Planar domination graphs

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Abstract

A graph $G$ is a domination graph if each induced subgraph of $G$ has a pair of vertices such that the open neighborhood of one is contained in the closed neighborhood of the other in the subgraph. No polynomial time algorithm or hardness result is known for the problem of deciding whether a graph is a domination graph. In this paper, it is shown that the class of planar domination graphs is equivalent to the class of planar weakly chordal graphs, and thus, can be recognized in polynomial time.

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1. Introduction

In this paper, graphs are assumed to be undirected and simple. For a graph $G = (V, E)$ and $v \in V$, the (open) neighborhood of $v$, denoted by $N(v)$, is the set $\{u \in V | uv \in E\}$, and the closed neighborhood of $v$, denoted by $N[v]$, is $N(v) \cup \{v\}$. We use $n$ to denote the cardinality of $V$. The complement of a graph $G$ is denoted by $\overline{G}$.

A domination graph is an undirected graph $G$ such that each induced subgraph $H$ of $G$ with more than one vertex contains a pair of vertices $u, v$ such that $N(u) \subseteq N[v]$ in $H$ [1]. In the case that $N(u) \subseteq N[v]$, we say $v$ dominates $u$, and $v$ is a dominating
vertex while $u$ is a dominated vertex. Two vertices $u$ and $v$ are said to be a comparable pair if either $u$ dominates $v$ or $v$ dominates $u$.

A hole is a chordless cycle on five or more vertices. An antihole is the complement of a hole. A graph is weakly chordal if it contains no holes and no antiholes [7]. One can see, from the definition of the class, that the complement of a domination graph is also a domination graph. Furthermore, a domination graph cannot contain a chordless cycle on five or more vertices. Thus, every domination graph is weakly chordal. Several subclasses of weakly chordal graphs have been shown to be contained in the class of domination graphs. This is the case for brittle graphs and (house, hole)-free graphs [2], and also trapezoid graphs and tolerance graphs [12].

Hayward [7] proved that the class of weakly chordal graphs and the class of domination graphs are distinct by constructing a graph on 24 vertices that is weakly chordal but not a domination graph. Rusu and Spinrad [12] later showed that there are an infinite number of weakly chordal graphs that are minimal (with respect to inclusion) nondomination graphs. It is interesting that Rusu and Spinrad [12] also defined a class of graphs that lies strictly between domination and weakly chordal graphs.

While weakly chordal graphs can be recognized in polynomial time [9,14], neither a polynomial time algorithm nor a hardness result has been established for the recognition of domination graphs. In this paper, we prove that the class of planar domination graphs is equivalent to the class of planar weakly chordal graphs, and thus, can be recognized in polynomial time. Furthermore, we prove that if $G$ is a planar domination graph, then $G$ either is a clique or has two nonadjacent dominated vertices.

2. Background

A graph is chordal if it does not contain any chordless cycle on four or more vertices. A vertex of a graph is simplicial if its neighborhood induces a clique. The following is a well-known characterization of chordal graphs due to Dirac [3].

**Theorem 1** (Dirac [3]). A graph $G$ is a chordal graph if and only if each induced subgraph $H$ of $G$ either is a clique or has two nonadjacent simplicial vertices of $H$.

Since a simplicial vertex is either dominated by each of its neighbors or has no neighbors, and thus is trivially dominated, all chordal graphs are domination graphs. Also, the complements of chordal graphs (called co-chordal graphs) are domination graphs. The domino (two chordless cycles on four vertices that share a common edge) is a domination graph that is neither chordal nor co-chordal.

A comparable pair in a domination graph may be adjacent or nonadjacent. Restricting the definition of domination graphs yields characterizations of chordal and co-chordal graphs. To see this, recall that a simplicial vertex is dominated by its neighbors, and observe that a chordless cycle on four or more vertices does not have an adjacent comparable pair. Then the following two corollaries follow from Dirac’s theorem.
Corollary 2. A graph $G$ is a chordal graph if and only if every induced subgraph $H$ of $G$ that has at least one edge has an adjacent comparable pair of $H$.

Corollary 3. A graph $G$ is a co-chordal graph if and only if every induced subgraph $H$ of $G$ that has at least two nonadjacent vertices has a nonadjacent comparable pair of $H$.

