Discretization methods for optimal control problems with state constraints

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Abstract

We consider an optimal control problem described by nonlinear ordinary differential equations, with control and state constraints, including pointwise state constraints. Since this problem may have no classical solutions, it is also formulated in relaxed form. The classical control problem is then discretized by using the implicit midpoint scheme, while the controls are approximated by (not necessarily continuous) piecewise linear classical controls. We first study the behavior in the limit of properties of discrete optimality, and of discrete admissibility and extremality. We then apply a penalized gradient projection method to each discrete classical problem, and also a corresponding progressively refining combined discretization-optimization method to the continuous classical problem, thus reducing computing time and memory. We prove that accumulation points of sequences generated by these methods are admissible and extremal in some sense for the corresponding discrete or continuous, classical or relaxed, problem. For nonconvex problems whose solutions are nonclassical, we show that we can apply the above methods to the problem formulated in Gamkrelidze relaxed form. Finally, numerical examples are given.

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1. Introduction

The scope of this work is to present a combined discretization-optimization approach to the numerical solution of optimal control problems with control and state constraints, using discrete classical controls as a tool, and examining the behavior in the limit of the methods in the two frameworks of classical and relaxation theory.

We consider an optimal control problem described by nonlinear ordinary differential equations, with control and state constraints, including pointwise state constraints. Since this problem may have no classical solutions, we also formulate it in relaxed form, using Young measures. The classical control problem is then discretized by using implicit midpoint schemes for the approximation of the states and adjoints, the midpoint integration rule for the approximation of the integrals involved in the derivatives of the functionals, while the controls are approximated by (not necessarily continuous) classical controls that are piecewise linear on successive pairs of intervals, in order to be uniquely defined by pairs of midpoint values. We have chosen here the midpoint scheme for simplicity of presentation and because it yields second-order approximations of the states and adjoints, if these are continuous and piecewise smooth, and gives a purely symmetric matching backward scheme for the adjoint discretization; also, the midpoint scheme is consistent with the Hamiltonian minimization (see [7]). A second equivalent possibility is to use the second-order trapezoidal explicit Runge–Kutta scheme, with (discontinuous) simple piecewise linear controls, but the Hamiltonian derivative w.r.t. the control involves then the computation of composite multivariable vector functions, which is tedious, especially for large systems. One can of course use higher order Runge–Kutta schemes, in which case it seems more efficient to use approximate non-matching discrete adjoints and functional derivatives, instead of the matching ones, which involve heavy computations (see [6,1]); but then, the behavior in the limit and relaxation parts of the theory are lost, due to the approximate derivatives. On the other hand, piecewise linear controls yield better approximations of the extremal (or optimal) control than piecewise constant ones, if this control is continuous and piecewise smooth, with possible a priori known discontinuity points, after an extra approximation procedure of the a priori unknown discontinuity points of the control derivative (see [1, Numerical Examples]). Discontinuous piecewise linear controls (free segments), as compared to continuous ones, simplify the discrete minimizations involving the Hamiltonian in the Algorithm. We first give various useful necessary conditions for optimality for the continuous classical and relaxed problems, and for the discrete problem. Next, we show that strong accumulation points in $L^2$ of sequences of optimal (resp. admissible and extremal) discrete controls are optimal (resp. admissible and weakly extremal) for the continuous classical problem, and that relaxed accumulation points of sequences of optimal (resp. admissible and extremal) discrete controls are optimal (resp. admissible and weakly extremal) for the continuous relaxed problem. We then apply a penalized gradient projection method to each discrete classical problem, and also a corresponding combined discretization-optimization method to the continuous classical problem, that progressively refines the discretization during the iterations, thus reducing computing time and memory, especially for large systems. We prove that accumulation points of sequences generated by the fixed discretization method are admissible and extremal for each discrete problem, and that strong classical (resp. relaxed) accumulation points of sequences of discrete controls generated by the progressively refining method are admissible and weakly extremal classical (resp. relaxed) for the continuous classical (resp. relaxed) problem. For nonconvex problems whose solutions are nonclassical, we show that we can apply the above methods to the problem formulated in Gamkrelidze relaxed form; using a standard procedure, the computed Gamkrelidze controls can then be approximated by classical ones. Finally, several
The outline of this paper is the following:

Section 2: Formulation of the continuous problem in classical and in relaxed form. Existence and necessary conditions for optimality.

Section 3: Formulation of the discrete classical problems. Existence and necessary conditions for discrete optimality.

Section 4: Behavior in the limit of discrete optimality, and of discrete admissibility and extremality.

Section 5: Combined discretization-optimization methods.

Section 6: Numerical examples.

2. The continuous optimal control problems

Consider the following optimal control problem. The state equation is given by

\[ y'(t) = f(t, y(t), w(t)), \quad \text{for } t \in I = [0, T], \quad y(0) = y^0, \]

where \( y(t) \in \mathbb{R}^d \), the constraints on the control \( w(t) \in U \), for \( t \in I \), where \( U \) is a compact subset of \( \mathbb{R}^d' \), the constraints on the state \( y = y_w \) are

\[ G_1(w) = \bar{g}_1(y(T)) + \int_0^T g_1(t, y(t), w(t)) \, dt = 0, \]
\[ G_2(w) = \bar{g}_2(y(T)) + \int_0^T g_2(t, y(t), w(t)) \, dt \leq 0, \]
\[ G_3(w)(s) = g_3(s, y(s)) \leq 0, \quad \text{for } s \in I, \]

where the vector functions \( \bar{g}_l, g_l \) take values in \( \mathbb{R}^{m_l}, l = 1, 2, \) and \( g_3 \) in \( \mathbb{R}^{m_3} \), and the cost functional to be minimized

\[ G_0(w) = \bar{g}_0(y(T)) + \int_0^T g_0(t, y(t), w(t)) \, dt. \]

The set of classical controls is defined by

\[ W = \{ w : I \to U \mid w \text{ measurable} \} \subset L^2(I, \mathbb{R}^{d'}), \]

and the set of relaxed controls (for the relevant theory, see [14,10]) by

\[ R = \{ r : I \to M_1(U) \mid r \text{ weakly measurable} \} \subset L^\infty_w(I, M(U)) \equiv L^1(I, C(U))^*, \]

where \( M(U) \) (resp. \( M_1(U) \)) is the set of Radon (resp. probability) measures on \( U \). The set \( W \) (resp. \( R \)) is endowed with the relative strong (resp. weak star) topology, and \( R \) is convex, metrizable and compact. If each classical control \( w(\cdot) \) is identified with its associated Dirac relaxed control \( r(\cdot) := \delta_{w(\cdot)}, \) then \( W \) may be considered as a subset of \( R \), and \( W \) is thus dense in \( R \). For a given \( \phi \in L^1(I; C(U) \otimes \mathbb{R}^n) \) (or
equivalently $\phi \in B(I, U; \mathbb{R}^p)$, where $B$ is the set of Caratheodory functions in the sense of Warga [14]) and $r \in R$, we use the notation

$$\phi(t, r(t)) = \int_U \phi(t, u)r(t)(du).$$

We can now define the relaxed problem. The state equation is

$$y'(t) = f(t, y(t), r(t)), \quad t \in I, \quad y(0) = y^0,$$

where $y = y_r$, the control constraint $r \in R$, and the state constraints and cost are defined as in the classical problem, but with $w$ replaced by $r$, with the above notation.

We define the norms $\|x\| = (\sum_{i=1}^p x_i^2)^{1/2}$, $\|x\|_1 = \sum_{i=1}^p |x_i|$, $\|x\|_\infty = \max_{i=1, \ldots, p} |x_i|$, in $\mathbb{R}^p$, and denote by $\|\cdot\|_{L^2}$, $\|\cdot\|_{L^1}$, $\|\cdot\|_{L^\infty}$, $\|\cdot\|_L$ the corresponding usual norms in $L^2(I)^p$, $L^1(I)^p$, $L^\infty(I)^p$, $C(I)^p$, respectively. We denote by $M(I) \equiv C(I)^*$ the set of finite regular measures on $I$, and by $\|\cdot\|_*$ the norm in $M(I)$ defined by $\|\mu\|_* = \int_I |\mu|(dt)$ (with $|\mu| = \mu$ if $\mu$ is positive). The order relations between vectors, functions or vector functions, are defined componentwise and/or pointwise.

We suppose in the sequel that the function $f$ is defined on $I \times \mathbb{R}^d \times U$, measurable for $y, u$ fixed, continuous for $t$ fixed, and satisfies

$$\|f(t, y, u)\| \leq \psi(t) + \beta\|y\|, \quad \text{for every } (t, y, u) \in I \times \mathbb{R}^d \times U,$$

with $\psi \in L^1(I)$, $\beta \geq 0$, and

$$\|f(t, y_1, u) - f(t, y_2, u)\| \leq L\|y_1 - y_2\|, \quad \text{for every } (t, y_1, y_2, u) \in I \times \mathbb{R}^d \times U.$$

The following theorem is standard (see [14]).

**Theorem 2.1.** For every relaxed (or classical, since $W \subset R$) control $r \in R$, the state equation has a unique absolutely continuous solution $y = y_r$. Moreover, there exists a constant $b$ such that $\|y_r\|_\infty \leq b$, for every control $r \in R$.

Let $B$ denote the closed ball in $\mathbb{R}^d$ with center 0 and radius $b$ (see Theorem 2.1). We suppose now in addition that the functions $g_l$, $l = 0, 1, 2$, are defined on $I \times B \times U$, measurable for fixed $y, u$, continuous for fixed $t$, and such that

$$\|g_l(t, y, u)\| \leq \zeta_l(t), \quad \text{for every } (t, y, u) \in I \times B \times U$$

with $\zeta_l \in L^1(I)$, and that the functions $\tilde{g}_l, \tilde{g}_{ly}$ are continuous on $B$. The results of the following theorem are proved in [14].

**Theorem 2.2.** The mappings $G_l : W \lor R \rightarrow \mathbb{R}^{m_l}$, $l = 0, 1, 2$, and $G_3 : W \lor R \rightarrow C(I)^{m_3}$ are continuous on $W \lor R$. If the relaxed problem is feasible, then it has a solution.

Note that in the classical problem we have $y'(t) \in f(t, y(t), U)$ (velocity set), while in the relaxed problem $y'(t) \in \text{co}(f(t, y(t), U))$. The classical problem may have no classical solution, and since $W \subset R$, we have in general

$$c_R := \min_{\text{constraints on } r} G_0(r) \leq \inf_{\text{constraints on } w} G_0(w) := c_W,$$
where the equality holds, in particular, if there are no state constraints, since $W$ is dense in $R$. Since usually approximation methods slightly violate the state constraints, approximating an optimal relaxed control by a relaxed or a classical control, hence the relaxed optimal cost $c_R$, is not a drawback in practice (see [14, p. 248]). Note also that approximating sequences of classical controls may converge to relaxed ones.

