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Generalization of Szasz operators involving Brenke type polynomials

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ABSTRACT

The purpose of this paper is to give a generalization of Szasz operators defined by means of the Brenke type polynomials. We obtain convergence properties of our operators with the help of Korovkin's theorem and the order of convergence by using a classical approach, the second modulus of continuity and Peetre's K-functional. An explicit example with our operators including Gould–Hopper polynomials which generalize Szasz operators in a natural way is given.

1. Introduction

In approximation theory, the positive approximation processes discovered by Korovkin play a central role and arise in a natural way in many problems connected with functional analysis, harmonic analysis, measure theory, partial differential equations and probability theory. The most useful examples of such operators are Szasz operators.

Szasz [1] defined the following linear positive operators:

$$S_n(f;x) := e^{-nx}\sum_{k=0}^{\infty}\frac{(nx)^k}{k!}f\left(\frac{k}{n}\right) \tag{1.1}$$

where $x \geq 0$ and $f \in C[0,\infty)$ whenever the above sum converges. Guided by this work, many authors have investigated several interesting properties of the operators (1.1).

Later, Jakimovski and Leviatan [2] obtained a generalization of Szasz operators by means of the Appell polynomials. Let $g(z) = \sum_{k=0}^{\infty} a_k z^k$ $(a_0 \neq 0)$ be an analytic function in the disc $|z| < R$, $(R > 1)$ and suppose that $g(1) \neq 0$. The Appell polynomials $p_k(x)$ have generating functions of the form

$$g(u)e^{ux} = \sum_{k=0}^{\infty} p_k(x)u^k. \tag{1.2}$$

Under the assumption that $p_k(x) \geq 0$ for $x \in [0,\infty)$, Jakimovski and Leviatan introduced the linear positive operators $P_n(f;x)$ via

$$P_n(f;x) := \frac{e^{-nx}}{g(1)}\sum_{k=0}^{\infty} p_k(nx) f\left(\frac{k}{n}\right) \tag{1.3}$$

and gave the approximation properties of these operators.

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Case 1. For $g(z)=1$, with the help of (1.2) we easily find $p_k(x)=\frac{x^k}{k!}$ and from (1.3) we meet again the Szasz operators given by (1.1).

Then, Ismail [3] presented another generalization of Szasz operators (1.1) and Jakimovski and Leviatan operators (1.3) by using Sheffer polynomials. Let $A(z)=\sum_{k=0}^{\infty}a_kz^k$ $(a_0\neq 0)$ and $H(z)=\sum_{k=1}^{\infty}h_kz^k$ $(h_1\neq 0)$ be analytic functions in the disc $|z|<R$ $(R>1)$ where a_k and h_k are real. The Sheffer polynomials $p_k(x)$ have generating functions of the type

$$A(t)e^{xH(t)}=\sum_{k=0}^{\infty}p_k(x)t^k,\quad |t|<R. \tag{1.4}$$

Using the following assumptions:

$$\begin{aligned}&\text{(i) for } x\in[0,\infty),\quad p_k(x)\geq 0,\\ &\text{(ii) } A(1)\neq 0 \quad\text{and}\quad H'(1)=1,\end{aligned} \tag{1.5}$$

Ismail investigated the approximation properties of the linear positive operators given by

$$T_n(f;x):=\frac{e^{-nxH(1)}}{A(1)}\sum_{k=0}^{\infty}p_k(nx)f\left(\frac{k}{n}\right),\quad \text{for } n\in\mathbb{N}. \tag{1.6}$$

Case 1. For $H(t)=t$, it can be easily seen that the generating functions (1.4) return to (1.2) and, from this fact, the operators (1.6) reduce to the operators (1.3).

Case 2. For $H(t)=t$ and $A(t)=1$, one can get the Szasz operators from the operators (1.6).

In this paper, we construct linear positive operators with the help of Brenke type polynomials. Brenke type polynomials [4] have generating functions of the form

$$A(t)B(xt)=\sum_{k=0}^{\infty}p_k(x)t^k \tag{1.7}$$

where A and B are analytic functions:

$$A(t)=\sum_{r=0}^{\infty}a_rt^r,\quad a_0\neq 0, \tag{1.8}$$

$$B(t)=\sum_{r=0}^{\infty}b_rt^r,\quad b_r\neq 0\ (r\geq 0) \tag{1.9}$$

and have the following explicit expression:

$$p_k(x)=\sum_{r=0}^{k}a_{k-r}b_rx^r,\quad k=0,1,2,\ldots. \tag{1.10}$$

We shall restrict ourselves to the Brenke type polynomials satisfying:

$$\begin{aligned}&\text{(i) } A(1)\neq 0,\quad \frac{a_{k-r}b_r}{A(1)}\geq 0,\quad 0\leq r\leq k,\ k=0,1,2,\ldots,\\ &\text{(ii) } B:[0,\infty)\longrightarrow(0,\infty),\\ &\text{(iii) (1.7) and the power series (1.8) and (1.9) converge for } |t|<R\ (R>1).\end{aligned} \tag{1.11}$$