A bipartite graph is chordal bipartite if it does not contain any chordless cycle on six or more vertices. It is easy to see that a graph is chordal bipartite if and only if it is both bipartite and weakly chordal. Golubic and Goss [6] defined an edge $xy$ in a bipartite graph to be bisimplicial if $N[x] \cup N[y]$ induces a complete bipartite graph, and proved that every chordal bipartite graph has a bisimplicial edge. Mahadev and Trotter [11] observed that if $xy$ is a bisimplicial edge, then either $xy$ is an isolated edge and $x$ and $y$ dominate each other, or any neighbor of $x$ dominates $y$ and vice versa. Since every induced subgraph of a chordal bipartite graph is also chordal bipartite, it follows that a graph is chordal bipartite if and only if it is both bipartite and domination. Mahadev and Trotter [11] used a reduction to bipartite graphs to show further that every hole-free comparability graph is a domination graph; an undirected graph is a comparability graph if it admits an acyclic transitive orientation. Since a comparability graph cannot contain an antihole, the class of hole-free comparability graphs is exactly the class of weakly chordal comparability graphs. It then follows that the class of weakly chordal comparability graphs is equivalent to the class of domination comparability graphs. $O(n^2)$ recognition and independent set algorithms for weakly chordal comparability graphs were given in [4].

Rusu and Spinrad [12] showed that there exist domination graphs that have only one dominated vertex. This is a structural property that separates domination graphs from chordal graphs [12], weakly chordal graphs, and even some superclasses of weakly chordal graphs in the following sense. Dirac’s theorem ensures that a chordal graph with at least two vertices always has at least two “special” vertices (i.e., simplicial vertices), and two nonadjacent special vertices when the graph is not a clique. Weakly chordal graphs with at least two edges always have at least two “special” edges (i.e., the co-pairs of Theorem 6 in Section 3), and have two independent special edges whenever possible [5]. A hierarchy of superclasses, $\mathcal{G}_k$, of the class of weakly chordal graphs can be defined as: $\mathcal{G}_k$, $4 \leq k \leq n - 1$, contains exactly those graphs having no holes of any size and no antiholes on $i$ vertices where $5 \leq i \leq k + 1$. In [5] it is shown that any graph in the class $\mathcal{G}_k$ with at least two edges always has two “special” edges (i.e., generalized co-pairs), and two independent special edges whenever possible.

A house is the complement of a chordless path on five vertices. Dahlhaus et al. [2] proved that the (house, hole)-free graphs are a subclass of domination graphs for which lexicographic breadth first search can be used to find a dominated vertex. In a (house, hole)-free graph with at least two vertices, lexicographic breadth first search can be applied twice to obtain two dominated vertices, and two nonadjacent dominated vertices when the graph is not a clique. We finally note that the classes of planar domination graphs and (house, hole)-free graphs are incomparable since a house is a planar domination graph and $K_{3,3}$ is (house, hole)-free.
3. Planar domination graphs

A two-pair \[8\] in a graph \(G\) is a pair of nonadjacent vertices \(x, y\) such that every chordless path between \(x\) and \(y\) has exactly two edges. We call the corresponding edge \(xy\) in \(\bar{G}\) a co-pair of \(\bar{G}\). Observe that if \(xy\) is a co-pair in a graph, then the vertices \(x\) and \(y\) cannot be the end vertices of a chordless path with three or more edges in the complement of the graph.

Hayward et al. \[8\] characterized weakly chordal graphs as follows: a graph \(G\) is weakly chordal if and only if every induced subgraph \(H\) of \(G\) either is a clique or contains a two-pair of \(H\). Since the complements of weakly chordal graphs are weakly chordal this can be restated as: a graph \(G\) is weakly chordal if and only if every induced subgraph \(H\) of \(G\) either has no edge or contains a co-pair of \(H\). Lemma 4 below, which is central to our result, holds for any 2-connected planar graph that has a co-pair.

We are now ready to develop the proof of the main result of the paper. We begin by proving in Lemma 4 that every 2-connected planar weakly chordal graph has a dominated vertex. We then show in Lemma 7 that any 2-connected planar weakly chordal graph has at least two dominated vertices. In Theorem 8, we prove that every planar weakly chordal graph has at least two dominated vertices. That the class of planar weakly chordal graphs is equivalent to planar domination graphs then follows as Theorem 9.

In a planar embedding of a graph, a cycle is embedded as a simple closed polygonal curve, which divides the plane into two regions, one interior and one exterior to the curve \[15,16\]. Note that a region may be a face or may contain several faces. The outer face of a planar embedding of a graph is the infinite face. All other faces are called interior faces. We use \(P_k\) to denote the chordless path on \(k\) vertices.

**Lemma 4.** If graph \(G\) is planar weakly chordal and 2-connected, then \(G\) has a dominated vertex. In particular, for any co-pair \(xy\) of \(G\), either \(x\) or \(y\) is a dominated vertex in \(G\).

