{h^2}{2} (\omega + \varepsilon v)) - \phi(x_0) - h D_x \phi(x_0)(u_0)}{h^2/2} \leq \eta + t$$

for all $v \in B_X$ and all $|t| \leq \varepsilon$. Hence, taking $\lim \inf$ on the left-hand side, the strict inequality $D^{(2)}_x \phi(x_0, u_0)(\omega) < \eta$ is obtained. However, to hit the converse we have to assume a stronger condition on $\phi$ at $(x_0, u_0)$.

**Definition 2.2.** Let $\phi: X \to \mathbb{R} \cup \{+\infty\}$. Given $x_0 \in \text{dom} \phi$ and $u_0 \in \text{dom} D_x \phi(x_0)$, the second-order semiderivative of $\phi$ at $(x_0, u_0)$ in the direction of $\omega \in X$ is the value of the limit (when it exists in $[-\infty, +\infty]$)

$$D^{(2)}_x \phi(x_0, u_0)(\omega) = \lim_{\gamma \to 0^+, \omega \to \omega} \frac{\phi(x_0 + h u_0 + \frac{h^2}{2} \omega') - \phi(x_0) - h D_x \phi(x_0)(u_0)}{h^2/2}.$$

$\phi$ is said to be **twice semidifferentiable at** $(x_0, u_0)$ **in the direction** $\omega$ **if** the preceding limit exists in $\mathbb{R}$. Finally, $\phi$ is called **twice semidifferentiable at** $(x_0, u_0)$ **if** it is twice differentiable in every direction $\omega \in X$.

**Remark 2.1.** In [13] a function $\phi$ is called twice semidifferentiable at $x_0 \in \text{dom} \phi$ for $\omega$ if it is semidifferentiable and the limit

$$d^2 \phi(x_0)(\omega) = \lim_{h \to 0^+, \omega \to \omega} \frac{\phi(x_0 + h \omega') - \phi(x_0) - h D_x \phi(x_0)(\omega')}{h^2/2}$$

exists in the extended real line $[-\infty, +\infty]$. Note that this definition is different from the previous one. So if $\phi$ is $C^2$ on a neighbourhood of $x_0$, then $d^2 \phi(x_0)(\omega) = 2\nabla^2 \phi(x_0)(\omega, \omega)$ and

$$D^{(2)}_x \phi(x_0, u_0)(\omega) = d^2 \phi(x_0)(u_0) + D_x \phi(x_0)(\omega).$$

This equality is not in general true. For instance, if we consider the step function

$$f(x) = \begin{cases} 0 & \text{if } x < 0 \\ 1 & \text{if } x \geq 0 \end{cases}$$

it follows that $f$ is neither contingently epidifferentiable ($D_x f(0)(\omega) = -\infty$, for any $\omega \leq 0$) nor semidifferentiable ($\lim \sup_{h \to 0^+, u \to 0}(f(hu) - 1)/h = 0$).
at zero. Nevertheless, \( f \) is twice semidifferentiable at \((0, 1)\) in the sense of Definition 2.2, because
\[
D_{(2)} f(0, 1)(\omega) = \lim_{h \to 0^+} \frac{f(h + \frac{h^2}{2} \omega') - f(0) - hD_1 f(0)(1)}{h^2/2} = 0
\]
for each \( \omega \in \mathbb{R} \).

Coming back to the computation of the second-order interior tangent set to \( \text{epi} \phi \), if \( \phi \) is twice semidifferentiable at \((x_0, u_0)\) it easily follows the desired equality
\[
A(\text{epi} \phi)((x_0, \phi(x_0)), (u_0, D_1 \phi(x_0)(u_0))) = \{ (\omega, \eta) : D_{(2)} \phi(x_0, u_0)(\omega) < \eta \}.
\]
Otherwise, i.e., if \( \phi(x_0) < \lambda \) or \( D_1 \phi(x_0)(u_0) < \beta \), under the same assumptions as before (\( \phi \) u.s.c. and semidifferentiable at \( x_0 \) for \( u_0 \)), then \( A(\text{epi} \phi)((x_0, \lambda), (u_0, \beta)) \) coincides with the whole space \( X \times \mathbb{R} \).