Now, in view of the above restrictions, we introduce the following linear positive operators including the Brenke type polynomials:

$$L_n(f;x):=\frac{1}{A(1)B(nx)}\sum_{k=0}^{\infty}p_k(nx)f\left(\frac{k}{n}\right) \tag{1.12}$$

where $x\geq 0$ and $n\in\mathbb{N}$.

Case 1. Let $B(t)=e^t$. In this case the operators (1.12) (resp. (1.7)) reduce to the operators given by (1.3) (resp. (1.2)).

Case 2. Let $B(t)=e^t$ and $A(t)=1$. We meet again the Szasz operators (1.1).

The purpose of this paper is to present a generalization of Szasz operators and operators given by (1.3) containing the Appell polynomials. Moreover, we give a suitable example with the operators (1.12) by using Gould–Hopper polynomials.

The structure of the paper is as follows. In Section 2, the convergence of the operators (1.12) is examined with the help of Korovkin's theorem. The order of approximation is established by means of a classical approach, the second modulus of continuity and Peetre's K-functional in Section 3. In the last section, operators including Gould–Hopper polynomials one of the Brenke type polynomials are given as an example.

2. Approximation properties of L_n operators

In this section, we give our main theorem with the help of the well-known Korovkin theorem.

Lemma 1. *For all $x \in [0, \infty)$, we have*

$$L_n(1; x) = 1 \tag{2.1}$$

$$L_n(s; x) = \frac{B'(nx)}{B(nx)}x + \frac{A'(1)}{nA(1)} \tag{2.2}$$

$$L_n(s^2; x) = \frac{B''(nx)}{B(nx)}x^2 + \frac{[A(1) + 2A'(1)]B'(nx)}{nA(1)B(nx)}x + \frac{A''(1) + A'(1)}{n^2A(1)}. \tag{2.3}$$

Proof. From the generating functions of the Brenke type polynomials given by (1.7), we obtain

$$\sum_{k=0}^{\infty} p_k(nx) = A(1)B(nx)$$

$$\sum_{k=0}^{\infty} kp_k(nx) = A'(1)B(nx) + nxA(1)B'(nx)$$

$$\sum_{k=0}^{\infty} k^2p_k(nx) = A''(1)B(nx) + 2nxA'(1)B'(nx) + (nx)^2A(1)B''(nx) + A'(1)B(nx) + nxA(1)B'(nx).$$

In view of these equalities, we get (2.1)–(2.3). □

Theorem 1. *Let*

$$\lim_{y\to\infty} \frac{B'(y)}{B(y)} = 1 \quad \textit{and} \quad \lim_{y\to\infty} \frac{B''(y)}{B(y)} = 1. \tag{2.4}$$

If $f \in C[0, \infty) \cap E$, then

$$\lim_{n\to\infty} L_n(f; x) = f(x),$$

and the operators L_n converge uniformly in each compact subset of $[0, \infty)$ where

$$E := \{f : \forall x \in [0, \infty), |f(x)| \le ce^{Ax}\ A \in \mathbb{R} \textit{ and } c \in \mathbb{R}^+\}.$$

Proof. According to (2.1)–(2.3), taking into account the equality (2.4) we find

$$\lim_{n\to\infty} L_n(s^i; x) = x^i, \quad i = 0, 1, 2.$$

The above convergence is verified uniformly in each compact subset of $[0, \infty)$. Applying Korovkin's theorem, we obtain the desired result. □

3. The order of approximation

We give the following lemmas and definitions which are used in this section.

Definition 1. Let $[a, b]$ be a closed interval and fix $f \in C[a, b]$. If $\delta > 0$, the modulus of continuity $\omega(f; \delta)$ of f is defined by

$$\omega(f; \delta) := \sup_{\substack{x,y\in[a,b] \\ |x-y|\le\delta}} |f(x) - f(y)|.$$

Definition 2. The second modulus of continuity of $f \in C_B[0, \infty)$ is defined by

$$\omega_2(f; \delta) := \sup_{0<t\le\delta} \|f(. + 2t) - 2f(. + t) + f(.)\|_{C_B}$$

where $C_B[0, \infty)$ is the class of real valued functions defined on $[0, \infty)$ which are bounded and uniformly continuous with the norm $\|f\|_{C_B} = \sup_{x\in[0,\infty)} |f(x)|$.

