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Graph-Encoded Maps

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An edge-colored graph model for the study of finite graphs embedded in surfaces (maps) is presented. This model brings to light the symmetry among three kinds of dualities on maps: (a) the usual duality that interchanges vertices and faces; (b) a duality that interchanges vertices and certain closed walks called zigzags; (c) a duality that interchanges faces and zigzags. Moreover, the orientability of the surface of a map is shown to be equivalent to the bipartiteness of the colored graph.

1. INTRODUCTION

The graphs that we use have a finite number of edges and vertices and may contain loops and multiple edges. A graph is considered primarily as a topological object in the usual sense. However, we may attach distinct labels to the edges and to the vertices of the graph and this enables us to view a graph combinatorially as an abstract system of incidence. Specifically, we identify each edge and each vertex with its label and so every label-edge has two label-vertices (which may coincide) "incident" to it.

The surfaces that we use are compact, connected and have no boundary. By a classical theorem in topology (see Chap. 2 of [2]) such surfaces are either the 2-sphere S^2, a connected sum of n 2-tori $T_n^2 = T^2 \# \cdots \# T^2$, or a connected sum of k projective planes $P_k^2 = P^2 \# \cdots \# P^2$.

A *map* M is a pair (G_M, S_M), where G_M is a graph, S_M is a surface and $S_M \backslash G_M$ is topologically equivalent to a collection of disjoint open discs; moreover, every point in S_M that belongs to more than one edge of G_M is one of its vertices.

Thus a map M can be described as a cellular embedding of G_M in S_M, where cellular means the above condition on the difference $S_M \backslash G_M$.

In a map M going around the boundary of a disc in $S_M \backslash G_M$ corresponds in

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G_M to the traversal of a closed walk. Each such closed path is called a *facial walk* of map M. A *face* of M is a disc in $S_M \backslash G_M$.

Now we describe what we mean by a zigzag walk in a map M. For what we want to do these paths play a role analogous to facial paths. A *zigzag walk* is a closed path in G_M which alternates choosing the rightmost and leftmost possibilities for the next edge at each vertex, until a transition from one edge to the next is encountered for the second time.

Observe that even if S_M is non-orientable, i.e., is a connected sum of projective planes, where the global concept of right and left is meaningless, a zigzag walk is well-defined because the concepts of right and left are used only locally in the definition. The zigzag walks are studied algebraically in [9], in the restricted case where S_M is the 2-sphere; there they are named left–right walks. They are also known as Petrie Polygons [3].

Figure 1 presents two distinct embeddings of the Petersen graph (with labeled edges) in the projective plane P^2. The two maps share a duality

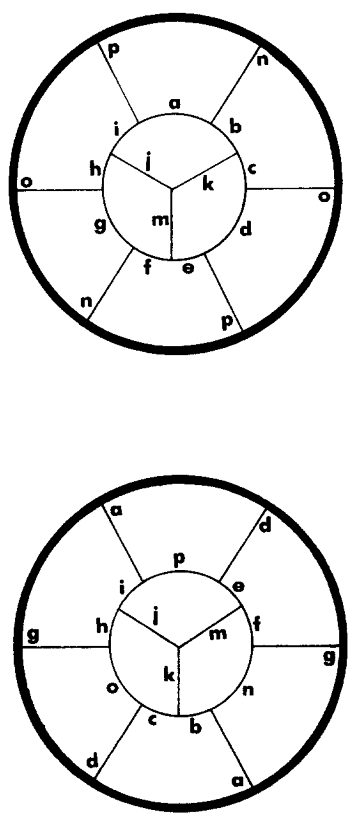

FIGURE 1

relation: their facial and zigzag walks are interchanged. For instance, the zigzag walk (p, a, b, c, d) in the first map corresponds to a facial walk in the second; on the other hand (h, j, m, f, g) is a facial walk in the first map and a zigzag walk in the second.