We close this section by showing the relationship between the semidifferentiability of the first and second orders.

**Lemma 2.3.** If \( \phi : X \to \mathbb{R} \cup \{ +\infty \} \) is twice semidifferentiable at \((x_0, u_0)\), then it is semidifferentiable at \( x_0 \) for \( u_0 \).

**Proof.** Let \( h_n \) and \( u_n \) be sequences such that \( h_n \to 0^+ \) and \( u_n \to u_0 \), respectively. We can suppose, without loss of generality, that the sequence \( h_n^{-1}\|u_n - u_0\| \) is convergent. Hence,
\[
\left| \frac{\phi(x_0 + h_n u_n) - \phi(x_0)}{h_n} - D_1 \phi(x_0)(u_0) \right| = \left| \frac{\phi(x_0 + h_n u_0 + \frac{h_n^2}{2} \omega_n) - \phi(x_0) - h_n D_1 \phi(x_0)(u_0)}{h_n^3/2} \right| \to 0,
\]
where \( \omega_n = 2h_n^{-1}(u_n - u_0) \) converges to \( \omega \in X \) as \( n \to +\infty \), by the assumptions made on \( h_n \) and \( u_n \). Finally, since \( \phi \) is twice semidifferentiable at \((x_0, u_0)\), by letting \( n \to +\infty \) in (9) we complete the proof.

**Lemma 2.4.** If \( \phi : X \to \mathbb{R} \cup \{ +\infty \} \) is contingently epidifferentiable at \( x_0 \in \text{dom} \phi \) and twice semidifferentiable at \((x_0, 0)\), then it is semidifferentiable at \( x_0 \). Moreover,
\[
D_1 \phi(x_0)(u) = \frac{D_{(2)} \phi(x_0, 0)(2u)}{2}
\]
for any \( u \in X \).

**Proof.** Let \( u \) be in \( X \). Clearly,
\[
\frac{\phi(x_0 + h u') - \phi(x_0)}{h} = \left( \frac{\phi(x_0 + h u' + \frac{h^2}{2} 2u') - \phi(x_0) - hD_1 \phi(x_0)(0)}{h/2} \right) \frac{1}{2}.
\]
Hence letting \( h \to 0^+ \) and \( u' \to u \), since \( D_1 \phi(x_0)(0) = 0 \), we have that \( \phi \) is semidifferentiable at \( x_0 \) for \( u \) and (10) holds.
3. LYAPUNOV FUNCTIONS

In this section we will assume that \( F : [0, +\infty] \times X^2 \rightarrow 2^X \) is a strict (i.e., \( \text{dom}(F) = [0, +\infty[ \times X^2 \)) u.s.c. set-valued map having closed convex values and linear growth; i.e.,

\[
F(t, y) \subseteq \alpha(t)(1 + \| y \|)B_X,
\]

for all \( (t, y) \in [0, +\infty[ \times X^2 \), with \( \alpha \in L^1_{\text{loc}}(0, +\infty) \).

A map \( V : X \rightarrow \mathbb{R} \) is said to be a Lyapunov function for (1)–(2) associated with (3) if and only if there exist \( x(\cdot), \) a solution of (1)–(2), and \( \beta(\cdot), \) a solution of (3) satisfying (5), such that \( (x(\cdot), \beta(\cdot)) \) is viable in \( \text{epi} V \); i.e., \( (x(t), \beta(t)) \in \text{epi} V \). Hence the compatibility conditions on \( x_0, u_0 \),

\[
x_0 \in \text{dom} V, \quad u_0 \in \text{dom} D_x V(x_0),
\]

are needed and they will be assumed throughout the remainder of this paper. The first theorem that we present is the next one, where a necessary condition on the initial states is obtained. Its proof follows easily from Theorem 2.3 in [5] and (7).

**Theorem 3.1.** Let \( V : X \rightarrow \mathbb{R} \). If \( V \) is a Lyapunov function for (1)–(2) associated with (3), then

\[
D^{(2)}_t V(x_0, u_0)(y) \leq -g(0, V(x_0), D_x V(x_0)(u_0))
\]

for some \( y \in F(0, x_0, u_0) \).