In order to state the various necessary conditions for optimality, we suppose in addition that the functions $f$, $g_l$, $f_y$, $f_u$, $g_{ly}$, $g_{lu}$ are defined on $I \times B' \times U'$, where $B'$ (resp. $U'$) is an open set containing $B$ (resp. $U$), measurable on $I$ for fixed $(y, u) \in B \times U$, continuous on $B \times U$ for fixed $t \in I$, and such that

$$\|f_y(t, y, u)\| \leq \zeta(t), \quad \|f_u(t, y, u)\| \leq \eta(t),$$

$$\|g_{ly}(t, y, u)\| \leq \zeta_{l1}(t), \quad \|g_{lu}(t, y, u)\| \leq \zeta_{l2}(t),$$

for every $(t, y, u) \in I \times B \times U$, with $\zeta, \eta, \zeta_{l1}, \zeta_{l2} \in L^1(I)$, and that the functions $\tilde{g}_{ly}$ are continuous. The results of the following theorem are proved using the techniques of [14].

**Theorem 2.3.** (i) If $U$ is convex, then, for $w, w' \in W$, the directional derivative of the mapping $G_l$, for $l = 0, 1, 2$, defined on $W$, is given by

$$DG_l(w, w' - w) = \lim_{x \to 0^+} \frac{G_l(w + x(w' - w)) - G_l(w)}{x} = \int_0^T [z_l(t)f_u(t, y(t), w(t)) + g_{lu}(t, y(t), w(t))][w'(t) - w(t)]\,dt,$$

where $y = y_w$, and the adjoint state $z_l = z_{lw}$, a row vector function ($l = 0$), or a matrix function ($l = 1, 2$), is defined by the adjoint equation

$$z_l'(t) = -z_l(t)f_y(t, y(t), w(t)) - g_{ly}(t, y(t), w(t)), \quad t \in I,$$

$$z_l(T) = \tilde{g}_{ly}(y(T)), \quad \text{with} \quad y = y_w,$$

where the controls are considered as purely classical. The directional derivative of $G_3 : W \to C(I)^{m_3}$, is given by the matrix function

$$DG_3(w, w' - w)(s) = g_{3y}(s, y(s))Z(s)^{-1} \int_0^s Z(t)f_u(t, y(t), w(t))[w'(t) - w(t)]\,dt, \quad s \in I,$$

where the matrix function $Z = Z_w$ satisfies the fundamental matrix equation

$$Z'(t) = -Z(t)f_y(t, y(t), w(t)), \quad t \in I,$$

$$Z(T) = E \quad (\text{identity matrix}).$$

(ii) For $r, r' \in R$, the directional derivative of the mapping $G_l$, for $l = 0, 1, 2$, defined on $R$, is given by

$$DG_l(r, r' - r) := \lim_{x \to 0^+} \frac{G_l(r + x(r' - r)) - G_l(r)}{x} = \int_I [z_l(t)f_y(t, y(t), r'(t) - r(t)) + g_l(t, y(t), r'(t) - r(t))][w'(t) - w(t)]\,dt.$$
where \( y = y_r \), and the relaxed adjoint \( z_l = z_{l_r} \) is defined by

\[
z_l(t) = -z_l(t) f_y(t, y(t), r(t)) - g_{ly}(t, y(t), r(t)), \quad t \in I,
\]

\[
z_l(T) = \bar{g}_{ly}(y(T)), \quad \text{with} \quad y = y_r.
\]

The directional derivative of \( G_3 : \mathbb{R} \to C(I)^{m_3} \) is given by

\[
DG_3(r, r' - r)(s) = g_{3y}(s, y(s))Z(s)^{-1} \int_0^s Z(t)f_u(t, y(t), r(t) - r(t)) \, dt, \quad s \in I,
\]

where \( Z = Z_r \) is defined as in (i), but with \( w \) replaced by \( r \).

(iii) The mappings

\[(w, w') \mapsto DG_l(w, w' - w) \text{ (resp. } (r, r') \mapsto DG_l(r, r' - r)), \quad l = 0, 1, 2, 3,
\]

are continuous on \( W \times W \) (resp. \( R \times R \)).

In the above notations of \( DG_l \), it is understood, depending on the notation used for the arguments, that the directional derivative is taken in the corresponding space (\( W \) or \( R \)) on which \( G_l \) is defined. Next, we give necessary conditions for optimality.

Theorem 2.4. (i) If \( U \) is convex and the control \( w \in W \) is optimal for the classical problem, then \( w \) is weakly extremal classical, i.e., there exist multipliers

\[
\lambda_0 \in \mathbb{R}, \lambda_1 \in \mathbb{R}^{m_1}, \lambda_2 \in \mathbb{R}^{m_2}, \lambda_3 \in [C(I)^{m_3}]^* \equiv M(I)^{m_3}
\]

with \( \lambda_0 \geq 0, \lambda_2 \geq 0, \lambda_3 \geq 0 \), \( \sum_{l=0}^2 \| \lambda_l \| + \| \lambda_3 \|_* = 1 \), where \( \| \lambda_3 \|_* = \sum_{j=1}^{m_3} \| \lambda_{j3} \|_{M_1} \),

such that

\[
\sum_{l=0}^2 \lambda_l DG_l(w, w' - w) + \int_0^T \lambda_3(ds)DG_3(w, w' - w)(s)
\]

\[
= \sum_{l=0}^2 \lambda_l \int_0^T \left[ z_l(t) f_u(t, y(t), w(t)) + g_{lu}(t, y(t), w(t)) \right] [w'(t) - w(t)] \, dt
\]

\[
+ \int_0^T \lambda_3(ds)g_{3y}(s, y(s))Z(s)^{-1} \int_0^s Z(t)f_u(t, y(t), w(t)) [w'(t) - w(t)] \, dt
\]

\[
= \int_0^T \left\{ \sum_{l=0}^2 \lambda_l [z_l(t) f_u(t, y(t), w(t)) + g_{lu}(t, y(t), w(t))]ight.
\]

\[
+ \left. \left( \int_t^T \lambda_3(ds)g_{3y}(s, y(s))Z(s)^{-1} \right) Z(t)f_u(t, y(t), w(t)) \right\} [w'(t) - w(t)] \, dt
\]

\[
\geq 0, \quad \text{for every } w' \in W,
\]
and

\[ \dot{\lambda}_2 G_2(w) = 0, \quad \int_0^T \dot{\lambda}_3(ds)G_3(w)(s) = 0 \quad \text{(transversality conditions).} \]

The above inequalities are equivalent to the pointwise weak classical minimum principle

\[
\left\{ \sum_{l=0}^{2} \dot{\lambda}_l[z_l(t)f_u(t, y(t), w(t)) + g_{lu}(t, y(t), w(t))] \right. \\
+ \left. \left[ \int_t^T \dot{\lambda}_3(ds)gz_3y(s, y(s))Z(s)^{-1} \right] Z(t)f_u(t, y(t), w(t)) \right\} w(t) \\
= \min_{u \in U} \left\{ \sum_{l=0}^{2} \dot{\lambda}_l[z_l(t)f_u(t, y(t), w(t)) + g_{lu}(t, y(t), w(t))] \\
+ \left[ \int_t^T \dot{\lambda}_3(ds)gz_3y(s, y(s))Z(s)^{-1} \right] Z(t)f_u(t, y(t), w(t)) \right\} u \}, \quad \text{for a.a. } t \in I.
\]

(ii) If the control \( r \in R \) is optimal for the relaxed problem, then \( r \) is extremal relaxed, i.e., there exist multipliers as in (i), such that

\[
\sum_{l=0}^{2} \dot{\lambda}_l DG_1(r, r' - r) + \int_0^T \dot{\lambda}_3(ds)DG_3(r, r' - r)(s) \\
= \sum_{l=0}^{2} \dot{\lambda}_l \int_0^T [z_l(t)f(t, y(t), r'(t) - r(t)) + g_l(t, y(t), r'(t) - r(t))] \, dt \\
+ \int_0^T \dot{\lambda}_3(ds)gz_3y(s, y(s))Z(s)^{-1} \int_0^s Z(t)f(t, y(t), r'(t) - r(t)) \, dt \\
= \int_0^T \left\{ \sum_{l=0}^{2} \dot{\lambda}_l[z_l(t)f(t, y(t), r'(t) - r(t)) + g_l(t, y(t), r'(t) - r(t))] \\
+ \left[ \int_t^T \dot{\lambda}_3(ds)gz_3y(s, y(s))Z(s)^{-1} \right] Z(t)f(t, y(t), r'(t) - r(t)) \right\} \, dt \geq 0, \quad \text{for every } r' \in R,
\]

and

\[ \dot{\lambda}_2 G_2(r) = 0, \quad \int_0^T \dot{\lambda}_3(ds)G_3(r)(s) = 0 \quad \text{(transversality conditions).} \]
The above inequalities are equivalent to the pointwise strong relaxed minimum principle

\[ \sum_{l=0}^{2} \lambda_l [ z_l(t) f(t, y(t), r(t)) + g_l(t, y(t), r(t)) ] + \left[ \int_t^T \lambda_3(s) g_{3y}(s, y(s)) Z(s)^{-1} \right] Z(t) f(t, y(t), r(t)) \]
\[ + \lambda_1(t) f(t, y(t), r(t)) \]
\[ = \min_{u \in U} \left\{ \sum_{l=0}^{2} \lambda_l [ z_l(t) f_u(t, y(t), u) + g_{lu}(t, y(t), u) ] + \left[ \int_t^T \lambda_3(s) g_{3y}(s, y(s)) Z(s)^{-1} \right] Z(t) f_u(t, y(t), u) \right\}, \text{ for a.a. } t \in I. \]

If \( U \) is convex, then this minimum principle implies the pointwise weak relaxed minimum principle

\[ \left\{ \sum_{l=0}^{2} \lambda_l [ z_l(t) f_u(t, y(t), r(t)) + g_{lu}(t, y(t), r(t)) ] + \left[ \int_t^T \lambda_3(s) g_{3y}(s, y(s)) Z(s)^{-1} \right] Z(t) f_u(t, y(t), r(t)) \right\} r(t) \]
\[ = \min_{\phi} \left\{ \sum_{l=0}^{2} \lambda_l [ z_l(t) f_u(t, y(t), r(t)) + g_{lu}(t, y(t), r(t)) ] + \left[ \int_t^T \lambda_3(s) g_{3y}(s, y(s)) Z(s)^{-1} \right] Z(t) f_u(t, y(t), r(t)) \right\} \phi(t, r(t)), \text{ for a.a. } t \in I, \]

where the minimum is taken over the set \( B(I, U; U) \) of Carathéodory functions (see [14]) \( \phi : I \times U \to U \), which in turn implies the global weak relaxed condition

\[ \int_0^T \left\{ \sum_{l=0}^{2} \lambda_l [ z_l(t) f_u(t, y(t), r(t)) + g_{lu}(t, y(t), r(t)) ] + \left[ \int_t^T \lambda_3(s) g_{3y}(s, y(s)) Z(s)^{-1} \right] Z(t) f_u(t, y(t), r(t)) \right\} [\phi(t, r(t)) - r(t)] \, dt \geq 0, \]

for every \( \phi \in B(I, U; U) \).

A control \( r \) satisfying this condition and the above transversality conditions is called weakly extremal relaxed.