Note that to obtain the zigzags from Fig. 1 when we cross the boundary of the disc (which is antipodally identified) the orientation is reversed and we go "left–left" or "right–right." This is a minor problem unavoidable in the planar representation of non-orientable surfaces and which is entirely eliminated by the edge-colored graph model which we present in the next section. From the model it is also apparent that given any map M we can constructively present another map M which realizes the duality between facial and zigzag walks. What is special about the example of Fig. 1 is that $S_M = S_{M^-}$ $(=p^2)$. Usually $S_M \neq S_{M^-}$. A symmetric set of three dualities on maps is presented in Section 3. In Section 4 we present a graph-theoretic counterpart for the orientability of surfaces.

2. Gems and Maps

The term "gem" is an acronym for "*graph-encoded map*." A *gem* M is a finite cubic graph C_M with a given proper 3-coloration of its edges in colors v_M, f_M and a_M, such that the components of the subgraph generated by the edges colored v_M and f_M are polygons with four edges each. Each such polygon is called an *M-square.*

A *bigon* (bicolored polygon) in a gem M is a polygon in C_M whose edges are colored (alternatively) with two of the three colors. The bigons are of three types:

(a) the M-squares, whose edges are painted with v_M and f_M;

(b) the *v-gons*, polygons whose edges are painted with v_M and a_M;

(c) the *f-gons*, polygons whose edges are painted with f_M and a_M.

The v-gons, the f-gons and the M-squares, as we show in the sequel, correspond in a natural way to, respectively, the vertices, faces and edges of a map which is obtained from the gem in a bijective correspondence. To obtain also the zigzag walks we consider the 4-regular graph Q_M obtained from a gem M by adding to C_M for each M-square a pair of edges painted with a fourth color z_M linking its opposite vertices. Q_M is used later.

Now we describe the bijective correspondence between gems and maps. To go from a gem M to the corresponding map $M^t = (G_M, S_M)$ first give labels to the vertices of C_M. Following that we consider as many closed discs as there are bigons in M, in such a way that every disc is bounded by a bigon in M (with labeled vertices). Observe that each edge of C_M appears twice in the boundary of the discs. A surface S_M is constructed if we identify the boundaries of the discs along the two occurrences of each edge so that vertices with the same label coincide. This gives us a map $N^t = (C_M, S_M)$. At this point we contract each disc bounded by a v-gon to a point. These contractions transform each disc corresponding to an M-square into a bounding digon. Finally, to obtain $M^t = (G_M, S_M)$ we thin each such bounding digon to an edge. See Fig. 2.