**Proof.** Let \(G\) be a planar weakly chordal graph that is 2-connected. We assume a planar embedding of \(G\). Since \(G\) is 2-connected, any embedding of \(G\) in the plane will be such that all facial boundaries are simple cycles.

Let \(xy\) be a co-pair in \(G\). Note that there cannot be a \(P_4\) \([u,x,y,w]\) in \(G\) as this implies that \(x\) and \(y\) are the end vertices of the \(P_4\) \([x,w,u,y]\) in \(\bar{G}\), which contradicts the fact that \(xy\) is a co-pair. This fact is used several times in the proof.

By way of contradiction, suppose neither \(x\) nor \(y\) is a dominated vertex in \(G\). Since \(xy\) is part of the bounding cycle of at least one interior face, we can divide the argument into two cases.

**Case A:** There is an interior face \(f\) bounded by a cycle with exactly three edges, one of which is the co-pair \(xy\).

Let \([x,y,c,x]\) be the cycle that forms the boundary of \(f\). Re-embed \(G\) so that \(f\) is the outer face. There must be a vertex \(a\) that is adjacent to \(x\), but not to \(y\), else \(y\) dominates \(x\). Likewise, there must be a vertex \(b\) that is adjacent to \(y\), but not to \(x\). It
must be the case that \( a \) and \( b \) are adjacent, otherwise we have the \( P_4 \) \([a, x, y, b]\) in \( G \), which implies \( x \) and \( y \) are at the ends of a \( P_4 \) in \( \tilde{G} \). Now \( c \) must be adjacent to one or both of \( a \) and \( b \), else the vertices \( \{x, a, b, y, c\} \) induce a house in \( G \) such that \( x \) and \( y \) are at the ends of the \( P_3 \) \([x, b, c, a, y]\) in \( \tilde{G} \). This gives rise to the following subcases.

Case A1: Vertex \( c \) is adjacent to both \( a \) and \( b \).

In this case, the region interior to the cycle \([x, y, c, x]\) must be divided into four subregions: \( r_0 \) is interior to \([x, y, b, a, x]\), \( r_2 \) is interior to \([x, a, c, x]\), \( r_3 \) is interior to \([a, b, c, a]\), and \( r_4 \) is interior to \([y, c, b, y]\). Fig. 1(a) illustrates these regions.

Vertex \( x \) cannot be dominated by \( b \), so there must be a vertex \( b' \) that is adjacent to \( x \), but not to \( b \). Vertex \( y \) must be adjacent to \( b' \), else we would have the \( P_4 \) \([b', x, y, b]\) in \( G \). Clearly, \( x \) cannot have a neighbor in \( r_3 \) or \( r_4 \), and \( y \) cannot have a neighbor in \( r_2 \). Thus, \( b' \) must be in \( r_0 \). Vertex \( a \) must also be adjacent to \( b' \), else the vertices \( \{x, a, b, y, b'\} \) induce a house in \( G \) such that \( x \) and \( y \) are at the ends of the \( P_5 \) \([x, b, b', a, y]\) in \( \tilde{G} \).

We now use \( r_1 \) to denote the region interior to \([x, y, b', x]\). Let \( r_5 \) be the region interior to \([x, b', a, x]\) and \( r_6 \) be the region interior to \([y, b, a, b', y]\). The resulting regions are illustrated in Fig. 1(b).

Vertex \( y \) cannot be dominated by \( a \), so there must be a vertex \( a' \) that is adjacent to \( y \), but not to \( a \). Vertex \( a' \) must be adjacent to \( x \), else we would have the \( P_4 \) \([a', y, x, a]\) in \( G \). Clearly, \( y \) cannot have a neighbor in \( r_2 \) or \( r_3 \), or \( r_5 \), and \( x \) cannot have a neighbor in \( r_4 \) or \( r_6 \). Thus, \( a' \) must be in \( r_1 \). As a consequence, \( a' \) cannot be adjacent to \( b \). But now, vertices \( \{x, a, b, y, a'\} \) induce a house in \( G \) such that \( x \) and \( y \) are at the ends of the \( P_5 \) \([x, b, a', a, y]\) in \( \tilde{G} \). This contradicts the fact that \( xy \) is a co-pair in \( G \).

Case A2: Vertex \( c \) is adjacent to exactly one of \( a \) and \( b \).

Suppose \( c \) is adjacent to \( b \). (The argument for \( c \) adjacent to \( a \) is identical.) In this case, the region interior to the cycle \([x, y, c, x]\) must be divided into three subregions: \( r_1 \) is interior to \([x, y, b, a, x]\), \( r_2 \) is interior to \([x, a, b, c, x]\), and \( r_3 \) is interior to \([y, c, b, y]\). Fig. 2(a) illustrates these regions.