**Proof.** The mappings \( G_l, l = 0, 1, 2, 3 \) are continuous on \( W \). Since \( DG_l \), for \( l = 0, 1, 2, 3 \), is continuous w.r.t. \((w, w')\) and linear w.r.t. \( w' - w \), \( G_l \) is \( p \)-differentiable for every integer \( p \) (in particular for \( p = m_1 \)), in the sense of [14]. The classical global condition and the transversality conditions follow then from the multiplier theorem V.3.2 in [14]. The strong relaxed necessary conditions follow similarly from the
general multiplier theorem V.2.3 in [14]. The equivalence of global and pointwise conditions in each case is standard (see [14]). Now, the strong relaxed minimum principle can be written in the compact form, for a.a. \( t \in I \), \( t \) fixed (and dropped)

\[
H(r) = \int_U H(u) r(du) \leq H(u), \quad \text{for every } u \in U.
\]

Let \( \phi : I \times U \to U \) be any Caratheodory function. Since \( U \) is convex here, we have

\[
\int_U H(u) r(du) \leq H(u + \epsilon(\phi(u) - u)), \quad \text{for every } u \in U, \ \epsilon \in [0, 1],
\]

hence

\[
\int_U H(u) r(du) \leq \int_U H(u + \epsilon(\phi(u) - u)) r(du).
\]

By the mean value theorem and the uniform continuity of \( H \) w.r.t. \( u \)

\[
0 \leq \int_U \frac{H(u + \epsilon(\phi(u) - u)) - H(u)}{\epsilon} r(du)
\]

\[
= \int_U H_u(u + \epsilon\mu(u)(\phi(u) - u)) [\phi(u) - u] r(du) \quad (0 \leq \mu(u) \leq 1)
\]

\[
= \int_U H_u(u) [\phi(u) - u] r(du) + z(\epsilon)
\]

where \( z(\epsilon) \to 0 \) as \( \epsilon \to 0 \), hence

\[
\int_U H_u(u) [\phi(u) - u] r(du) = H_u(r) [\phi(r) - r] \geq 0,
\]

for every Caratheodory function \( \phi \in B(I, U; U) \), a.e. in \( I \), which is the weak relaxed minimum principle. By integration, we obtain also the global weak relaxed condition

\[
\int_Q H_u(r) [\phi(r) - r] dx dt \geq 0. \quad \square
\]

3. The discrete problems

In the sequel, we suppose that the functions \( f, f_y, f_u, g_l, g_{ly}, g_{lu} \) are continuous in all their arguments. For each integer \( n \geq 0 \), let \( N^n \) be a positive integer with \( N^n = 2N^n \). We suppose that \( N^n \to \infty \) as \( n \to \infty \).

Set

\[
N = N^n, \quad N' = N^n, \quad h^n = T / N, \quad t^n_i = ih^n, \quad i = 0, \ldots, N, \quad I^n_i = [t^n_{i-1}, t^n_i], \quad i = 1, \ldots, N - 1, \quad I^n_N = [t^n_{N-1}, t^n_N].
\]

We first define the set of classical piecewise constant controls w.r.t. the \( I^n_i \)

\[
\tilde{W}^n = \{ \tilde{w}^n \in W | \tilde{w}^n(t) = \tilde{w}^n_i \in U \text{ on } I^n_i, \ i = 1, \ldots, N \}.
\]
and its subset $\tilde{W}^n$ of controls that are constant on the interior of each union of two successive intervals $I_{j-1} \cup I_j, j = 1, \ldots, N$. We then define the set of discrete controls $W^n$ as the set of (possibly discontinuous) classical controls that are linear (i.e., affine) on each union of two successive intervals $I_{j-1} \cup I_j, j = 1, \ldots, N$, uniquely defined by their values $\tilde{w}_i^n$ at the midpoints $\bar{t}_i^n = (t_{i-1}^n + t_i^n)/2$, and satisfy the constraints

$$\tilde{w}_{2j-1}^n - (\tilde{w}_{2j}^n - \tilde{w}_{2j-1}^n)/2 \in U, \quad \tilde{w}_{2j}^n + (\tilde{w}_{2j}^n - \tilde{w}_{2j-1}^n)/2 \in U, \quad j = 1, \ldots, N',$$

(which guarantee that $w^n(t) \in U, t \in I$, if $U$ is convex), and the (linear) Lipschitz constraints

$$\|\tilde{w}_{2j}^n - \tilde{w}_{2j-1}^n\|_\infty \leq L_0, \quad j = 1, \ldots, N'.$$

For a given discrete control $w^n \in W^n$, the discrete state $y^n = y^n_{w^n}$ is the solution of the implicit midpoint scheme

$$y_i^n = y_{i-1}^n + h^n f(\bar{t}_i^n, \bar{y}_i^n, \bar{w}_i^n), \quad i = 1, \ldots, N, \quad y_0^n = y^0,$$

with $\bar{y}_i^n = (y_{i-1}^n + y_i^n)/2$, $\bar{t}_i^n = (t_{i-1}^n + t_i^n)/2$.

**Remark.** Note that all the results in the sequel remain valid (with obvious simplifications) if we take $W^n := \tilde{W}^n$.

**Theorem 3.1.** For $h^n < 2/L$ and every $\tilde{w}^n \in \tilde{W}^n$, the corresponding discrete state $y^n = (y_0^n, \ldots, y_N^n)$ is uniquely defined and satisfies $\|y_i^n\| \leq b', i = 0, \ldots, N$.

**Proof.** The discrete scheme can be written in the form $y_i^n = F(y_i^n)$, where the mapping $F$, from $\mathbb{R}^d$ to $\mathbb{R}^d$, is defined by

$$F(y) = y_{i-1}^n + h^n f(\bar{t}_i^n, \bar{y}_i^n, \bar{w}_i^n).$$

For $y_1, y_2 \in \mathbb{R}^d$, we have

$$\|F(y_1) - F(y_2)\| \leq \frac{h^n}{2} L \|y_1 - y_2\|,$$

which shows that $F$ is a contraction for $h^n < 2/L$. Therefore $y_i^n$ is defined as the unique fixed point of $F$ in $\mathbb{R}^d$. On the other hand, since $f$ is continuous here, we remark that the Lipschitz continuity of $f$ implies that

$$\|f(t, y, u)\| \leq \|f(t, 0, u)\| + \|f(t, y, u) - f(t, 0, u)\| \leq C + L \|y\|.$$

Now, the discrete scheme yields by summation

$$y_i^n = y^0 + \sum_{j=1}^i h^n f(\bar{t}_j^n, \bar{y}_j^n, \bar{w}_j^n),$$

hence

$$\left(1 - L \frac{h^n}{2}\right) \|y_i^n\| \leq \left(1 - L \frac{h^n}{2}\right) \|y^0\| + Ch^n i + Lh^n \sum_{j=0}^{i-1} \|y_j^n\| \leq C' \|y^0\| + CT + Lh^n \sum_{j=0}^{i-1} \|y_j^n\|. $$
For $h^n < 2/L$, it then follows from the discrete Gronwall inequality (see [12]) that
\[ \|y^n_i\| \leq b', \quad i = 0, \ldots, N. \]

From now on, we suppose that $h^n < 2/L$, for every $n$. The discrete state equation can be solved numerically, for each $i$, by the standard predictor-corrector method, i.e., using the explicit Euler scheme as a predictor, and the contractive corrector iterative scheme (where only a few iterations are practically sufficient).

The discrete control constraint is $w^n \in W^n$. Defining the discrete mappings
\[ G^n_l(w^n) = \bar{g}_l(y^n_N) + h^n \sum_{i=1}^{N} g_l(\bar{r}_i^n, \bar{y}_i^n, \bar{w}_i^n), \quad l = 0, 1, 2, \]
\[ G^n_3(w^n)(j) = g_3(\bar{r}_i^n, \bar{y}_i^n), \quad j = 1, \ldots, N, \]
the discrete cost to be minimized is $G^n_0(w^n)$, and the discrete state constraints are either of the two following ones
- Case (a) $\|G^n_1(w^n)\| \leq \epsilon^n_1$ or
- Case (b) $G^n_1(w^n) = \epsilon^n_1$,

and
\[ G^n_2(w^n) \leq \epsilon^n_2, \]
\[ G^n_3(w^n)(j) \leq \epsilon^n_3, \quad j = 1, \ldots, N, \]
where the admissibility perturbations $\epsilon^n_l$ are positive numbers or vectors converging to zero (to be defined later).

**Theorem 3.2.** The mappings $w^n \rightarrow y^n$, $w^n \rightarrow G^n_l(w^n)$ are continuous on $W^n$. If any of the above discrete problems is feasible, then it has a solution.

**Proof (Sketch).** The continuity of the states $y^n = (y^n_i)$ is easily proved by induction on $i$, or by using the discrete Bellman–Gronwall inequality, and the continuity of the $G^n_l$ follow. \(\square\)

The proofs of the following two theorems essentially parallel the continuous case.

**Theorem 3.3.** If $U$ is convex, for $w^n$, $w'^n \in W^n$, the directional derivative of the mapping $G^n_l$ ($l = 0, 1, 2$), defined on $W^n$, is given by
\[ DG^n_l(w^n, w'^n - w^n) = h^n \sum_{i=1}^{N} [z^n_i f_{u}(\bar{r}_i^n, \bar{y}_i^n, \bar{w}_i^n) + g_{ly}(\bar{r}_i^n, \bar{y}_i^n, \bar{w}_i^n)](\bar{w}_i^n - \bar{w}_i'^n), \]
where the adjoint state $z^n_l = z^n_{lw^n}$ is defined by
\[ z^n_{l,i-1} = z^n_i + h^n[z^n_i f_{y}(\bar{r}_i^n, \bar{y}_i^n, \bar{w}_i^n) + g_{ly}(\bar{r}_i^n, \bar{y}_i^n, \bar{w}_i^n)], \quad i = N, \ldots, 1, \]
\[ z^n_{l,N} = \bar{g}_{ly}(y^n_N), \quad \text{with} \quad y^n = y^n_{w^n}. \]
The directional derivative of $G^n_3(j) : W^n \rightarrow \mathbb{R}^{m_3} (j = 1, \ldots, N)$ is given by

$$DG^n_3(w^n, w^m - w^n)(j) = h^n \sum_{i=1}^{N} z^n_{3i} f_u(\overline{t}^n_i, \overline{y}^n_i, \overline{w}^n_i) (w^m_i - \overline{w}^n_i),$$

where the adjoint $z^n_{3j} = z^n_{3w^n}$ is defined by

$$z^n_{3j} = z^n_{3i - 1} + h^n z^n_{3j} f_y(\overline{t}^n_j, \overline{y}^n_j, \overline{w}^n_j) + \sum_{i=1}^{N} \lambda^n_{3i} g_3(\overline{t}^n_i, \overline{y}^n_i, \overline{w}^n_i),$$

$$z^n_{3N} = 0, \quad \text{with } y^n = y^n_w.$$

The mappings $(w^n, w^m) \mapsto DG_l(w^n, w^m - w^n), l = 0, 1, 2, 3,$ are continuous on $W^n \times W^n$.

We now state the discrete necessary conditions for optimality.