To go from a map $M^t = (G_M, S_M)$ to the corresponding gem M we reverse

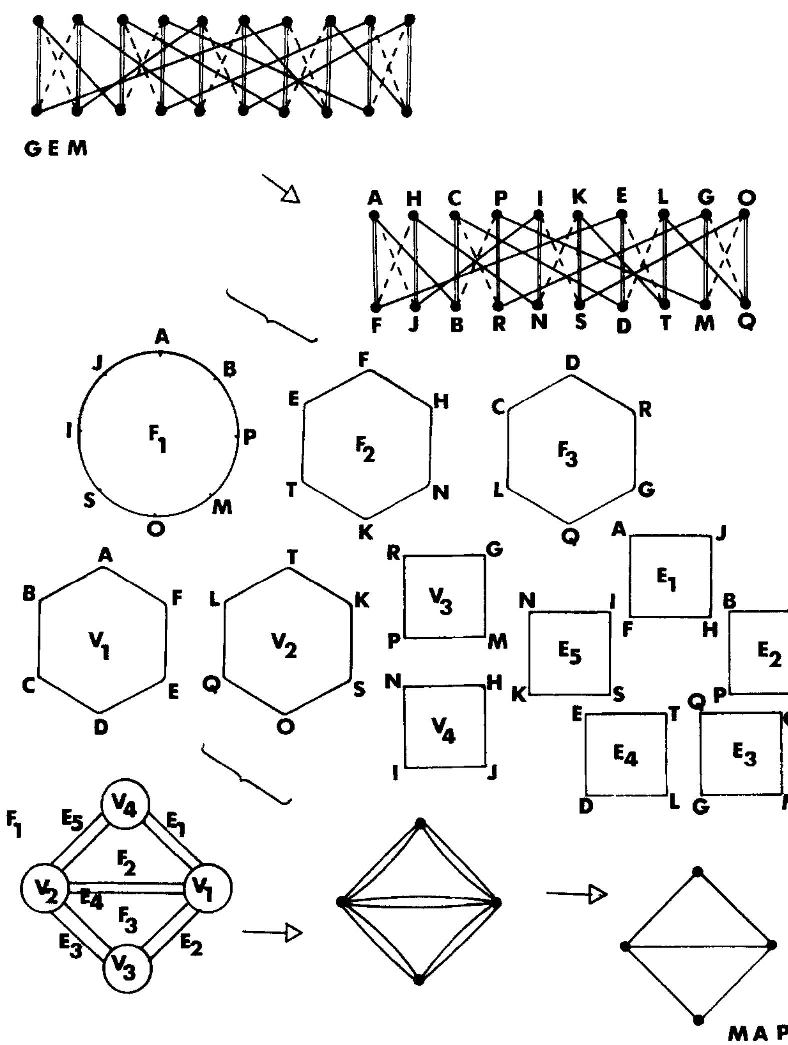

FIGURE 2

the process just described. Thus we start by replacing each edge by a bounding digon in S_M. Following that we replace each vertex by a disc in such a way as to obtain a cubic graph C_M embedded in S_M forming a map N^t. There are three kinds of faces in N^t:

(i) the ones which correspond to the faces of M^t;

(ii) the ones which correspond to the vertices of G_M;

(iii) the ones which correspond to the edges of G_M (and which are now bounded by polygons with four edges).

Paint with a_M the edges of C_M which separate faces of types (i) and (ii); paint with v_M the edges of C_M which separate faces of types (ii) and (iii); finally, paint with f_M the edges of C_M which separate faces of types (iii) and (i). This 3-coloration of the edges of C_M makes it into a gem M. Clearly, the map M^t can be recovered from gem M by considering its bigons and repeating the process to go from M to M^t.

The above considerations establish a bijective correspondence ${}^t: M \leftrightarrow M^t$ between gems and maps.

At first we obtained the above correspondence indirectly by considering the permutation model for a map which appears in [10]. In [4] the connection between gems and the permutation model, as well as between gems and maps described above, is presented. While a previous version of this paper was being refereed the editors pointed out to us that there is a brief mention of what we call gems on page 45 of [8].

The principal advantage of the gem model is that it naturally replaces the two-dimensional structure of a map by a colored one-dimensional structure. Thus we free ourselves from the burden of the representation of the surfaces by planar diagrams which make use (see Figs. 1 and 3) of handles, cross caps and stereographic projections. It also cleans some definitions: consider the definition of zigzag walk and compare it with the corresponding concept, z-gon, which we now describe.

Recall that Q_M is the 4-regular graph obtained from gem M by adding to C_M edges of a fourth color z_M which are going to form diagonals for each M-square. A *z-gon* in a gem M is a polygon in Q_M whose edges are colored with a_M and z_M. We have the following table of correspondences, whose justifications are straightforward consequences of the correspondence between gems and maps:

Map M^t	Gem M
vertex	v-gon
face	f-gon
edge	M-square
zigzag	z-gon

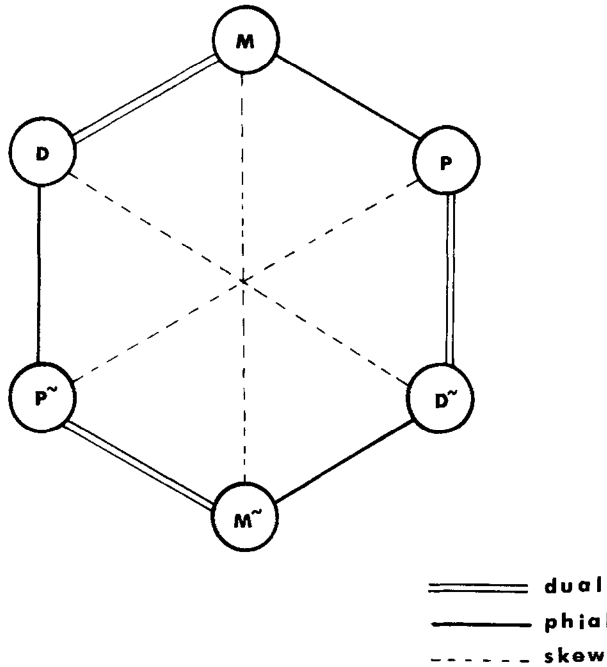

FIGURE 3

3. DUALITIES

In this section we use the gem model to present three natural dualities which are interpreted in the context of maps. Two of these dualities are not directly apparent from the maps themselves and as far as we know they have not appeared in the literature before.