As in Case A1, vertex \( x \) cannot be dominated by \( b \), so there must be a vertex \( b' \) in region \( r_1 \) that is adjacent to \( x, y \) and \( a \), but not adjacent to \( b \). Clearly, \( c \) cannot be adjacent to \( b' \). Thus, the graph induced by \( \{x, y, c, a, b, b'\}\) in \( G \) is completely
determined and it is the complement of a $P_6$; $x$ and $y$ are at the ends of the $P_6$ $[x,b,b',c,a,y]$ in $\bar{G}$. This contradicts the fact that $xy$ is a co-pair in $G$. The graph is illustrated in Fig. 2(b).

**Case B:** There is an interior face $f$ bounded by a cycle with more than three edges, one of which is the co-pair $xy$.

Let $[x,y,b,\ldots,a,x]$ be the cycle that forms the boundary of $f$. Re-embed $G$ so that $f$ is the outer face.

Suppose that $y$ is adjacent to $a$. Vertex $x$ cannot be dominated by $y$, so there must be a vertex $w$ that is adjacent to $x$, but not to $y$. Since $[x,y,b,\ldots,a,x]$ is the boundary of the outer face, vertex $w$ must be interior to the region bounded by $[a,x,y,a]$, and hence cannot be adjacent to $b$. Also, $x$ cannot be adjacent to $b$. But then $[w,x,y,b]$ is a $P_4$ in $G$, which contradicts the fact that $xy$ is a co-pair in $G$. Thus, it must be the case that $y$ is not adjacent to $a$. A symmetric argument implies that $x$ is not adjacent to $b$. It follows that $a$ must be adjacent to $b$, else we would have the $P_4$ $[a,x,y,b]$ in $G$.

Let $r_0$ denote the region interior to the cycle $[x,y,b,a,x]$. The embedding of $G$ is illustrated in Fig. 3(a). Now any neighbor of $x$, other than $a$ and $y$, and any neighbor of $y$, other than $b$ and $x$, must be in $r_0$, since $[a,x,y,b]$ is part of the boundary of the outer face. We now have an argument similar to that of Case A1.
Vertex $x$ cannot be dominated by $b$, so there must be a vertex $b'$ in $r_0$ that is adjacent to $x$, but not to $b$. It follows that $y$ must be adjacent to $b'$, else we would have the $P_4 [b', x, y, b]$ in $G$. Vertex $a$ must also be adjacent to $b'$, else the vertices $\{x, a, b, y, b'\}$ induce a house in $G$ such that $x$ and $y$ are at the ends of the $P_5 [x, b, b', a, y]$ in $G$.

We now use $r_1$ to denote the region interior to $[x, y, b', x]$. Let $r_5$ be the region interior to $[x, b', a, x]$ and $r_6$ be the region interior to $[y, b, a, b', y]$. These regions are illustrated in Fig. 3(b).

Vertex $y$ cannot be dominated by $a$, so there must be a vertex $a'$ that is adjacent to $y$, but not to $a$. Vertex $a'$ must be adjacent to $x$, else we would have the $P_4 [a', y, x, a]$ in $G$. Clearly, $y$ cannot have a neighbor in $r_5$, and $x$ cannot have a neighbor in $r_6$. Thus, $a'$ must be in $r_1$. As a consequence, $a'$ cannot be adjacent to $b$. But now, vertices $\{x, a, b, y, a'\}$ induce a house in $G$ such that $x$ and $y$ are at the ends of the $P_5 [x, b, a', a, y]$ in $G$. This contradicts the fact that $xy$ is a co-pair in $G$. □

Since simplicial vertices are dominated vertices, Dirac’s theorem implies that if a graph $G$ is chordal, then $G$ either is a clique or has two nonadjacent dominated vertices. Lemma 5 is an analogous statement for co-chordal graphs. Note that Corollary 3 is a characterization of co-chordal graphs, while the conclusion of Lemma 5 also holds for graphs that are not co-chordal.

Lemma 5. If graph $G$ is co-chordal, then $G$ either is a clique or has two nonadjacent dominated vertices.