**Remark 3.1.** Note that \( y \mapsto D^{(2)}_t V(x_0, u_0)(y) \) is l.s.c., because its epigraph is closed. Therefore if the compact set \( F(0, x_0, u_0) \) is contained in the domain of \( D^{(2)}_t V(x_0, u_0) \), then the infimum of this function is attained and (13) is the same as

\[
\inf_{y \in F(0, x_0, u_0)} D^{(2)}_t V(x_0, u_0)(y) \leq -g(0, V(x_0), D_x V(x_0)(u_0)).
\]

**Remark 3.2.** The statement of Theorem 2.3 in [5] is not true when \( F \) is an almost u.s.c. set-valued map (see Example 4.1 in [12]), so (13) cannot be verified under that assumption. However, if \( \alpha \) is assumed to be continuous at zero, that assertion can be obtained (see Theorem 4.2 in [12]).
3.1. Local Results

Condition (13) is not, however, sufficient to ensure that $V$ will be a Lyapunov function. A first result of this kind is the following.

**Theorem 3.2.** Let $V: X \to \mathbb{R}$ u.s.c. at $x_0$ and twice semidifferentiable at $(x_0, u_0)$. If

$$D^{(2)}_t V(x_0, u_0)(y) < -g(0, V(x_0), D_t V(x_0)(u_0))$$

for all $y \in F(0, x_0, u_0)$, then there exist a solution $x(\cdot)$ of (1)–(2) and a solution $\beta(\cdot)$ of (3) satisfying (5) and $T > 0$ such that $V(x(t)) \leq \beta(t)$, for all $t \in [0, T]$.

**Proof.** From Theorem 4.1 in [5], a sufficient condition ensuring the existence of a solution of (1)–(3), (5) which is locally viable in $\text{epi} V$ is

$$F(0, x_0, u_0) \times \{-g(0, V(x_0), D_t V(x_0)(u_0))\}$$

$$\subset AI^{(2)}_{\text{epi} V}((x_0, V(x_0)), u_0, D_t V(x_0)(u_0)).$$

On the other hand, since we are under the required hypotheses (see Lemma 2.3) we have the representation of the second-order interior tangent set to $\text{epi} V$ previously obtained (see p. 344), and the above inclusion is equivalent to (15).

**Example 3.1.** In the particular case where $V$ is assumed to be $C^2$, the equality

$$D^{(2)}_t V(x_0, u_0)(y) = (\nabla V(x_0), y) + 2\nabla^2 V(x_0)(u_0, u_0)$$

holds, with $\langle \cdot, \cdot \rangle$ being the usual inner product in $X$. So (14) can be rewritten as

$$\sigma_{F(0, x_0, u_0)}(-\nabla V(x_0)) \geq 2\nabla^2 V(x_0)(u_0, u_0) + g(0, V(x_0), D_t V(x_0)(u_0)),$$

where $\sigma_{F(0, x_0, u_0)}(-\nabla V(x_0)) = \sup_{y \in F(0, x_0, u_0)}(-\nabla V(x_0), y)$ is the support function of the set $F(0, x_0, u_0)$ and (15) is the same as

$$-\sigma_{F(0, x_0, u_0)}(\nabla V(x_0)) > 2\nabla^2 V(x_0)(u_0, u_0)$$

$$+ g(0, V(x_0), D_t V(x_0)(u_0)).$$

Theorem 3.2 provides a characterization of the local Lyapunov functions for (1)–(3). One way to extend the solutions of these problems satisfying (4), when $V$ is twice semidifferentiable on $X^2$, consists in assuming the condition

$$D^{(2)}_t V(x, u)(y) < -g(t, V(x), D_t V(x)(u)),$$

(17)
for all \( y \in F(t, x, u) \) and all \( (t, x, u) \in [0, +\infty) \times X^2 \). Then we can use Theorem 3.5 in [12] to get nonextending functions \( x(\cdot) \), a solution of (1)–(2), and \( \beta(\cdot) \), a solution of (3) satisfying (4) on \( [0, \delta[ \) with either \( \delta = +\infty \) or

\[
((x(\delta), \beta(\delta)), (x'(\delta), \beta'(\delta))) \not\in \overline{\mathcal{G}(epi V)}.
\]