Definition 3 (*Ditzian and Totik [5]*). Peetre's K-functional of the function $f \in C_B[0,\infty)$ is defined by

$$K(f;\delta) := \inf_{g\in C_B^2([0,\infty))} \{\|f-g\|_{C_B} + \delta\|g\|_{C_B^2}\}$$

where

$$C_B^2[0,\infty) := \{g \in C_B[0,\infty) : g',\ g'' \in C_B[0,\infty)\}$$

and the norm $\|g\|_{C_B^2} := \|g\|_{C_B} + \|g'\|_{C_B} + \|g''\|_{C_B}$. It is clear that the following inequality:

$$K(f;\delta) \le M\{\omega_2(f;\sqrt{\delta}) + \min(1,\delta)\|f\|_{C_B}\}$$

is valid, for all $\delta > 0$. The constant M is independent of f and δ.

Lemma 2 (*Gavrea and Rasa [6]*). *Let $g \in C^2[0,\infty)$ and $(P_n)_{n\ge 0}$ be a sequence of linear positive operators with the property $P_n(1;x) = 1$. Then*

$$|P_n(g;x) - g(x)| \le \|g'\|\sqrt{P_n((s-x)^2;x)} + \frac{1}{2}\|g''\|P_n((s-x)^2;x). \tag{3.1}$$

Lemma 3 (*Zhuk [7]*). *Let $f \in C[a,b]$ and $h \in (0, \frac{b-a}{2})$. Let f_h be the second-order Steklov function attached to the function f. Then the following inequalities are satisfied:*

$$\begin{aligned} &\text{(i) } \|f_h - f\| \le \frac{3}{4}\omega_2(f;h)\\ &\text{(ii) } \|f_h''\| \le \frac{3}{2h^2}\omega_2(f;h). \end{aligned} \tag{3.2}$$

Lemma 4. *For $x \in [0,\infty)$, we have*

$$L_n((s-x)^2;x) = \frac{B''(nx) - 2B'(nx) + B(nx)}{B(nx)}x^2 + \frac{A(1)B'(nx) + 2A'(1)[B'(nx) - B(nx)]}{nA(1)B(nx)}x + \frac{A''(1) + A'(1)}{n^2A(1)}.$$

Proof. From the linearity property of L_n operators, we can write

$$L_n((s-x)^2;x) = L_n(s^2;x) - 2xL_n(s;x) + x^2L_n(1;x).$$

By virtue of Lemma 1, the proof is completed. □

The rate of convergence will be calculated using the following four theorems.

Theorem 2. *Let $f \in C[0,\infty) \cap E$. The L_n operators verify the following inequality:*

$$|L_n(f;x) - f(x)| \le 2\omega(f;\sqrt{\lambda_n(x)})$$

where

$$\begin{aligned} \lambda := \lambda_n(x) = L_n((s-x)^2;x) &= \frac{B''(nx) - 2B'(nx) + B(nx)}{B(nx)}x^2 \\ &\quad + \frac{A(1)B'(nx) + 2A'(1)[B'(nx) - B(nx)]}{nA(1)B(nx)}x + \frac{A''(1) + A'(1)}{n^2A(1)}. \end{aligned} \tag{3.3}$$

Proof. Using (2.1) and the properties of the modulus of continuity, we deduce

$$\begin{aligned} |L_n(f;x) - f(x)| &\le \frac{1}{A(1)B(nx)}\sum_{k=0}^{\infty} p_k(nx)\left|f\left(\frac{k}{n}\right) - f(x)\right| \\ &\le \left\{1 + \frac{1}{A(1)B(nx)}\frac{1}{\delta}\sum_{k=0}^{\infty} p_k(nx)\left|\frac{k}{n} - x\right|\right\}\omega(f;\delta). \end{aligned} \tag{3.4}$$

By considering the Cauchy–Schwarz inequality, in view of Lemma 4 we get

$$\sum_{k=0}^{\infty} p_k(nx)\left|\frac{k}{n}-x\right| \le \sqrt{A(1)B(nx)}\left\{\sum_{k=0}^{\infty} p_k(nx)\left|\frac{k}{n}-x\right|^2\right\}^{\frac{1}{2}}$$
$$= A(1)B(nx)\left\{\frac{B''(nx)-2B'(nx)+B(nx)}{B(nx)}x^2 + \frac{A(1)B'(nx)+2A'(1)[B'(nx)-B(nx)]}{nA(1)B(nx)}x + \frac{A''(1)+A'(1)}{n^2A(1)}\right\}^{1/2}.$$