**Theorem 3.4.** If $U$ is convex, and if $w^n$ is optimal for the discrete problem (with state constraints Case (b)), then it is discrete extremal classical, i.e., there exist multipliers

$$\lambda^n_0 \in \mathbb{R}, \quad \lambda^n_1 \in \mathbb{R}^{m_1}, \quad \lambda^n_2 \in \mathbb{R}^{m_2}, \quad \lambda^n_3 \in \mathbb{R}^{m_3 N},$$

with $\lambda^n_0 \geq 0, \quad \lambda^n_1 \geq 0, \quad \lambda^n_3 \geq 0, \quad \sum_{l=0}^{2} \| \lambda^n_l \| + h^n \sum_{j=1}^{N} \| \lambda^n_{3j} \|_1 = 1,$

such that

$$\sum_{l=0}^{2} \lambda^n_l DG^n_l(w^n, w^m - w^n) + h^n \sum_{j=1}^{N} \lambda^n_{3j} DG^n_3(w^n, w^m - w^n)(j) \geq 0, \quad \text{for every } w^m \in W^n,$$

and

$$\lambda^n_0 [G^n_2(w^n) - \varepsilon^n_0] = 0, \quad \lambda^n_3(j) [G^n_3(w^n)(j) - \varepsilon^n_3] = 0, \quad j = 1, \ldots, N.$$

Defining the complete Hamiltonian

$$H(t, y, z, u) = zf(t, y, u) + \sum_{l=0}^{2} \lambda^n_l g_l(t, y, u),$$

and the complete discrete adjoint

$$z^n_{i-1} = z^n_i + h^n \left\{ z^n_i f_y(\overline{t}^n_i, \overline{y}^n_i, \overline{w}^n_i) + \sum_{l=0}^{2} \lambda^n_l g_l(\overline{t}^n_i, \overline{y}^n_i, \overline{w}^n_i) \right\} + \lambda^n_{3i} g_3(\overline{t}^n_i, \overline{y}^n_i, \overline{w}^n_i),$$

$$i = N, \ldots, 1,$$

$$z^n_N = \sum_{l=0}^{2} \lambda^n_l (gf_l)(y^n_N),$$
the inequality condition of Theorem 3.4 can be written as
\[ \sum_{i=1}^{N} H_u(\bar{r}_i^n, \bar{y}_i^n, \bar{z}_i^n, \bar{w}_i^n)(\bar{w}_i^n - \bar{w}_i^n) \geq 0, \quad \text{for every } w^n \in W^n, \]
which is equivalent to the double-piecewise minimum principle
\[
\begin{align*}
H_u(\bar{r}_{2j-1}^n, \bar{y}_{2j-1}^n, \bar{z}_{2j-1}^n, \bar{w}_{2j-1}^n) &+ H_u(\bar{r}_{2j}^n, \bar{y}_{2j}^n, \bar{z}_{2j}^n, \bar{w}_{2j}^n) \\
&= \min_{u,v}[H_u(\bar{r}_{2j-1}^n, \bar{y}_{2j-1}^n, \bar{z}_{2j-1}^n, \bar{w}_{2j-1}^n)u + H_u(\bar{r}_{2j}^n, \bar{y}_{2j}^n, \bar{z}_{2j}^n, \bar{w}_{2j}^n)v], \quad j = 1, \ldots, N',
\end{align*}
\]
where the minimum is taken subject to the constraints
\[ u, v \in U, \quad u - (v - u)/2 \in U, \quad v + (v - u)/2 \in U. \]
For a given control \( w^n \in W^n \), we also define the solution \( Z^n = Z^n_{w^n} \) of the discrete matrix scheme
\[
\begin{align*}
Z_{i-1}^n &= Z_i^n + h^n \tilde{Z}_i f_y(\bar{r}_i^n, \bar{y}_i^n, \bar{w}_i^n), \quad i = N - 1, \ldots, 1, \\
Z_N^n &= E, \quad \text{with } \tilde{Z}_i^n = (Z_{i-1}^n + Z_i^n)/2, \quad y^n = y_{w^n}.
\end{align*}
\]

4. Behavior in the limit

In this section we will study the behavior in the limit of properties of discrete optimality, and of discrete admissibility and extremality.

**Proposition 4.1.** Let \( \phi \) be a continuous function defined on \( I \times \mathbb{R}^p \times U \times U \). If \( z_k \to z \) uniformly, \( w_k \to w \) in \( L^2(I) \) strongly, and \( r_k \to r \) in \( R \), then
\[
\int_0^T \phi(t, z_k(t), w_k(t), r_k(t)) \, dt \to \int_0^T \phi(t, z(t), w(t), r(t)) \, dt, \quad \text{as } k \to \infty,
\]

**Proof.** We write
\[
\begin{align*}
C_k &= \int_0^T [\phi(t, z_k(t), w_k(t), r_k(t)) - \phi(t, z(t), w(t), r(t))] \, dt = A_k + B_k, \\
A_k &= \int_0^T [\phi(t, z_k(t), w_k(t), r_k(t)) - \phi(t, z(t), w(t), r(t))] \, dt, \\
B_k &= \int_0^T [\phi(t, z(t), w(t), r_k(t)) - \phi(t, z(t), w(t), r(t))] \, dt.
\end{align*}
\]
By Egorof’s theorem, \( w_k \to w \) a.e. in \( I \), for a subsequence \( k \in K \). Since the integrand in \( A_k \) is bounded in \( L^\infty \), by Lebesgue’s Dominated Convergence Theorem, we have \( A_k \to 0 \), as \( k \to \infty \), \( k \in K \), and since \( r_k \to r \) in \( R \), we have \( B_k \to 0 \). Therefore \( C_k \to 0 \), as \( k \to \infty \), \( k \in K \), and since the limit 0 is unique, this holds also for the whole sequence. \( \square \)
Remark. Note that for any sequence \((w^n \in W^n)\), from the Lipschitz constraints imposed on the discrete controls, we have

\[
|w^n(t) - \bar{w}^n(t)| = |w^n(t) - w^n(t^n)| \leq \frac{h^n}{2} L_0, \quad t \in I^n_i,
\]

hence

\[
\|w^n - \bar{w}^n\|_{\infty} \to 0, \quad \text{as } n \to \infty.
\]

It follows that \(w^n \to w\) if and only if \(\bar{w}^n \to w\), in \(L^2\) strongly or weakly. It also follows from the definition of the weak star convergence in \(R\) that \(w^n \to r\) in \(R\) \((w^n)\) considered as a sequence in \(R\) if and only if \(\bar{w}^n \to r\) in \(R\) (take a continuous test function \(\phi(x, t, u)\) to pass to the limit, and then use the density of these functions in the set of Carathéodory functions).

Define the piecewise constant functions

\[
\bar{y}^n(t) = \left( y^n_{i-1} + y^n_i \right)/2, \quad t \in I^n_i, \text{ } i = 1, \ldots, N,
\]

and the piecewise linear functions

\[
\hat{y}^n(t) = y^n_{i-1} + (t - t^n_{i-1}) f(\bar{t}^n_i, \bar{y}^n_i, \bar{w}^n_i), \quad t \in I^n_i, \text{ } i = 1, \ldots, N.
\]

We also set

\[
b'' = \max(b, b'), \quad D = \{(t, y, u) \mid t \in I, \|y\| \leq b'', u \in U\}, \quad M = \max_D \|f(t, y, u)\|.
\]

**Theorem 4.1** (Consistency). (i) Let \((w^n \in W^n)\) be a sequence such that \(w^n \to w\) in \(L^2\) strongly. Then \(w \in W, \hat{y}^n \to y, \bar{y}^n \to y\) uniformly, where \(y = y_w, \text{ and}\)

\[
G^n_l(w^n) \to G^l(w), \quad l = 0, 1, 2, 3, \text{ as } n \to \infty.
\]

(ii) Let \((w^n \in W^n \subset R)\) be a sequence such that \(w^n \to r\) in \(R\). Then \(\hat{y}^n \to y, \bar{y}^n \to y\) uniformly, where \(y = y_r, \text{ and}\)

\[
G^n_l(w^n) \to G^l(r), \quad l = 0, 1, 2, 3, \text{ as } n \to \infty.
\]

**Proof.** We give the proof of (i) only, the proof of (ii) is similar. Let \(\varepsilon > 0\). Since \(f\) is uniformly continuous on the compact set \(D\), there exists \(\delta > 0\) such that

\[
\|f(t_1, y_1, u_1) - f(t_2, y_2, u_2)\| \leq \varepsilon, \quad \text{for } |t_1 - t_2| \leq \delta, \|y_1 - y_2\| \leq \delta, \|u_1 - u_2\| \leq \delta.
\]

Now, choose

\[
h^n \leq 2 \min(\delta, \delta/M, \delta/L_0).
\]

By construction of \(\hat{y}^n\), we have

\[
\|\hat{y}^n(t_1) - \hat{y}^n(t_2)\| \leq M|t_1 - t_2|, \quad \text{for } t_1, t_2 \in I,
\]

and

\[
\|\hat{y}^n(t) - y^0\| \leq MT, \quad t \in I,
\]
which show that \((\hat{y}^n)\) is a bounded sequence of equicontinuous functions. Now, we have
\[
\|\hat{y}^n(t) - \bar{y}^n_i\| \leq Mh^n / 2 \leq \delta, \quad t \in I^*_i, \; i = 1, \ldots, N.
\]

We write
\[
\hat{y}^n(t) = f(t, \hat{y}^n(t), \bar{w}^n(t)) + \bar{x}^n(t), \quad t \in I^*_i, \; i = 1, \ldots, N,
\]
where
\[
\bar{x}^n(t) = f(t, \bar{y}^n_i, \bar{w}^n_i) - f(t, \hat{y}^n(t), w^n(t)), \quad t \in I^*_i, \; i = 1, \ldots, N,
\]
and due to the Lipschitz condition on the discrete controls
\[
\|\bar{x}^n(t)\| \leq \varepsilon, \quad t \in I.
\]

Therefore, \(\bar{x}^n \to 0\) uniformly. By integration, we have
\[
\hat{y}^n(t) = y^0 + \int_0^t \left[ f(s, \hat{y}(s), \bar{w}(s)) + \bar{x}(s) \right] ds.
\]

By Ascoli’s theorem, there exists a subsequence \((\hat{y}^n)_{n \in K}\) and \(y \in C(I)^d\) such that \(\hat{y}^n \to y\) uniformly. We have
\[
\hat{y}^n(t) = y^0 + \int_0^t \left[ f(s, \hat{y}(s), w^n(s)) - f(s, y(s), w^n(s)) \right] ds
\]
\[
+ \int_0^t \left[ f(s, y(s), w^n(s)) - f(s, y(s), w(s)) \right] ds
\]
\[
+ \int_0^t f(s, y(s), w(s)) ds + \int_0^t \bar{x}(s) ds.
\]

Since \(w^n \to w\) in \(L^2\), by Egorof’s theorem we can suppose that \(w^n \to w\) a.e. in \(I\), for a subsequence, which shows that \(w \in W\). By the uniform continuity of \(f\) for the first integral, and using Proposition 4.1 for the second, we can pass to the limit in this equation and obtain
\[
y(t) = y^0 + \int_0^t f(s, y(s), w(s)) ds,
\]
i.e., \(y = y_w\). The convergence of the original sequence follows from the uniqueness of the limit \(y\). It follows easily that also \(\bar{y}^n \to y\) uniformly, that \(G^n_l(w^n) \to G_l(w), l = 0, 1, 2\), and, by the uniform continuity of \(g\), that the sequence of piecewise constant functions corresponding to \((G^n_3(w^n))\) converges uniformly to \(G_3(w)\). \(\square\)

The following theorem is proved similarly to Theorem 4.1, and using this theorem.