Note that this is not a good way to extend the solutions satisfying (4) to the full interval \([0, +\infty[\), because as is well-known \( \overline{\mathcal{G}(epi V)} \) is not, in general, a locally compact set and \( V(x(t)) \leq \beta(t), \ 0 \leq t < \delta \), implies

\[
((x(t), \beta(t)), (x'(t), \beta'(t))) \in \mathcal{G}(epi V).
\]

Therefore \( ((x(\delta), \beta(\delta)), (x'(\delta), \beta'(\delta))) \in \overline{\mathcal{G}(epi V)} \), but if this point does not belong to the graph of \( T_{epi V} \) then one cannot ensure that \( x(\cdot) \) and \( \beta(\cdot) \) could be extended while satisfying (4), as the next example shows.

**Example 3.2.** Let us consider the u.s.c. set-valued map

\[
F(x) = \begin{cases} 
  x & \text{if } x > 0 \\
  [0, 1] & \text{if } x = 0 \\
  1 - x & \text{if } x < 0.
\end{cases}
\]

Let us also consider the maps \( V(x) = |x| \) and \( g(x) = -(x + 1 + r) \), where \( r > 0 \). It is easy to check the equality

\[
D^2_{f}V(x, u)(y) = \begin{cases} 
  y & \text{if } x > 0 \text{ or } x = 0, u > 0 \\
  |y| & \text{if } x = 0, u = 0 \\
  -y & \text{otherwise}.
\end{cases}
\]

Hence \( D^2_{f}V(x, u)(y) < -g(x) \) for any \( (x, u) \in \mathbb{R}^2 \) and any \( y \in F(x) \), and (17) is satisfied. If we take \( x_0 > 0 \) and \( u_0 \) such that \( x_0 + u_0 + 1 + r < 0 \), then the unique solution of

\[
\beta''(t) = \beta(t) + (1 + r) \quad \beta(0) = x_0, \quad \beta'(0) = u_0
\]

is \( \beta(t) = \frac{1}{2}((x_0 + u_0 + 1 + r)e^t + (x_0 - u_0 + 1 + r)e^{-t}) - (1 + r) \). Obviously, \( \beta(t) \to -\infty \) as \( t \to +\infty \), and therefore the inequality \( |x(t)| \leq \beta(t) \) is not satisfied for any solution \( x(\cdot) \) of

\[
x''(t) \in F(x(t)) \quad x(0) = x_0, \quad x'(0) = u_0
\]

on the full interval \([0, +\infty[\).
3.2. Global Results

To overcome that difficulty we consider a closed set contained in \( \mathcal{G}(T_{\text{epi}} V) \). This set is given by the graph of a set-valued map defined as

\[
R(x, \lambda) = \begin{cases} 
\text{epi} D_1 V(x) \cap (H(x) \times \mathbb{R}) & \text{if } x \in B \\
\emptyset & \text{otherwise}
\end{cases}
\]  

for any \((x, \lambda) \in \text{epi} V\), where \( B \subset X \) is a nonempty closed set and \( H: B \to 2^X \) is a strict set-valued map with closed graph satisfying the compatibility condition

\[
\mathcal{G}(H) \subseteq \mathcal{G}(T_B).
\]   

This map has nonempty values on \( \text{epi} V \cap (B \times \mathbb{R}) \) if \( D_1 V(x)(u) < +\infty \) is assumed for every \((x, u) \in \mathcal{G}(H)\), and its graph will be contained in \( \mathcal{G}(T_{\text{epi}} V) \) by assuming that \( V \) is u.s.c. Furthermore, since

\[
\mathcal{G}(R) = \{(x, \lambda), (u, \mu) : (x, u) \in \mathcal{G}(H), V(x) \leq \lambda, D_1 V(x)(u) \leq \mu\},
\]

this graph is closed whenever \( V \) and \( \Gamma(\cdot, \cdot) = D_1 V(\cdot)(\cdot) \) are l.s.c. maps.