By using the last inequality in (3.4), we obtain

$$|L_n(f;x)-f(x)| \le \left\{1+\frac{1}{\delta}\sqrt{\lambda_n(x)}\right\}\omega(f;\delta) \tag{3.5}$$

where $\lambda_n(x)$ is given by (3.3). With the inequality (3.5), on choosing $\delta = \sqrt{\lambda_n(x)}$, we obtain the desired result. □

Theorem 3. *For $f \in C[0,a]$, the following inequality:*

$$|L_n(f;x)-f(x)| \le \frac{2}{a}\|f\|h^2 + \frac{3}{4}(a+2+h^2)\omega_2(f;h)$$

is satisfied where

$$h := h_n(x) = \sqrt[4]{L_n((s-x)^2;x)}$$

and the second modulus of continuity of $f \in C[a,b]$ is given by

$$\omega_2(f;\delta) := \sup_{0<t\le\delta}\|f(.+2t)-2f(.+t)+f(.)\|$$

with the norm $\|f\| = \max_{x\in[a,b]}|f(x)|$.

Proof. Let f_h be the second-order Steklov function attached to the function f. By virtue of the identity (2.1), we have

$$\begin{aligned}|L_n(f;x)-f(x)| &\le |L_n(f-f_h;x)| + |L_n(f_h;x)-f_h(x)| + |f_h(x)-f(x)| \\ &\le 2\|f_h-f\| + |L_n(f_h;x)-f_h(x)|.\end{aligned} \tag{3.6}$$

Taking into account the fact that $f_h \in C^2[0,a]$, it follows from Lemma 2 that

$$|L_n(f_h;x)-f_h(x)| \le \|f_h'\|\sqrt{L_n((s-x)^2;x)} + \frac{1}{2}\|f_h''\|L_n((s-x)^2;x). \tag{3.7}$$

Combining the Landau inequality and Lemma 3, we can write

$$\begin{aligned}\|f_h'\| &\le \frac{2}{a}\|f_h\| + \frac{a}{2}\|f_h''\| \\ &\le \frac{2}{a}\|f\| + \frac{3a}{4}\frac{1}{h^2}\omega_2(f;h).\end{aligned}$$

From the last inequality, (3.7) becomes, on taking $h = \sqrt[4]{L_n((s-x)^2;x)}$,

$$|L_n(f_h;x)-f_h(x)| \le \frac{2}{a}\|f\|h^2 + \frac{3a}{4}\omega_2(f;h) + \frac{3}{4}h^2\omega_2(f;h). \tag{3.8}$$

Substituting (3.8) in (3.6), Lemma 3 hence gives the proof of the theorem. □

Remark 1. In Theorem 3, we give a proof for $h \in (0, \frac{a}{2})$. For the special case $B(t) = e^t$, $A(t) = 1$ and $x = 0$, one can deduce that $h = 0$ from the equality $h := h_n(x) = \sqrt[4]{L_n((s-x)^2;x)}$. The inequality obtained in Theorem 3 still remains true when $h = 0$.

Theorem 4. *Let $f \in C_B^2[0,\infty)$. Then*

$$|L_n(f;x)-f(x)| \le \gamma\|f\|_{C_B^2}$$

where

$$\gamma := \gamma_n(x) = \left[\frac{B''(nx) - 2B'(nx) + B(nx)}{B(nx)}x^2 + \frac{(B'(nx) - B(nx))(nA(1) + 2A'(1)) + A(1)B'(nx)}{nA(1)B(nx)}x + \frac{A''(1) + (n+1)A'(1)}{n^2A(1)}\right].$$

Proof. Using the Taylor expansion of f, the linearity property of the operators L_n and (2.1), it follows that

$$L_n(f; x) - f(x) = f'(x)L_n(s - x; x) + \frac{1}{2}f''(\eta)L_n((s - x)^2; x), \quad \eta \in (x, s). \tag{3.9}$$