Consider a gem M and its associated 4-regular, 4-edge-colored graph Q_M. By leaving out of Q_M edges of one of the three colors v_M, f_M, z_M and permuting the other two, we get a set of six gems symbolized, defined, and named as follows:

Symbol	Definition	Name
M	$(Q_M \backslash z_M, v_M, f_M)$	(reference) gem
$D_M = D$	$(Q_M \backslash z_M, f_M, v_M)$	*dual* gem
$P_M = P$	$(Q_M \backslash v_M, z_M, f_M)$	*phial* gem
$M^{\sim}$	$(Q_M \backslash f_M, v_M, z_M)$	*skew* gem
$D^{\sim}$	$(Q_M \backslash v_M, f_M, z_M)$	*skew–dual* gem
$P^{\sim}$	$(Q_M \backslash f_M, z_M, v_M)$	*skew–phial* gem

We observe that in the definition of a gem X above the first entry is C_X, the second is v_X and the third is f_X. We note also that $a_X = a_M$ for every X in $\{M, D, P, M^{\sim}, D^{\sim}, P^{\sim}\}$.

It is an easy observation that the set of six maps, as above, associated with an arbitrary map M is closed under the following three operations, which are involutions:

(a) take the dual;

(b) take the phial;

(c) take the skew.

Note that from the definitions we have:

(i) the dual gem maintains the z-gons and interchanges the v-gons and f-gons;

(ii) the phial gem maintains the f-gons and interchanges the v-gons and z-gons;

(iii) the skew gem maintains the v-gons and interchanges the f-gons and the z-gons.

Thus one class of object is maintained and the other two are interchanged, while there is a natural correspondence between the squares. The interchanges of the v- and f-gons by dual gems, of the v- and z-gons by phial gems, and of the f- and z-gons by skew gems are called *gem dualities*. The complete structure of the gem dualities is presented in Fig. 3.

From the structure of the gem dualities we can see, for instance, that the skew gem is equal to the dual of the phial of the dual gem; it is also equal to the phial of the dual of the phial gem. We can observe also the perfect abstract symmetry of the three dualities.

The proof of the next two theorems, which are the main results of this section, are made very simple by the gem model. In the statements of these theorems, since there is no global way to distinguish clockwise from counter-clockwise in non-orientable surfaces, we talk about the *pair* of inverse cyclic sequences of edges around a vertex of a map. For symmetry we also talk about pairs of inverse closed walks when referring to the facial and zigzag walks.

THEOREM 1. *Given any map $M^t = (G_M, S_M)$, there exists a map $(M^t)^{\sim}$, called the* skew *of M^t, which is another embedding of G_M, possibly in another surface, and satisfies the following conditions*:

(i) *the sets of pairs of inverse cyclic sequences of the edges around the vertices of G_M are the same in M^t and $(M^t)^{\sim}$*;

(ii) *the sets of pairs of inverse cyclic sequences of the edges around the facial and zigzag walks of* M^t *and* $(M^t)^{\sim}$ *are interchanged.*

Proof. Given map M^t, consider gem M and obtain gem $M^{\sim}$. Take $(M^t)^{\sim}$ to be the map $(M^{\sim})^t$. ∎

THEOREM 2. *Given any map* $M^t = (G_M, S_M)$, *there exists another map* $P^t = (G_P, S_P)$ *called the* phial *of* M^t, *possibly in another surface, such that the edge-sets of* G_M *and* G_P *are the same and satisfy the following conditions*:

(i) *the sets of pairs of inverse cyclic sequences of the edges around the facial walks are the same in* M^t *and* P^t;