**Proposition 3.1.** Let \( V: \Omega \subseteq X \to \mathbb{R} \) be a semidifferentiable function (\( B \subseteq \Omega \) open) such that \( \Gamma(\cdot, \cdot) = D_1 V(\cdot)(\cdot) \) is also semidifferentiable on \( \Lambda \) (\( \mathcal{G}(H) \subseteq \Lambda \) open). Given \((x, \lambda), (u, \mu) \in \mathcal{G}(R)\) and

\[
((z, \eta), (\omega, r)) \in T_{\mathcal{G}(R)}((x, \lambda), (u, \mu)),
\]

the following statements are satisfied:

(i) If \( V(x) = \lambda \) and \( D_1 V(x)(u) = \mu \), then (22) holds if and only if

\[
(z, \omega) \in T_{\mathcal{G}(H)}(x, u), \quad D_1 V(x)(z) \leq \eta, \quad D_1 \Gamma(x, u)(z, \omega) \leq r.
\]

(ii) If \( V(x) < \lambda \) and \( D_1 V(x)(u) = \mu \), then (22) holds if and only if

\[
(z, \omega) \in T_{\mathcal{G}(H)}(x, u), \quad D_1 \Gamma(x, u)(z, \omega) \leq r.
\]

(iii) If \( V(x) = \lambda \) and \( D_1 V(x)(u) < \mu \), then (22) holds if and only if

\[
(z, \omega) \in T_{\mathcal{G}(H)}(x, u), \quad D_1 V(x)(z) \leq \eta.
\]

(iv) If \( V(x) < \lambda \) and \( D_1 V(x)(u) < \mu \), then (22) holds if and only if

\[
(z, \omega) \in T_{\mathcal{G}(H)}(x, u).
\]
Proof. Let \(((z, \eta), (\omega, r)) \in T_{\gamma(R)}((x, \lambda), (u, \mu))\). From the sequential characterization of Bouligand cones there are sequences \(h_n \to 0^+, z_n \to z\), \(\eta_n \to \eta\), \(\omega_n \to \omega\), and \(r_n \to r\) satisfying
\[
((x, \lambda), (u, \mu)) + h_n ((z_n, \eta_n), (\omega_n, r_n)) \in \mathcal{G}(R).
\]
Hence, \(x + h_n z_n \in B\) and \(u + h_n \omega_n \in H(x + h_n z_n)\), which implies \((z, \omega) \in T_{\gamma(H)}(x, u)\). Furthermore,
\[
V(x + h_n z_n) \leq \lambda + h_n \eta_n
\]
and
\[
D_t V(x + h_n z_n)(u + h_n \omega_n) \leq \mu + h_n r_n
\]
hold. We shall distinguish four cases:

- If \(V(x) < \lambda\) and \(D_t V(x)(u) < \mu\), then given \((z, \omega) \in T_{\gamma(H)}(x, u)\) we get sequences \(h_n \to 0^+, z_n \to z\), and \(\omega_n \to \omega\) satisfying \((x, u) + h_n(z_n, \omega_n) \in \mathcal{G}(H)\). On the other hand, by Lemma 2.2, we have the inequalities \(V(x + h_n z_n) \leq \lambda + h_n \eta\) and \(D_t V(x + h_n z_n)(u + h_n \omega_n) \leq \mu + h_n r\) for each \(\eta, r \in \mathbb{R}\) and \(h_n\) small enough. Hence (iv) is proved.

- If \(V(x) = \lambda\) and \(D_t V(x)(u) < \mu\), by (23) we have that \((z, \omega) \in \text{epi } D_t V(x)\) for each \((z, \eta), (\omega, r) \in T_{\gamma(R)}((x, \lambda), (u, \mu))\). Conversely, given a point \((z, \omega)\) in \(T_{\gamma(H)}(x, u)\) with \(D_t V(x)(z) \leq \eta\), there exist \(h_n \to 0^+, (z_n, \eta_n) \to (z, \eta)\), and \(\omega_n \to \omega\) satisfying \(u + h_n \omega_n \in H(x + h_n z_n)\) and
\[
V(x + h_n z_n) - V(x) \leq h_n \eta_n.
\]
Here we use the semidifferentiability of \(V\). Finally, for all \(r \in \mathbb{R}\) and \(h_n\) small enough,
\[
D_t V(x + h_n z_n)(u + h_n \omega_n) \leq \mu + h_n r
\]
since \(\Gamma\) is continuous. So (iii) is proved.