**Theorem 4.2** (Consistency). If \((w^n \in W^n)\) is a sequence such that \(w^n \to w\) in \(L^2\) strongly (resp. \(w^n \to r\) in \(R\)), then \(\hat{z}^n \to z, \hat{Z}^n \to Z, \hat{\bar{Z}}^n \to \bar{Z}\) uniformly, where \(z = z_w, Z = Z_w\) (resp.
$z_l = z_{lr}$, $Z = Z_r$). If $(w^n \in W^n)$, $(w'^n \in W'^n)$ are sequences such that $w^n \to w$, $w'^n \to w'$ in $L^2$ strongly, then

$$DG^n_l (w^n, w'^n - w^n) \to DG_l(w, w' - w).$$

**Lemma 4.1.** For a given discrete control $w^n \in W^n$, let $\tilde{y}^n$ be the corresponding solution of the continuous state equation. We have

$$\|\hat{\tilde{y}}^n - \tilde{y}^n\|_{L^\infty} \leq \eta^n \to 0, \quad \|\bar{\tilde{y}}^n - \tilde{y}^n\|_{L^\infty} \leq \eta^n \to 0,$$

where $\eta^n$ is independent of the involved control $w^n$.

**Proof.** We have (see proof of Theorem 4.1)

$$\hat{\tilde{y}}^n(t) = y^0 + \int_0^t f(s, \hat{\tilde{y}}(s), w^n(s)) \, ds + \beta^n(t),$$

where

$$\|\beta^n\|_{L^\infty} \leq c \int_0^T \|z^n(t)\| \, dt \to 0, \quad \text{as } n \to \infty.$$

Consequently

$$\|\hat{\tilde{y}}^n(t) - \bar{\tilde{y}}^n(t)\| \leq \int_0^t \|f(s, \hat{\tilde{y}}^n, w^n) - f(s, \bar{\tilde{y}}^n, w^n)\| \, ds + \|\beta^n\|_{L^\infty} \leq L \int_0^t \|\hat{\tilde{y}}^n - \bar{\tilde{y}}^n\| \, ds + \|\beta^n\|_{L^\infty}.$$

By Gronwall’s inequality

$$\|\hat{\tilde{y}}^n(t) - \bar{\tilde{y}}^n(t)\|_{L^\infty} \leq c' \|\beta^n\|_{L^\infty} \leq \eta^n \to 0.$$

The second convergence follows easily. □

It can be shown that, if in addition $f$ is Lipschitz continuous w.r.t. $(t, y)$, then

$$\|\hat{\tilde{y}}^n - \tilde{y}^n\|_{L^\infty} \leq c h^n, \quad \|\bar{\tilde{y}}^n - \tilde{y}^n\|_{L^\infty} \leq c h^n,$$

and if $f$ is $C^2$ in $(t, y, u)$, then

$$\max_{i=0, \ldots, N} \|y^n_i - \tilde{y}(t^n_i)\| \leq c (h^n)^2,$$

with $c$ independent of $n$ and the involved control $w^n$.

For $w, w' \in W$, let $\psi$ be the solution of the linearized continuous classical state equation

$$\dot{\psi}(t) = f_y(t, y(t), w(t)) \psi(t) + f_u(t, y(t), w(t))[w'(t) - w(t)], \quad \psi(0) = 0.$$

For $w^n, w'^n \in W^n$, let $\psi^n$ be the solution of the corresponding linearized discrete classical state equation

$$\psi^n_{i-1} = \psi^n_i + f_y(t^n_i, \tilde{y}^n_i, \tilde{w}^n_i) \psi^n_i + f_u(t^n_i, \tilde{y}^n_i, \tilde{w}^n_i)(\tilde{w}^n_i - \tilde{w}^n_i), \quad \psi^n_0 = 0.$$
For \( w^n, w^m \in W^n \), let \( \tilde{\phi}^n \) denote the solution of the corresponding linearized continuous classical state equation. We have, similarly to Lemma 4.1 and using also this lemma
\[
\| \tilde{\psi}^n - \tilde{\psi}^n \|_{L^\infty} \leq \zeta^n \to 0,
\]
where \( \zeta^n \) is independent of the involved controls \( w^n, w^m \).

For \( w^n \in W^n \), let \( \tilde{Z}^n \) be the continuous fundamental matrix solution corresponding to \( w^n \). We have similarly
\[
\| \tilde{Z}^n - \tilde{Z}^n \|_{L^\infty} \leq \zeta^n \to 0.
\]

**Proposition 4.2 (Control approximation).** (i) For every \( w \in W \), there exists a sequence \( (w^n \in W^n) \) that converges to \( w \) in \( L^2 \) strongly.

(ii) For every \( r \in R \), there exists a sequence \( (w^n \in W^n) \) that converges to \( r \) in \( R \).

**Proof.** The result (i), but with \( W^n \) replaced by its subset \( \tilde{W}^m \) (see Section 2), is proved in [8], and (i) follows. The result (ii) is proved in [2], see also [9]. \( \square \)

We suppose in the sequel that each considered continuous classical or relaxed problem is feasible. The following theorem is a theoretical result concerning the behavior in the limit of optimal discrete controls.

**Theorem 4.3.** In the presence of state constraints, we suppose that the sequences \( (\bar{\phi}^i) \) in the discrete state constraints (Case (a)) converge to zero as \( n \to \infty \) and satisfy
\[
\|G^i_1(\tilde{w}^n)\| \leq \bar{\psi}^i_1, \quad G^i_2(\tilde{w}^n) \leq \bar{\psi}^i_2, \bar{\psi}^i_2 \geq 0, \quad G^i_3(\tilde{w}^n)(i) \leq \bar{\psi}^i_3, \quad i = 1, \ldots, N, \bar{\psi}^i_3 \geq 0,
\]
for every \( n \), where \( (\tilde{w}^n \in W^n) \) is some sequence converging in \( L^2 \) (resp. in \( R \)) to an optimal control, if it exists (resp. which exists) \( \tilde{w} \in W \) (resp. \( \tilde{r} \in R \)) of the classical (resp. relaxed) problem. For each \( n \), let \( w^n \) be optimal for the discrete problem (Case (a)). Then every accumulation point of \( (w^n) \) in \( L^2 \) (resp. in \( R \)) is optimal for the continuous classical (resp. relaxed) problem.

**Proof.** The proof is similar to that of Theorem 4.1 in [3], using here Theorem 4.1. \( \square \)

Now consider the discrete problems with state constraints (Case (b)). We shall construct a sequence of perturbations \( (\bar{\phi}^i_n) \) converging to zero and such that the discrete problem is feasible for every \( n \). For each \( n \), let \( w^m \in W^n \) be a solution of the minimization problem without state constraints
\[
c^n = \min_{w^n \in W^n} \left\{ \sum_{j=1}^{m_1} [G^i_{j1}(w^n)]^2 + \sum_{j=1}^{m_2} [\max(0, G^i_{j2}(w^n))]^2 + h^n \sum_{i=1}^{N} \sum_{j=1}^{m_3} [\max(0, G^i_{j3}(w^n)(i))]^2 \right\}.
\]

Define the sequences of vectors \( (\bar{\phi}^i_1), (\bar{\phi}^i_2), (\bar{\phi}^i_3) \) by
\[
\bar{\phi}^i_1 = G^i_{j1}(w^m), \quad j = 1, \ldots, m_1,
\]
\[
\bar{\phi}^i_2 = \max(0, G^i_{j2}(w^m)), \quad j = 1, \ldots, m_2,
\]
\[
\bar{\phi}^i_3 = \max_{i=1,\ldots,N} [\max(0, G^i_{j3}(w^m)(i))], \quad j = 1, \ldots, m_3,
\]
Theorem 4.4. For each $n$ above minimum feasibility (there exists a subsequence continuous classical problem Suppose that and weakly extremal relaxed for the continuous relaxed problem piecewise constant functions which imply a fortiori that $\tilde{c}_n \rightarrow 0$, hence $\hat{c}_1^n \rightarrow 0$, $\hat{c}_2^n \rightarrow 0$, $\hat{c}_3^n \rightarrow 0$. Now, $(\theta^n(\cdot) = g_3(\cdot, \tilde{y}_n(\cdot)))$, where $y^n$ corresponds to $w^n$, is a sequence of equicontinuous functions, which is bounded in $L^\infty$. Therefore, there exists a subsequence $(\theta^n)_{n \in K}$ that converges uniformly to some function $\theta$; hence the sequence of piecewise constant functions

$$(\theta^n(\cdot) = G_3^n(w^n(\cdot)) = g_3(\tilde{w}(\cdot), \tilde{y}_n(\cdot)))_{n \in K}$$

converges also uniformly to $\theta$. Since $\hat{c}_3^n = \| \tilde{\theta}^n \|_{L^2}^2 \rightarrow 0$, we must have $\theta = 0$, hence $\| \tilde{\theta}^n \|_{L^\infty} \rightarrow 0$, $\hat{c}_3^n \rightarrow 0$, as $n \rightarrow \infty$, $n \in K$, and this holds also for the whole sequences, by the uniqueness of the limit. Now, choosing the perturbations $\phi_l^n$, $l = 1, 2, 3$, as above, the discrete problem (Case (b)) is clearly feasible for every $n$, with $\hat{c}_l^n \rightarrow 0$, $l = 1, 2, 3$. We suppose in the sequel that the perturbations $\phi_l^n$ are chosen as in the above minimum feasibility procedure. Note that we often find $c^n = 0$, for large $n$, due to sufficient discrete controllability, in which case we take $\hat{c}_l^n = 0$, $l = 1, 2, 3$.

The following theorem addresses the behavior in the limit of extremal discrete controls.

**Theorem 4.4.** For each $n$, let $w^n$ be admissible and extremal for the discrete problem (Case (b)). Then every accumulation point $v$ of $(w^n)$ in $L^2$ (if it exists) is admissible and weakly extremal classical for the continuous classical problem, and every accumulation point $v$ in $R$ (which always exists) is admissible and weakly extremal relaxed for the continuous relaxed problem.