Proof. Since $G = (V, E)$ is co-chordal, its complement has a simplicial vertex $s$. In $G$, $V - N[s]$ induces an independent set and $s$ dominates any vertex in $V - N[s]$. If $|V - N[s]| \geq 2$, we are done. If $|V - N[s]| = 0$, then $s$ is adjacent to all of $V - \{s\}$ and $s$ dominates any vertex in $V - \{s\}$. In this case, either $G$ is a clique or there are at least two nonadjacent dominated vertices in $V - \{s\}$. Suppose $|V - N[s]| = 1$ and let $u$ be the single vertex in $V - N[s]$. If $u$ is adjacent to all of $N(s)$, then $u$ dominates $s$, and we are done. If $u$ is not adjacent to some vertex $w$ in $N(s)$, then $s$ dominates $w$, and $u$ and $w$ are the desired nonadjacent pair of dominated vertices. □

As a consequence of the structural characterization for the graph classes $\mathcal{G}_k$ given in [5] (and described in Section 2), we have a stronger characterization of weakly chordal graphs than that of Hayward et al. It is this stronger result that we use in the proof of Lemma 7. A pair of edges in a graph are independent if they are not incident on a common vertex.

Theorem 6 (Eschen and Sritharan [5]). A graph $G$ is weakly chordal if and only if each induced subgraph $H$ of $G$ (with at least one edge) either has no pair of independent edges and every vertex of the nontrivial connected component of $H$ is incident on a co-pair of $H$, or has two independent co-pairs of $H$.

Lemma 7. If graph $G$ is planar weakly chordal and 2-connected, then $G$ either is a clique or has two nonadjacent dominated vertices.
Proof. Let $G$ be a planar weakly chordal graph that is 2-connected. If a weakly chordal graph has no pair of independent edges, then its complement is a chordal graph. Thus, if $G$ has no pair of independent edges, $G$ is co-chordal and the lemma follows from Lemma 5. If $G$ has a pair of independent edges, then, by Theorem 6, $G$ has two independent co-pairs. By Lemma 4, these independent co-pairs yield two nonadjacent dominated vertices. \hfill \square

In the following theorem we show that 2-connectedness is not required for the existence of two nonadjacent dominated vertices.

**Theorem 8.** If graph $G$ is planar weakly chordal, then $G$ either is a clique or has two nonadjacent dominated vertices.

**Proof.** The proof is by induction on the number of vertices. The theorem obviously holds for all graphs with one or two vertices. Let $G=(V,E)$ be an arbitrary planar weakly chordal graph with at least three vertices. Assume that the theorem holds for all planar weakly chordal graphs with fewer vertices than $G$.

In the case that $G$ is disconnected, the theorem holds for each connected component of $G$. If a vertex is dominated in a component of $G$, it is dominated in $G$. Hence, $G$ has at least two nonadjacent dominated vertices.

Now assume that $G$ is connected. If $G$ is not 2-connected, $G$ has a cut vertex $x$. Let $A$ be the vertex set of a connected component of the graph induced by $V \setminus \{x\}$. Let $G_1$ be the subgraph induced by $A \cup \{x\}$, and $G_2$ be the subgraph induced by $V \setminus A$. Both $G_1$ and $G_2$ have at least two vertices and fewer vertices than $G$. By the induction hypothesis, $G_1$ is either a clique or has two nonadjacent dominated vertices. In either case, note that $G_1$ has a dominated vertex that is different from $x$, and that this vertex is dominated in $G$ also. Likewise, for $G_2$. It follows that $G$ has two nonadjacent dominated vertices. \hfill \square

**Theorem 9.** The class of planar weakly chordal graphs is equivalent to the class of planar domination graphs.

**Proof.** As noted previously the class of domination graphs is contained in the class of weakly chordal graphs. Theorem 8 ensures that every planar weakly chordal graph has a dominated vertex. Since every induced subgraph of a planar weakly chordal graph is also planar weakly chordal, the theorem follows. \hfill \square

We note that planar domination graphs can be recognized in polynomial time by first testing whether the given graph is planar, and then testing whether it is weakly chordal. Planar graphs can be recognized in linear time \cite{10,13}, and the most efficient weakly chordal graph recognition algorithm runs in $O(m^3)$ time \cite{9}. Recognition of planar domination graphs in $O(n^2)$ time is achieved by using the well-known fact that a planar graph with $n$ vertices can have at most $3n-6$ edges \cite{16}. Simply count the number of edges on input and if more than $3n-6$ edges are detected terminate with
false, otherwise continue to test whether the input graph with $m = O(n)$ edges is planar and weakly chordal.

4. Conclusions

We have shown that the class of planar domination graphs is equivalent to the class of planar weakly chordal graphs. This implies that planar domination graphs can be recognized in quadratic time, while the recognition problem for domination graphs remains open. Furthermore, we prove that every planar domination graph either is a clique or has two nonadjacent dominated vertices. We leave as an open question whether weakly chordal graphs on other surfaces, such as the torus, are necessarily domination graphs.

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