- Let us consider \(V(x) < \lambda\) and \(D_t V(x)(u) = \mu\), then by (24)
\[
\frac{D_t V(x + h_n z_n)(u + h_n \omega_n) - D_t V(x)(u)}{h_n} \leq r_n
\]
and therefore \(D_t \Gamma(x, u)(z, \omega) \leq r\) if \((z, \eta), (\omega, r) \in T_{\gamma(R)}((x, \lambda), (u, \mu))\). The converse follows from the very definition of the Bouligand tangent cone and the assumed hypotheses.

- Finally, (i) is achieved by combining previous arguments. \(\blacksquare\)

Theorems 3.1 and 3.2 in [12] can be reformulated by assuming that \(\mathcal{G}(T_K)\) is not closed, but contains a closed set. The next lemma will be very useful to get the main result of this paper, Theorem 3.3. Its proof easily comes from [12], so it is omitted.
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**Lemma 3.1.** Let $G: [0, \delta[ \times \mathcal{C}(T_K) \to 2^X$ be an u.s.c. set-valued map having closed convex values, with $K \subset X$ a nonempty closed set. It is given that $R: K \to 2^X$, a set-valued map with closed graph, if the following hypotheses are satisfied:

1. $\mathcal{C}(R) \subseteq \mathcal{C}(T_K)$.
2. There exists $\gamma \in L^1_{\text{loc}}(0, \delta)$ with $G(t, y) \subseteq \gamma(t)(1 + \|y\|)B_X$, for all $(t, y)$ in $[0, \delta[ \times \mathcal{C}(R)$.
3. For any $(t, x, u) \in [0, \delta[ \times \mathcal{C}(R)$,
   \[ G(t, x, u) \cap DR(x, u)[u] \neq \emptyset. \]  

Then the problem

\[ x''(t) \in G(t, x(t), x'(t)) \quad x(0) = x_0, \quad x'(0) = u_0 \]  

has a viable solution in $K$ defined on $[0, \delta]$, i.e., satisfying $x(t) \in K$, $0 \leq t < \delta$, for each initial condition $(x_0, u_0)$ in $\mathcal{C}(R)$.

**Remark 3.3.** Note that in the previous lemma the case $\delta = +\infty$ is allowed. Moreover, its statement remains valid when assumption (ii) is replaced with

\[ G(t, y) \cap \gamma(t)(1 + \|y\|)B_X \neq \emptyset. \]  

It also remains true under the hypotheses

\[ \langle x, u \rangle + \sup_{y \in F(t, x, u)} \langle u, y \rangle \leq \varphi(t, \|x\|)^2, \]  

for almost every $t \in [0, \delta]$ and any $(x, u) \in \mathcal{C}(R)$, where $\varphi$ is a Carathéodory map and $r'(t) = 2\varphi(t, r(t))$ has a bounded maximal solution on $[0, \delta]$ satisfying $r(0) = \|x_0, u_0\|^2$. Finally, note that (i)–(iii) yield a result stronger than the existence of a solution of (25) viable in $K$.

**Theorem 3.3.** Let $V: \Omega \subseteq X \to \mathbb{R}$ be a semidifferentiable function ($B \subset \Omega$ open) such that $\Gamma(\cdot, \cdot) = D_1V(\cdot)(\cdot)$ is semidifferentiable on $\Lambda$ ($\mathcal{C}(\Lambda) \subset \Omega$ open). Let $R$ be defined as in (20). Then if $(x_0, u_0) \in \mathcal{C}(\Lambda)$ and if

\[ F(t, x, u) \cap DH(x, u)[u] \neq \emptyset \]  

and

\[ \inf_{y \in F(t, x, u) \cap DH(x, u)[u]} D_1\Gamma(x, u)(u, y) \leq -g(t, \lambda, D_1V(x)(u)) \]  

hold for all $(t, x, u) \in [0, +\infty[ \times \mathcal{C}(\Lambda)$ and all $(x, \lambda) \in \text{epi} V \cap (B \times \mathbb{R})$, there are a solution $x(\cdot)$ of (1) and a solution $\beta(\cdot)$ of (3) satisfying (2) and (5), respectively, and such that $V(x(t)) \leq \beta(t)$ for all $t \geq 0$. 