**Proof.** Suppose that $w^n \rightarrow v$ in $L^2$, for a subsequence. Since $w^n$ is admissible, it follows from Theorem 4.1 that $w$ is admissible for the continuous classical problem. Now, using the solution $\psi^n$ of the linearized
state equation, the discrete relaxed necessary optimality inequality can also be written as
\[
\sum_{l=0}^{2} \lambda_t^n h^n \sum_{l=1}^{N} \left[ z_t^n f_u(\tilde{t}_l^n, \tilde{y}_l^n, \tilde{w}_l^n) + g_{lu}(\tilde{t}_l^n, \tilde{y}_l^n, \tilde{w}_l^n) \right](\tilde{w}_l^n - \tilde{w}_l^n) \\
+ h^n \sum_{l=1}^{N} \lambda_3^n(i) g_{3y}^n(\tilde{t}_l^n, \tilde{y}_l^n) \tilde{\psi}_l^n \geq 0, \quad \text{for every } w^n \in W^n,
\]
or equivalently, in continuous form
\[
\sum_{l=0}^{2} \lambda_t^n \int_0^T \left[ \tilde{z}_t^n(t) f_u(\tilde{t}_l^n(t), \tilde{y}_l^n(t), \tilde{w}_l^n(t)) + g_{lu}(\tilde{t}_l^n(t), \tilde{y}_l^n(t), \tilde{w}_l^n(t)) \right](\tilde{w}_l^n(t) - \tilde{w}_l^n(t)) \, dt \\
+ \int_0^T \lambda_3^n(s) g_{3y}^n(\tilde{t}_l^n(s), \tilde{y}_l^n(s)) \tilde{\psi}_l^n(s) \, ds \geq 0, \quad \text{for every } w^n \in W^n.\]

Since \((\lambda_t^n)\) is bounded in \(L^1(I)^{m_3}\) and \((g_{3y}^n)\) is bounded in \(L^{\infty}(I)^{m_3}\), from the above approximation of \(\tilde{\psi}_l^n\) by \(\tilde{\psi}_l^n\), we have
\[
\sum_{l=0}^{2} \lambda_t^n \int_0^T \left[ \tilde{z}_t^n(t) f_u(\tilde{t}_l^n(t), \tilde{y}_l^n(t), \tilde{w}_l^n(t)) + g_{lu}(\tilde{t}_l^n(t), \tilde{y}_l^n(t), \tilde{w}_l^n(t)) \right](\tilde{w}_l^n(t) - \tilde{w}_l^n(t)) \, dt \\
+ \int_0^T \lambda_3^n(s) g_{3y}^n(\tilde{t}_l^n(s), \tilde{y}_l^n(s)) \tilde{\psi}_l^n(s) \, ds + e^n \geq 0, \quad \text{for every } w^n \in W^n,
\]
where \(e^n \to 0\). From the known property of solutions of linear differential systems, we have
\[
\sum_{l=0}^{2} \lambda_t^n \int_0^T \left[ \tilde{z}_t^n(t) f_u(\tilde{t}_l^n(t), \tilde{y}_l^n(t), \tilde{w}_l^n(t)) + g_{lu}(\tilde{t}_l^n(t), \tilde{y}_l^n(t), \tilde{w}_l^n(t)) \right](\tilde{w}_l^n(t) - \tilde{w}_l^n(t)) \, dt \\
+ \int_0^T \lambda_3^n(s) g_{3y}^n(\tilde{t}_l^n(s), \tilde{y}_l^n(s)) \tilde{\psi}_l^n(s) \, ds + e^n \geq 0, \quad \text{for every } w^n \in W^n.
\]

Now let any \(w' \in W\) and let \((w^n \in W^n)\) be a sequence converging to \(w'\) in \(L^2\) strongly. Since
\[
\sum_{l=0}^{2} \| \lambda_t^n \| + h^n \sum_{i=0}^{N} \| \lambda_3^n(i) \|_1 = 1,
\]
the sequences \((\lambda_t^n), l = 0, 1, 2,\) are bounded, and \((\lambda_3^n)\) is bounded in \(L^1\), hence in \(M(I)\). Taking subsequences, we can suppose that \(\lambda_t^n \to \lambda_t, l = 0, 1, 2,\) and that \(\lambda_3^n \to \lambda\) in \(M(I)^{m_3} = [C(I)^{m_3}]^*\) weak star. Since \(\tilde{y}_l^n \to y = y_w,\) we have \(\tilde{y}_l^n \to y,\) uniformly, by Theorem 4.1 and Lemma 4.1. We have \(\tilde{Z}_n \to Z = Z_w,\) \(\tilde{z}_l^n \to z_{lw}\) (Theorem 4.2), and also \(\tilde{Z}_n \to Z,\) uniformly. The involved integrals in \(t\) depending on \(s\) converge for each fixed \(s\) and are equicontinuous in \(s\), hence the convergence is uniform.
Since all these uniform limits are continuous, we can pass to the limit (using also Proposition 4.1) in the above inequality and obtain the continuous classical global extremality condition (see Theorem 2.4)

\[
\sum_{l=0}^{2} \lambda_l \int_0^T \left[ z_l(t) f_u(t, y(t), v(t)) + g_{l u}(t, y(t), v(t)) \right] (w'(t) - v(t)) \, dt \\
+ \int_0^T \lambda_3 (ds) g_3(y(s), y(t), v(t)) Z(s)^{-1} \left( \int_0^s Z(t) f_u(t, y(t), v(t)) \, dt \right) (w'(t) - v(t)) \, dt \geq 0,
\]

for every \( w' \in W \).

Suppose now that \( w_n \to v \) in \( R ((w^n) \) considered as a sequence in \( R \), for a subsequence. Let \( \phi : I \times U \to U \) be any continuous function. By the discrete optimality inequality, we have as above, setting here \( \bar{w}^n(t) = \phi(\bar{r}^n, \bar{w}^n(t)) \)

\[
\sum_{l=0}^{2} \lambda_l \int_0^T \left[ z_l(t) f_u(t, y(t), v(t)) + g_{l u}(t, y(t), v(t)) \right] (\phi(\bar{r}^n(t), \bar{w}^n(t)) - \bar{w}^n(t)) \, dt \\
+ \int_0^T \lambda_3 (s) g_3(\bar{r}^n(s), \bar{y}^n(s)) Z(s)^{-1} \left( \int_0^s Z(t) f_u(t, y(t), v(t)) \, dt \right) (\phi(\bar{r}^n(t), \bar{w}^n(t)) - w^n(t)) \, dt \, ds + e^n \geq 0.
\]

Passing to the limit, we obtain

\[
\sum_{l=0}^{2} \lambda_l \int_0^T \left[ z_l(t) f_u(t, y(t), v(t)) + g_{l u}(t, y(t), v(t)) \right] (\phi(t, v(t)) - v(t)) \, dt \\
+ \int_0^T \lambda_3 (ds) g_3(y(s), y(t), v(t)) Z(s)^{-1} \left( \int_0^s Z(t) f_u(t, y(t), v(t)) \, dt \right) (\phi(t, v(t)) - v(t)) \, dt,
\]

for every such \( \phi \).

Now let \( \phi : I \times U \to U \) be any Caratheodory function, or equivalently (since \( I \) is compact) \( \phi \in L^1(I; C(U; U)) \), and let \( (\phi_k) \) be a sequence in \( C(I; C(U; U)) = C(I \times U; U) \) converging to \( \phi \) in \( L^1(I; C(U; U)) \). By Egorof’s theorem, for a subsequence, \( \phi_k \to \phi \) a.e. in \( I \), with values in \( C(U; U) \), hence a.e. in \( I \times U \), with values in \( U \). Replacing \( \phi \) by \( \phi_k \) in the above inequality, by Lebesgue’s dominated convergence theorem, the third integral in \( t \) converges for each fixed \( s \), hence (as above) uniformly, and we can pass to the limit in the inequality to obtain the global weak relaxed extremality condition.

On the other hand, in both convergence cases, the discrete transversality conditions can be written

\[
\lambda_2^n [G_2^n (w^n) - \varepsilon_2^n] = 0,
\]

\[
h^n \sum_{i=1}^{N} \lambda_3^n (i) [g_3(\bar{r}_i^n, \bar{y}_i^n) - \varepsilon_3^n] = \int_0^T \lambda_3^n (s) [g_3(\bar{r}_i^n(s), \bar{y}_i^n(s)) - \varepsilon_3^n] \, ds = 0.
\]
Passing to the limit, we obtain the continuous conditions

\[ \lambda_2 G_2(v) = 0, \quad \int_0^T \lambda_3(ds)g_3(s, y(s)) = 0. \]

Finally, since \( \lambda_n \geq 0 \), we have

\[ \sum_{l=0}^2 \| \lambda_l \| + \int_0^T \sum_{j=1}^{m_3} [1 \cdot \lambda_j(ds)] \, ds = 1. \]

By the above convergences of the \( \lambda_l, l = 0, 1, 2, 3 \), we obtain in the limit

\[ \sum_{l=0}^2 \| \lambda_l \| + \int_0^T \sum_{j=1}^{m_3} [1 \cdot \lambda_j(ds)] = \sum_{l=0}^2 \| \lambda_l \| + \| \lambda_3 \| = 1. \]

5. Combined discretization-optimization methods

Let \((M^n_l), l = 1, 2, 3\), be nonnegative increasing sequences such that \(M^n_l \to \infty\) as \(m \to \infty\), and define the penalized discrete functionals

\[ G^{nm}(w^n) = G^n_0(w^n) + \frac{1}{2} \left\{ M^n_1 \sum_{j=1}^{m_1} |G^n_{1j}(w^n)|^2 + M^n_2 \sum_{j=1}^{m_2} [\max(0, G^n_{2j}(w^n))]^2 \\
+ M^n_3 \sum_{i=1}^N \sum_{j=1}^{m_3} [\max(0, G^n_{3j}(w^n)(i))]^2 \right\}. \]

Let \( \gamma \geq 0 \), \( b, c \in (0, 1) \), and let \((\beta^m), (\zeta^m)\) with \( \zeta^m \leq 1\), be positive decreasing sequences that converge to zero. The algorithm described below contains various options. In the case of the progressively refining version, we suppose that either \( N(n+1) = N(n) \) or \( N(n+1) = \mu N(n) \), for some integer \( \mu \geq 2 \). In this case, we have \( W^n \subset W^{n+1} \), and thus a control \( w^n \in W^n \) may be considered also as belonging to \( W^{n+1} \), and therefore the computation of states, adjoints and functional derivatives for this control, but with the possibly finer discretization \( n+1 \), makes sense. If \( \gamma > 0 \), we have a penalized gradient projection method, and if \( \gamma = 0 \), a conditional gradient method.

**Algorithm.**

Step 1: Set \( k = 0, m = 1 \), choose a value of \( n \) and an initial control \( w_0^{n1} \in W^n \).

Step 2: Find \( v_k^{nm} \in W^n \) such that

\[ e_k = DG^{nm}(w_k^n, v_k^{nm} - w_k^{nm}) + (\gamma/2) \| v_k^{nm} - \bar{w}_k^{nm} \|_{L^2}^2 \]

\[ = \min_{v_k^{nm} \in W^n} [DG^{nm}(w_k^n, v^n - w_k^{nm}) + (\gamma/2) \| v_k^{nm} - \bar{w}_k^{nm} \|_{L^2}^2], \]

and set \( d_k = DG^{nm}(w_k^n, v_k^{nm} - w_k^{nm}) \).
Theorem 5.1. Algorithm in Step 2.

Step 2. Let \( m \) be a fixed discretization, and replace the discrete functionals \( I_n \) by the corresponding optimality conditions, with \( |e_k| \leq \beta^m \), set \( w_n^m = w_k^m, v_n^m = v_k^m, e^m = e_k, d^m = d_k, m = m + 1, [n = n + 1] \), and go to Step 2.

Step 4 (Armijo step search): Find the lowest integer value \( s \in \mathbb{Z} \), say \( \bar{s} \), such that \( z = c^{\bar{s}} \bar{e}^m \in (0, 1] \) and \( z \) satisfies the inequality
\[
G^m(w_k^m + z_k(v_k^m - w_k^m)) - G^m(w_k^m) \leq z_k \bar{e}_k,
\]
and then set \( \bar{s} = c^{\bar{s}} \bar{e}^m \).

Step 5: Set \( w_{k+1}^m = w_k^m + z_k(v_k^m - w_k^m), k = k + 1 \), and go to Step 2.

In the above Algorithm, we consider two cases:

Case A: “\( n = n + 1 \)” is skipped in Step 3: \( n \) is a constant integer chosen in Step 1, i.e., we choose a fixed discretization, and replace the discrete functionals \( G^n_l \) by the perturbed ones.

Case B: “\( n = n + 1 \)” is not skipped in Step 3: we have a progressively refining discrete method, i.e., \( n \to \infty \) (see proof of Theorem 5.1 below), in which case we can take \( n = 1 \) in Step 1, hence \( n = m \) in the Algorithm.