(ii) *the sets of pairs of inverse cyclic sequences of the edges around the vertices and around the zigzag walks of* M^t *and* P^t *are interchanged.*

Proof. Given map M^t consider gem M and obtain gem P. Map P^t is the required map. ∎

Thus the gem dualities induce, as we made clear, the *map dualities*. The map duality associated with pairs of dual gems is the usual duality that interchanges vertices and faces while maintaining the zigzag walks. In Fig. 4 we present the dualities in the case that M^t is the cube map. The dual map D^t is, in this case, the usual octahedron map. The phial map P^t is a special embedding of a square with each edge replaced by three edges linking the same vertices, in a non-orientable surface of connectivity 4, as shown by the two non-orientable handles labeled A and B. Finally, the skew map $(M^t)^{\sim}$ is a special embedding of the 1-skeleton of a cube in the familiar torus.

In Chapter 2 of [4] the map dualities are used in the study of a class Ω of graphs which are called *map-rich* and which satisfy the following property: If G_0 is in Ω then there exist G_1 and G_2 in Ω with the same edge-set as G_0 such that, for $i = 0, 1, 2$, each circuit in G_i is expressible as the symmetric difference of a coboundary in G_{i+1} and a coboundary in G_{i+2} (subscripts mod 3). The three graphs are G_M, G_D and G_P, where M^t is a special kind of map characterized by the fact that the cycle space (mod 2) of G_M is generated by the sum of the coboundary spaces of G_D and G_P. The class Ω is shown to include the planar and the complete graphs. The Petersen graph with all the edges doubled is shown not to be in Ω. A topological characterization of Ω, similar to Whitney's [12] characterization of abstract duality in graphs by means of planarity, is given. Loosely speaking it is the following: $G \in \Omega$ if and only if G has a "special" type of embedding in a surface determined by parameters in G itself.

To conclude this paper we show in the next section the counterpart for gems M of the orientability of S_M: bipartiteness of C_M.

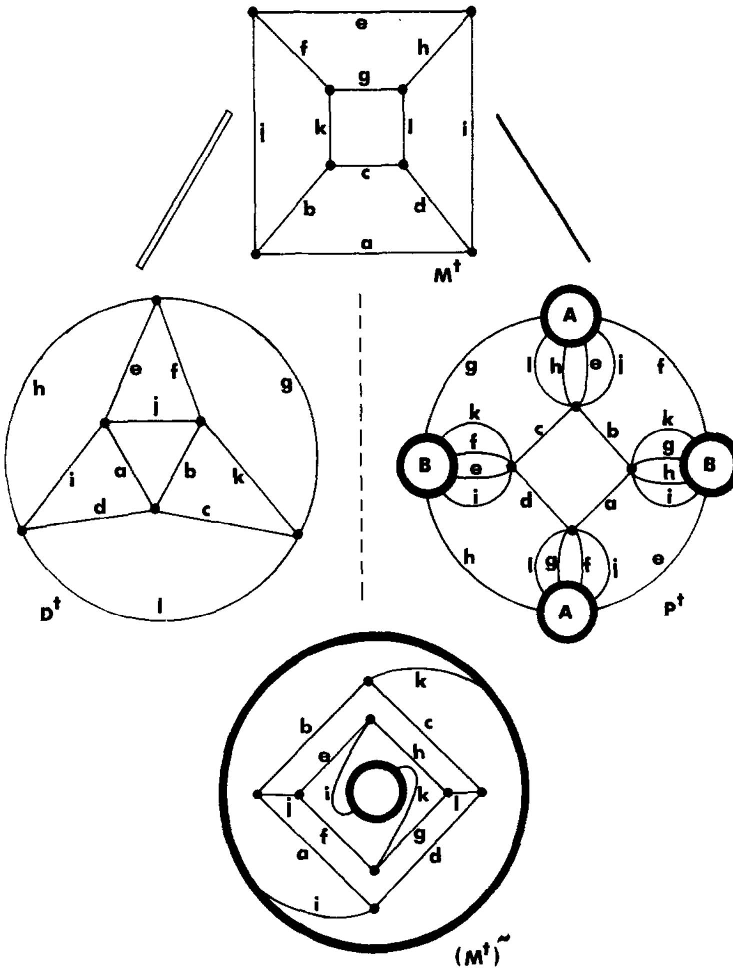

FIGURE 4

4. ORIENTABILITY AND BIPARTITENESS

Consider a surface S. It is well known (see [2] or [6]) that S is characterized by a number $\chi(S)$, called the Euler characteristic of S, and by specifying whether or not S is orientable. Moreover, it is also well known that this information is provided by any cellular embedding of an arbitrary graph G in S as follows:

(a) $\chi(S) = f(G, S) + v(G) - e(G)$, where $f(G, S)$ denotes the number of faces of the embedding of G in S, $v(G)$ and $e(G)$ are, respectively, the number of vertices and the number of edges of G. It is a fact that $\chi(S)$ is a topological invariant of S and so independent of the graph G and of how it is embedded in S.

(b) To define orientability, let us put an arrow on each edge of G so as to give an arbitrary but fixed orientation to it. The surface S is *orientable* if it is possible to assign an orientation for the traversal of each facial walk such that each edge (which appears twice in the facial walks) occurs once as a direct edge and once as a reverse edge. Here, again, orientability is a topological property of S which does not depend on G and on how it is embedded. See [6] or [2].

Observe that from a gem M we can obtain easily $\chi(S_M)$, where S_M is the surface of the associated map $M^t = (G_M, S_M)$. Consider the map $N^t = (C_M, S_M)$. The number of its faces is the number of bigons in M. We also have the number of vertices and edges of C_M, whence $\chi(S_M)$ is determined. The corollary of the next theorem shows that the orientability of S_M is also easily "readable" from the gem M. To state the theorem we first define the concept of faithful embedding of a cubic 3-edge-colored graph.

Let G be a cubic, properly 3-edge-colored graph. The *faithful* embedding of G is the unique embedding of G in which the faces are bounded by the polygons of G which are colored with two colors, the *bigons* of G. The uniqueness of such an embedding is given by the uniqueness of the quotient topology formed by the identifications of the boundaries of the discs bounded by the bigons of G. See Section 2.11 of [7]. We have the following theorem:

THEOREM 3. *Let G be a cubic, properly 3-edge-colored graph. The surface of the faithful embedding of G is orientable if and only if G is bipartite.*

Proof. Call the three colors a, b, c. Assume first that the surface of the faithful embedding is orientable. Consider disjoint discs bounded by the bigons of G. Since the embedding is orientable we can embed all these discs in the plane such that for any given edge we can "slide" one of the discs whose boundary contains one occurrence of the edge, so as to "glue" it with its second occurrence (in the boundary of another disc) with the correct orientation, without the need of "turning over" one of the discs. Therefore, after identification along an edge, it has an end in which the cyclic sequence of colors of edges around it is of type $a - b - c$ clockwise, and an end where it is $a - b - c$ counterclockwise. Also, if two edges are incident to the same vertex, then this vertex is of type clockwise or counterclockwise for both edges. Thus we can partition the vertices of G into clockwise and counterclockwise vertices. If we note that each edge of G links a clockwise vertex to a counterclockwise one, it follows that G is bipartite, as desired.

Conversely, assume that G is bipartite. Orient the edges of G consistently from one class of vertices to the other, such that every vertex of one class is a source and every vertex in the other class is a sink. Traverse each *ab*-bigon of G such that the a edges are direct edges and the b edges are reverse edges. Traverse each *bc*-bigon of G such that the b edges are direct edges and the c edges are reverse edges. Finally, traverse each *ca*-bigon of G such that the c edges are direct edges and the a edges are reverse edges. Since in the facial walks of the faithful embedding of G each edge appears once as a direct edge and once as a reverse edge, it follows that the faithful embedding of G is in an orientable surface. ∎

COROLLARY. *The surface S_M of the map M^t, which corresponds to the gem M, is orientable if and only if C_M is bipartite.*

Proof. The surface S_M is the surface of the faithful embedding of C_M. Thus the corollary is a direct consequence of Theorem 3. ∎

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