Proof. By Lemma 3.1, it suffices to show that

\[(\{(u, \mu)\} \times (F(t, x, u) \times \{-g(t, \mu, \mu)\})) \cap \mathcal{T}_{\mathcal{G}}(\{(x, \lambda), (u, \mu)\}) \neq \emptyset\]

for any \((x, \lambda), (u, \mu) \in \mathcal{G}(R)\) and any \(t \geq 0\). This statement is equivalent, from Proposition 3.1, to the existence of \(y \in F(t, x, u)\) such that \((u, y) \in T_{\mathcal{G}}(H, x, u)\), i.e., \(y\) belongs to \(DH(x, u)(u)\), and satisfies

\[D_x \Gamma(x, u)(u, y) \leq -g(t, \lambda, D_x V(x)(u))\]

for every \(V(x) \leq \lambda\). Finally, by using the lower semicontinuity of the epiderivative and the compactness of the values taken by \(F\) we get the proof. \(\blacksquare\)

Remark 3.4. From the proof of the preceding theorem, it follows that

\[(x(t), \beta(t)), (x'(t), \beta'(t)) \in \mathcal{G}(R), \quad t \geq 0,\]

which implies that \((x(t), x'(t)) \in \mathcal{G}(H)\) and \(x(t) \in B\). Hence, \(x'(t) \in T_B(x(t))\). This fact leads us to assume (21). On the other hand, from Theorem 4.1 in [12] it easily comes that Lemma 3.1 remains true when \(F\) is assumed to be almost u.s.c. Therefore Theorem 3.3 is also valid for this kind of set-valued map.

Example 3.3. Let us revisit Example 3.2 and consider the closed interval \(B = [a, +\infty[\), where \(0 < a < x_0\), and the set-valued map \(H(x) = [a - x, +\infty[\), \(x \in B\). Obviously \(x_0 \in B\) and \(\mathcal{G}(H)\) is a closed set contained in the graph of \(T_B\). Furthermore, (28) is satisfied for any \((x, u) \in \mathcal{G}(H)\) and, since \(D_x \Gamma(x, u)(u, y) = y\) for any \((x, u) \in \mathcal{G}(H)\) and any \(y \in X\), (29) is easily achieved. Therefore we are under the hypotheses of Theorem 3.3, and a sufficient condition ensuring the existence of a solution \(x(\cdot)\) of (19) satisfying

\[|x(t)| \leq \frac{x_0 + u_0 + 1 + r}{2} e^t + \frac{x_0 - u_0 + 1 + r}{2} e^{-t} - (1 + r),\]

for all \(t \geq 0\), consists in taking \(a - x_0 \leq u_0\) \((0 < a < x_0)\).

4. Solutions with exponential decay

In this section, as an application of the previous results, we obtain conditions ensuring the existence of a solution of (1) having exponential decay. Let us consider the closed set

\[B = \{x \in X : a_i \leq x_i, \quad \forall i\},\]
where $0 < a_i$, and let $V(x) = \|x\|$. Obviously $V$ is $C^2$ on the open set $X \setminus \{0\}$ and $B \subset X \setminus \{0\}$. Furthermore, the following relationships are satisfied for any $(x, u) \in B \times X$ and any $(z, y) \in T_B(x) \times X$:

$$D_1 V(x)(u) = \frac{\langle x, u \rangle}{\|x\|}, \quad D_1 \Gamma(x, u)(z, y) = -\frac{\langle x, u \rangle^2}{\|x\|^3} + \frac{\langle x, y \rangle}{\|x\|} + \frac{\|u\|^2}{\|x\|}.$$  \hfill (30)

$\Gamma(\cdot, \cdot)$ being equal to $D_1 V(\cdot)(\cdot)$.