The progressively refining version has the advantage of reducing computing time and memory, and also of avoiding the computation of minimum feasibility perturbations (see Section 4). It is justified by the fact that a finer discretization becomes more crucial as the iterate gets closer to an extremal control.

A (continuous strongly or weakly, classical or relaxed, or a discrete) extremal control is called abnormal if there exist multipliers as in the corresponding optimality conditions, with \( \lambda_0 = 0 \) (or \( \lambda^0 = 0 \)). A control is admissible and abnormal extremal in very exceptional, degenerate, situations.

Define the sequences of multipliers
\[
\lambda_1^n = M_1^n G_1^n(w_n^m), \quad \lambda_2^n = M_2^n \max(0, G_2^n(w_n^m)), \\
\lambda_3^n(i) = M_3^n \max(0, G_3^n(w_n^m)(i)) = M_3^n \max(0, g_3(i^n_1, \bar{s}_1^n)),
\]
where \( \max \) denotes a vector of max values, and \( w_n^m \) is defined in Step 3 of the Algorithm.

Theorem 5.1. (i) In Case B, let \( (w_n^m) \) be a subsequence (if it exists) of the sequence generated by the Algorithm in Step 3 that converges to some \( w \in W \) in \( L^2 \) strongly as \( m \to \infty \) (hence \( n \to \infty \)). If the sequences \( (\lambda_1^n), l = 1, 2, 3, (\lambda_3^n) \) in \( L^1(I)^m_3 \), are bounded, then \( w \) is admissible and extremal classical for the continuous classical problem.

(ii) In Case B, let \( (w_n^m) \) be a subsequence of the sequence generated by the Algorithm in Step 3 that converges to some \( r \in R \) as \( m \to \infty \) (hence \( n \to \infty \)). If the sequences \( (\lambda_1^n), l = 1, 2, 3, (\lambda_3^n) \) in \( L^1(I)^m_3 \), are bounded, then \( r \) is admissible and weakly extremal relaxed for the continuous relaxed problem.

(iii) In Case A, let \( (w_n^m \in W^m) \), \( n \) fixed, be a subsequence generated by the Algorithm in Step 3 that converges to some \( w^n \in W^n \) as \( m \to \infty \). If the sequences \( (\lambda_1^n), l = 1, 2, 3, \) are bounded, then \( w^n \) is admissible and extremal for the fixed discrete problem.

(iv) In any of the three above convergence cases (i), (ii) or (iii), suppose that the (discrete or continuous) limit problem has no admissible, abnormal extremal, controls. If the limit control is admissible, then the sequences of multipliers are bounded, and this control is extremal as above.

Proof. We shall first show that \( m \to \infty \) in the Algorithm. Suppose on the contrary, in both Cases A, B, that \( m \), hence \( n \), remains constant after a finite number of iterations in \( k \), and so we drop here the indices
Therefore, we must have $e_n \to \infty$ for all $n$. Let us show that then $e_k \to 0$. Since $W^n$ is compact, let $(w_k)_{k \in K}$, $(v_k)_{k \in K}$ be subsequences of the generated sequences in Steps 2 and 5 such that $w_k \to \tilde{w}$, $v_k \to \tilde{v}$, in $W^n$, as $k \to \infty$, $k \in K$. By Step 2, $d_k \leq e_k < 0$ for every $k$, hence

$$e = \lim_{k \to \infty} e_k = DG(\tilde{w}, \tilde{v} - \tilde{w}) + \frac{\gamma}{2} \|\tilde{v} - \tilde{w}\|_{L^2}^2 \leq 0,$$

$$d = \lim_{k \to \infty} d_k = DG(\tilde{w}, \tilde{v} - \tilde{w}) \leq \lim_{k \to \infty} e_k = e \leq 0.$$

Suppose that $e < 0$, hence $d < 0$. The function $\Phi(z) = G(w + z(w' - w))$ is continuous on $[0, 1]$. Since the directional derivative $DG(w, w' - w)$ is linear w.r.t. $w' - w$, $\Phi$ is differentiable on $(0, 1)$ and has derivative

$$\Phi'(z) = DG(w + z(w' - w), w' - w).$$

Using the mean value theorem, we thus have, for each $z \in (0, 1)$

$$G(w_k + z(v_k - w_k)) - G(w_k) = zDG(w_k + z'(v_k - w_k), v_k - w_k),$$

for some $z' \in (0, z)$. Therefore, for $z \in [0, 1]$, by Theorem 3.3, we have

$$G(w_k + z(v_k - w_k)) - G(w_k) = z(d + \varepsilon_{zk}),$$

where $\varepsilon_{zk} \to 0$ as $k \to \infty$, $k \in K$, and $z \to 0^+$. We have

$$d_k = d + \eta_k,$$

where $\eta_k \to 0$ as $k \to \infty$, $k \in K$, and since $b \in (0, 1)$

$$d + \varepsilon_{zk} \leq b(d + \eta_k) = bd_k,$$

for $z \in (0, z')$, for some $z' > 0$, and $k \geq k'$, $k \in K$. Hence

$$G(w_k + z(v_k - w_k)) - G(w_k) \leq zbd_k \leq zbe_k,$$

for $z \in (0, z']$, $k \geq k'$, $k \in K$. It follows from the choice of the Armijo step $z_k$ in Step 4 that we must necessarily have $z_k \geq cz'$ for $k \geq k'$, $k \in K$. Hence

$$G(w_{k+1}) - G(w_k) = G(w_k + z_k(v_k - w_k)) - G(w_k) \leq z_k be_k \leq cz'be_k \leq cz'be_k/2,$$

for $k \geq k'$, $k \in K$. It follows that $G(w_k) \to -\infty$ as $k \to \infty$, $k \in \mathbb{N}$, since the whole sequence is decreasing by construction, which contradicts the fact that $G(w_k) \to G(\tilde{w})$, as $k \to \infty$, $k \in K$, by Theorem 3.2. Therefore, we must have $e = 0$, and $\varepsilon_k \to e = 0$, for the whole sequence, by uniqueness of the limit. But Step 3 then implies that $m \to \infty$, which is a contradiction. Therefore, $m \to \infty$. This shows also that $n \to \infty$ in Case B.

(i) Let $(w^{nm})$ be a subsequence (same notation) of the sequence generated in Step 3 that converges in $L^2$ to an accumulation point $w \in W$, as $n, m \to \infty$. Suppose that the sequences $(\lambda^{nm}_l)$ are bounded
and, up to subsequences, that $\lambda_i^{nm} \to \lambda_i$, $i = 1, 2$, and $\lambda_3^{nm} \to \lambda_3$ in $M(I)^m$ weak star. By Theorem 4.1, we have

$$0 = \lim_{m \to \infty} \frac{\lambda_1^{nm}}{M_1^m} = \lim_{m \to \infty} G_1^n(w^{nm}) = G_1(w),$$

$$0 = \lim_{m \to \infty} \frac{\lambda_2^{nm}}{M_2^m} = \lim_{m \to \infty} \left[ \max(0, G_2^n(w^{nm})) \right] = \max(0, G_2(w)),$$

$$0 = \lim_{m \to \infty} \frac{\|x_3^{nm}\|_{L^1}}{M_3^m} = \lim_{m \to \infty} \frac{1}{M_3^m} h^n \sum_{i=1}^N \lambda_3^{nm} = \lim_{m \to \infty} \left[ h^n \sum_{i=1}^N \max(0, G_3^n(w^{nm})(i)) \right]$$

$$= \int_0^T \max(0, G_3(w)) \, ds,$$

which show that $w$ is admissible. Now, let any $w' \in W$ and let $(w^n \in W^n)$ be a subsequence converging to $w'$ in $L^2$ strongly. By Steps 2 and 3, we have

$$DG_0^n(w^n, w' - w^n) + (\gamma/2)\|w' - w^n\|_{L^2}^2$$

$$= DG_0^n(w^n, w' - w^n) + \lambda_1 G_1^n(w^{nm}, w' - w^n) + \lambda_2 G_2^n(w^{nm}, w' - w^n)$$

$$+ \sum_{i=1}^N \lambda_3^{nm} G_3^n(w^{nm}, w' - w^n)(i) + (\gamma/2)\|w' - w^n\|_{L^2}^2 \geq c^n.$$
By the definition of $\lambda_{3j}^{nm}$, we have
\[ \lambda_{3j}^{nm}(s) = 0, \quad s \in S_j', \quad \text{for } m \geq m'. \]

It follows that
\[ \eta_j^{nm} = \left| \int_0^T \lambda_{3j}^{nm}(s) g_3 j(\tilde{r}^n(s), \tilde{y}^{nm}(s)) \, ds \right| = \left| \int_{S_c} \lambda_{3j}^{nm}(s) g_3 j(\tilde{r}^n(s), \tilde{y}^{nm}(s)) \, ds \right| \]
\[ \leq 2c \| \lambda_{3j}^{nm} \|_{L^1(S_c)} \leq 2c \| \lambda_{3j}^{nm} \|_{L^1(I)} \leq 2c, \quad \text{for } m \geq m'. \]

Therefore, by the involved weak star and uniform convergences
\[ 0 = \lim_{m \to \infty} \eta_j^{nm} = \left| \int_0^T \lambda_{3j}(ds) g_3 j(s, y(s)) \right|, \quad j = 1, \ldots, m_3, \]
or equivalently (since $\lambda_3 \geq 0$, see below, and $g_3(s, y(s)) \leq 0$)
\[ \int_0^T \lambda_3(ds) g_3(s, y(s)) = 0. \]

On the other hand, since $\lambda_{3j}^{nm} \geq 0$ and $\lambda_{3j}^{nm} \to \lambda_3$ weak star, we have
\[
1 + \sum_{l=0}^{2} \| \lambda_l^{nm} \| + N \sum_{i=0}^{N} \| \lambda_3^{nm}(i) \|_1 = 1 + \sum_{l=0}^{2} \| \lambda_l^{nm} \| + \int_0^T \left\{ \sum_{j=1}^{m_3} [1 \cdot \lambda_{3j}^{nm}(s)] \right\} \, ds \\
\to 1 + \sum_{l=0}^{2} \| \lambda_l \| + \int_0^T \left\{ \sum_{j=1}^{m_3} [1 \cdot \lambda_{j3}(ds)] \right\} = 1 + \sum_{l=0}^{2} \| \lambda_l \| + \| \lambda_3 \| = a \neq 0.
\]

We clearly have also $\lambda_0 = 1$, $\lambda_2 \geq 0$, and since $\lambda_{3j}^n \geq 0$ and $\lambda_{3j}^n \to \lambda_3$ in $M(I)^{m_3}$ weak star, we have also $\lambda_3 \geq 0$. Dividing all multipliers by $a$, $w$ is thus extremal classical.