Taking into account the fact that

$$L_n(s - x; x) = \frac{B'(nx) - B(nx)}{B(nx)}x + \frac{A'(1)}{nA(1)} \geq 0$$

for $x \leq s$, by combining Lemmas 1 and 4 in (3.9) we are led to

$$\begin{aligned}
|L_n(f; x) - f(x)| &\leq \left[\frac{B'(nx) - B(nx)}{B(nx)}x + \frac{A'(1)}{nA(1)}\right]\|f'\|_{C_B} + \frac{1}{2}\left[\frac{B''(nx) - 2B'(nx) + B(nx)}{B(nx)}x^2\right. \\
&\quad \left. + \frac{A(1)B'(nx) + 2A'(1)\left[B'(nx) - B(nx)\right]}{nA(1)B(nx)}x + \frac{A''(1) + A'(1)}{n^2A(1)}\right]\|f''\|_{C_B} \\
&\leq \left[\frac{B''(nx) - 2B'(nx) + B(nx)}{B(nx)}x^2 + \frac{(B'(nx) - B(nx))(nA(1) + 2A'(1)) + A(1)B'(nx)}{nA(1)B(nx)}x\right. \\
&\quad \left. + \frac{A''(1) + (n+1)A'(1)}{n^2A(1)}\right]\|f\|_{C_B^2}
\end{aligned}$$

which completes the proof. □

Theorem 5. *Let* $f \in C_B[0, \infty)$. *Then*

$$|L_n(f; x) - f(x)| \leq 2M\{\omega_2(f; \sqrt{\delta}) + \min(1, \delta)\|f\|_{C_B}\}$$

where

$$\delta := \delta_n(x) = \frac{1}{2}\gamma_n(x)$$

and $M > 0$ *is a constant independent of the function* f *and of* δ. *Note that* $\gamma_n(x)$ *is defined as in Theorem 4.*

Proof. Let $g \in C_B^2[0, \infty)$. Theorem 4 allows us to write

$$\begin{aligned}
|L_n(f; x) - f(x)| &\leq |L_n(f - g; x)| + |L_n(g; x) - g(x)| + |g(x) - f(x)| \\
&\leq 2\|f - g\|_{C_B} + \left[\frac{B''(nx) - 2B'(nx) + B(nx)}{B(nx)}x^2\right. \\
&\quad \left. + \frac{(B'(nx) - B(nx))(nA(1) + 2A'(1)) + A(1)B'(nx)}{nA(1)B(nx)}x + \frac{A''(1) + (n+1)A'(1)}{n^2A(1)}\right]\|g\|_{C_B^2} \\
&= 2[\|f - g\|_{C_B} + \delta\|g\|_{C_B^2}].
\end{aligned} \tag{3.10}$$

The left-hand side of inequality (3.10) does not depend on the function $g \in C_B^2[0, \infty)$, so

$$|L_n(f; x) - f(x)| \leq 2K(f; \delta). \tag{3.11}$$

By using the relation between Peetre's K-functional and the second modulus of smoothness, (3.11) becomes

$$|L_n(f; x) - f(x)| \leq 2M\{\omega_2(f; \sqrt{\delta}) + \min(1, \delta)\|f\|_{C_B}\}. \quad \square$$

Remark 2. Note that in Theorems 2–5 when $n \to \infty$, then λ, h, γ and δ tend to zero under the assumption (2.4).

4. Example

Gould–Hopper polynomials [8] have generating functions of the form

$$e^{ht^{d+1}} \exp(xt) = \sum_{k=0}^{\infty} g_k^{d+1}(x, h) \frac{t^k}{k!} \tag{4.1}$$

and the explicit representation

$$g_k^{d+1}(x, h) = \sum_{s=0}^{\left[\frac{k}{d+1}\right]} \frac{k!}{s!(k-(d+1)s)!} h^s x^{k-(d+1)s} \tag{4.2}$$

where, as usual, [.] denotes the integer part. The Gould–Hopper polynomials $g_k^{d+1}(x, h)$ are d-orthogonal polynomial sets of Hermite type [9]. The notion of d-orthogonality was introduced by Van Iseghem [10] and Maroni [11].

From (4.1), it is clear that the Gould–Hopper polynomials are the Brenke type polynomials with

$$A(t) = e^{ht^{d+1}} \quad \text{and} \quad B(t) = e^t.$$

Under the assumption $h \geq 0$, the restrictions (1.11) and condition (2.4) for the operators L_n given by (1.12) are satisfied. Then the explicit form of the L_n operators including the Gould–Hopper polynomials is

$$L_n^*(f; x) = e^{-nx-h} \sum_{k=0}^{\infty} \frac{g_k^{d+1}(nx, h)}{k!} f\left(\frac{k}{n}\right). \tag{4.3}$$

It is worthy of note that for $h = 0$ we obtain $g_k^{d+1}(nx, 0) = (nx)^k$ and the operators (4.3) lead to the well-known Szasz operators.

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