Let us also consider the set-valued map

$$H(x) = \begin{cases} \{u \in X : u_i - a_i + x_i \geq 0, \forall i\} & \text{if } x \in B, \\ \emptyset & \text{otherwise}. \end{cases}$$

Clearly $\emptyset(H)$ is closed and it is contained in the graph of $T_B$, because as is well-known

$$T_B(x) = \begin{cases} \{u \in X : u_i \geq 0, \forall i \in I(x)\} & \text{if } I(x) \neq \emptyset, \\ X, & \text{if } I(x) = \emptyset, \end{cases}$$

where $I(x) = \{i : a_i = x_i\}$ is the set of active constraints at $x$.

We are, therefore, under the assumptions of Theorem 3.3, and then if (28) and

$$\inf_{y \in F(t, x, u) \cap \partial \mathcal{H}(x, u)(u)} \frac{\langle x, y \rangle + \|u\|^2}{\|x\|} = \frac{\langle x, u \rangle^2}{\|x\|^3} \leq \frac{\langle x_0, u_0 \rangle^2}{\|x_0\|^3} \|x\|$$  \hfill (31)

are satisfied for every $(t, x, u) \in [0, +\infty[ \times \emptyset(H)$ there exists a solution $x(\cdot)$ of (1)–(2) ($(x_0, u_0) \in \emptyset(H)$), such that $x(t) \in B$ and

$$\|x(t)\| \leq \|x_0\| \exp\left(\frac{\langle x_0, u_0 \rangle t}{\|x_0\|^2}\right)$$  \hfill (32)

for all $t \geq 0$.

We will now rewrite (28) for this particular case. For that purpose we compute the contingent derivative of $H$ by using a classical result for sets given by inequality constraints (see e.g. [2, 5, or 11]) and obtain

$$T_{\emptyset(H)}(x, u) = \{(z, y) : z_i \geq 0, \forall i \in I(x) \text{ and } z_j + y_j \geq 0, \forall j \in J(x, u)\}$$

if $I(x) \neq \emptyset$ or $J(x, u) \neq \emptyset$, and $T_{\emptyset(H)}(x, u) = X$ otherwise, $J(x, u)$ being equal to $\{j : u_j + x_j - a_j = 0\}$. Hence,

$$DH(x, u)(u) = \begin{cases} X, & \text{if } J(x, u) = \emptyset, \\ \{y \in X : y_j - x_j - a_j \geq 0, \text{ if } j \in J(x, u)\}, & \text{otherwise}. \end{cases}$$

Therefore (28) is satisfied if and only if

$$\sup_{y \in F(t, x, u)} y_j \geq x_j - a_j$$  \hfill (33)

whenever $u_j = a_j - x_j$, for any $(t, x, u) \in [0, +\infty[ \times \emptyset(H)$.

Thus, if both inequalities (31) and (33) hold, then there exists a solution of (1)–(2) satisfying (32). This means that we have the desired solution which has exponential decay by taking initial conditions $(x_0, u_0) \in \emptyset(H)$ such that $I(x_0) \neq \emptyset$ and $\langle x_0, u_0 \rangle < 0$.  

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4.1. An Application to Control Systems

Let us consider a control system with a second-order dynamics,

\[
x''(t) = f(x(t), x'(t), \omega(t)), \quad \omega(t) \in [-1, 1],
\]

where \( f : \mathbb{R}^3 \to \mathbb{R} \) is a continuous function having linear growth \( (|f(x, u, \omega)| \leq \alpha(1 + |(x, u)|)). \) Let us also assume that

\[
\sup_{\omega \in [-1, 1]} f(x, a - x, \omega) \geq x - a, \quad \forall x \geq a,
\]

and

\[
\inf_{\omega \in [-1, 1]} f(x, u, \omega) \leq \frac{u_0^2}{x_0}, \quad \forall (x, u) \in [a, +\infty] \times [a - x, +\infty].
\]

Then by the previous statements, if \( 0 < a < x_0 \) and \( a - x_0 < u_0 < 0 \), there exists a relaxed solution of (34) (see e.g. [1] or [10]), with \( x(0) = x_0, \quad x'(0) = u_0 \) and having exponential decay.

REFERENCES