(ii) Let $(w^{nm})$ be a subsequence of the sequence generated in Step 3, that converges in $R$ to an accumulation point $r$, as $n, m \to \infty$. The admissibility of $r$ is proved as in (i). Suppose as in (i) that $\lambda_{l}^{nm} \to \lambda_l$, $l = 1, 2$, and $\lambda_{3j}^{nm} \to \lambda_3$ in $M(I)^{m_3}$ weak star. As in (i), we have
\[
DG^{nm}(w^{nm}, w^m - w^{nm}) + (\gamma/2)\| \tilde{w}^{nm} - \tilde{w}^{nm} \|^2_{L^2} \\
= DG_0^n(w^{nm}, w^m - w^{nm}) + \lambda_1^m DG_1^n(w^{nm}, w^m - w^{nm}) + \lambda_2^m DG_2^n(w^{nm}, w^m - w^{nm}) \\
+ \sum_{i=1}^{N} \lambda_3^m(i) DG_3^n(w^{nm}, w^m - w^{nm})(i) + (\gamma/2)\| \tilde{w}^m - \tilde{w}^{nm} \|^2_{L^2} \geq \varepsilon^m,
\]
for every $w^m \in W^n$. Choosing any continuous function $\phi : I \times U \to U$ and setting $\tilde{w}^n(t) = \phi(\tilde{r}^n, \tilde{w}^n(t))$, we can pass to the limit in this inequality as in the second part of the proof of Theorem 4.4.
and find
\[
\sum_{l=0}^{2} \int_{0}^{T} \left[ z_l(t) f_u(t, y(t), r(t)) + g_{lk}(t, y(t), r(t)) [\phi(t, r(t)) - r(t)] \right] dt \\
+ \int_{0}^{T} \lambda_3(s) g_3(s, y(s)) Z(s)^{-1} \left( \int_{0}^{s} Z(t) f_u(t, y(t), r(t)) [\phi(t, r(t)) - r(t)] dt \right) \\
+ (\gamma/2) \int_{0}^{T} \|\phi(t, r(t)) - r(t)\|^2 dt \geq 0,
\]
for every such \( \phi \),

which implies, by an argument similar to (i), the same inequality, but without the last integral term, and with multipliers as in the optimality conditions. The weak relaxed inequality condition follows then by density. The transversality conditions are proved similarly to (i).

(iii) Similarly to the proof of (i), passing here to the limit as \( m \to \infty \), for \( n \) fixed, we find that \( w^n \) is admissible, the condition
\[
DG_0^n(w^n, w^n - w^n) + \lambda_1^n DG_1^n(w^n, w^n - w^n) + \lambda_2^n DG_2^n(w^n, w^n - w^n) \\
+ \sum_{i=1}^{N} \lambda_{3i}^n DG_3^n(w^n, w^n - w^n)(i) \geq 0, \quad \text{for every } w^n \in W^n,
\]
the discrete transversality conditions (since, by the continuity of \( G^n_2, G^n_3 \), we have \( \lambda_{2m}^n = 0 \) if \( G^n_2(w^n) - \varepsilon_2^n < 0 \), and \( \lambda_{3i}^n = 0 \) if \( G^n_3(w^n)(i) - \varepsilon_3^n < 0 \), for \( m \) sufficiently large)
\[
\lambda_2^n [G^n_2(w^n) - \varepsilon_2^n] = 0, \\
\lambda_{3i}^n [G^n_3(w^n)(i) - \varepsilon_3^n] = 0, \quad i = 1, \ldots, N,
\]
and
\[
\lambda_0^n = \lambda^n_0 = 1, \quad \lambda_2^n \geq 0, \quad \lambda_3^n \geq 0, \quad 1 + \sum_{l=0}^{2} \|\lambda_l^n\| + h^n \sum_{i=0}^{N} \|\lambda_3^n(i)\|_1 \neq 0,
\]
which show that \( w^n \) is extremal for the discrete problem.

(iv) In any of the above cases (i), (ii) or (iii), suppose that the limit control is admissible and that the limit problem has no admissible, abnormal extremal, controls. Suppose that the multipliers are not all bounded. Then, dividing the corresponding inequality resulting from Step 2 by the greatest multiplier norm and passing to the limit for a subsequence, we readily see that we obtain an extremality inequality where the first multiplier is zero, and that the limit control is abnormal extremal, a contradiction. Therefore, the sequences of multipliers are bounded, and by (i), (ii), or (iii), this limit control is extremal as above. □

One can easily see that Theorem 5.1 remains valid if \( e_k \) is replaced by \( d_k \) in Step 4 of the Algorithm.

When directly applied to nonconvex optimal control problems whose solutions are nonclassical relaxed controls, the classical methods yield often very poor convergence. For this reason, we describe here an alternate approach that uses Gamkrelidze controls in classical form. For simplicity, we consider only the case without state constraints and the pure gradient methods. We suppose that \( U \) is a convex and compact
subset of $\mathbb{R}^{d'}$. Consider the relaxed problem, with state equation
\[ y'(t) = f(t, y(t), r(t)) \quad \text{in} \quad I, \quad y(0) = y^0, \]
control constraint $r \in R$, and cost functional
\[ G(r) = g_f(y(T)). \]
For each $t$ fixed, the vector $f(t, y(t), r(t))$ in $\mathbb{R}^d$ belongs to the convex hull of the set $f(t, y(t), U) \subset \mathbb{R}^d$.
Hence, we can write
\[ f(t, y(t), r(t)) = \sum_{j=1}^{d+1} v_j(t) f(t, y(t), w_j(t)), \quad \text{with} \ 0 \leq v_j(t) \leq 1, \quad \sum_{j=1}^{d+1} v_j(t) = 1, \]
and by Filippov's selection theorem (see [14]), we can suppose that these functions $v_j, w_j$ are measurable.
Conversely, for given such functions $v_j, w_j$, the corresponding Gamkrelidze control $r = \sum_{j=1}^{d+1} v_j(t) \delta_{w_j(t)}$ is a relaxed control that yields the same velocity vector and cost. Therefore, the above relaxed control problem is equivalent to the following classical one, with state equation
\[ y'(t) = \sum_{j=1}^{d+1} v_j(t) f(t, y(t), w_j(t)) \quad \text{in} \quad I, \quad y(0) = y^0, \]
controls $v = (v_j), w = (w_j)$, control constraints
\[ \sum_{j=1}^{d+1} v_j(t) = 1, \quad 0 \leq v_j(t) \leq 1, \quad w_j(t) \in U, \quad j = 1, \ldots, d + 1, \]
and cost functional $G(v, w) = g_f(y(T))$. We can therefore apply the gradient methods described above to this problem. The main disadvantage of this approach is that the dimension of the control space is rapidly increased. It can therefore be successfully applied for relatively small dimensions $d, d'$. The Gamkrelidze relaxed controls computed thus can then be approximated by piecewise classical controls using a standard procedure, see [4]. In the general case, i.e., if $U$ is not convex, one can use relaxed methods to solve such strongly nonconvex problems, see [4,5].

6. Numerical examples

(a) Let $I = [0, 1]$, and define the reference state and control
\[ \bar{y}_1(t) = e^{-t}, \quad \bar{y}_2(t) = e^{-2t}, \quad \bar{y}_3(t) = e^{-3t}, \quad \bar{w}(t) = \begin{cases} -1, & t \in [0, 0.25), \\ -0.8 + 1.8s^2(2 - s), & t \in [0.25, 1] \end{cases} \]
with $s = (t - 0.25)/0.75$. Consider the following problem, with state equations
\[ y'_1 = -y_1 + y_3 - e^{-3t} + \sin y_1 - \sin \bar{y}_1 + w_1 - \bar{w}, \]
\[ y'_2 = y_1 - 2y_2 - e^{-t} + w_2 - \bar{w}, \]
Fig. 1. Example (a): Optimal control (first component).

\[ y_3' = y_2 - 3y_3 - e^{-2t} + w_3 - \bar{w}, \]
\[ y_1(0) = y_2(0) = y_3(0) = 1, \]

control constraint set \( U = [-1, 1] \), and cost functional

\[ G_0(w) := \frac{1}{2} \int_0^1 \left\{ \sum_{i=1}^3 \left[ (y_i - \bar{y}_i)^2 + (w_i - \bar{w})^2 \right] \right\} dt. \]

The optimal control and state are clearly \( w^* := (\bar{w}, \bar{w}, \bar{w}) \) and \( y^* := (\bar{y}_1, \bar{y}_2, \bar{y}_3) \). The Algorithm, without penalties, was applied to this example, with \( \gamma = 0.5 \), fixed step size \( h^n := h = 1/256 \), and zero initial control. After 12 iterations in \( k \), we obtained the results

\[ G_0^n(w_k^n) = 2.415 \cdot 10^{-11}, \quad e_k = -4.861 \cdot 10^{-11}, \]
\[ e_k = 1.605 \cdot 10^{-6}, \quad \eta_k = 7.163 \cdot 10^{-5}, \quad \zeta_k = 2.174 \cdot 10^{-5}, \]

where \( e_k \) was defined in Step 2 of the Algorithm, \( e_k \) is the max error for the state at the end points of the intervals, \( \eta_k \) the max error for the control at the end points of the double intervals, and \( \zeta_k \) the max error for the control at the midpoints of the intervals. Fig. 1 shows the first component of the last computed control.

(b) With the same state equations, cost and parameters as in Example (a), but with constraint set \( U = [-0.7, 0.3] \), the control constraints being now strictly active, we obtained after 12 iterations the results

\[ G_0^n(w_k^n) = 0.103032255750910, \quad e_k = -6.489 \cdot 10^{-18}, \]

and the first component of the control shown in Fig. 2.

(c) With the same state equations, cost, and parameters as in Example (a), but with constraint set \( U = [-0.95, 1] \), additional pointwise state constraints

\[ G_3(w)(t) = 0.8 - 0.4t - y_1(t) \leq 0, \quad t \in I, \]
feasibility perturbation $\varepsilon_1^n = 0$, and applying here the Algorithm with penalties, we obtained after 99 iterations in $k$ the results

$$G^n_0(w_k^{nm}) = 1.520111433102930 \cdot 10^{-3}, \quad e_k = -9.804 \cdot 10^{-7},$$

$$\theta_k := \max_{i=1,...,N} \left[ \max(0, 0.8 - 0.4t_i^n - y_{1i,k}^{nm}) \right] = 6.434 \cdot 10^{-5},$$

and the first component of the control and state shown in Figs. 3 and 4.

(d) Consider the nonconvex problem, with state equations

$$y'_1 = -y_1 + w, \quad y'_2 = 0.5(y_1 - \bar{y})^2 - w^2, \quad y_1(0) = 1, \quad y_2(0) = 0,$$

where $\bar{y}(t) = e^{-t}$, control constraint set $U = [-1, 1]$, and cost $G(w) = y_2(1)$. The unique optimal relaxed control is clearly $r^*(t) = (\delta_{-1} + \delta_1)/2$, with optimal state $y^* = \bar{y}$ and optimal cost $G(r^*) = -1$. Note that the optimal relaxed cost can be approximated as closely as desired with a classical control, but cannot be attained for such a control. Since here the set $f(t, y, U)$ is a continuous arc in $\mathbb{R}^2$, hence a connected set in $\mathbb{R}^2$, the Gamkrelidze formulation involves only three controls $v, u, w$

$$y'_1 = -y_1 + vu + (1 - v)w, \quad y'_2 = \frac{1}{2}(y_1 - \bar{y})^2 - vu^2 - (1 - v)w^2,$$

$$y_1(0) = 1, \quad y_2(0) = 0$$
with \( v \in [0, 1] \) and \( u, w \in [-1, 1] \). Applying the conditional gradient method (i.e., with \( \gamma = 0 \)), which yielded better convergence for this special problem, without penalties, and initial controls \( v_0 = 0.6 + 0.3t \), \( u_0 = -0.6 - 0.3t \), \( w_0 = 0.7 + 0.2t \), we obtained after 12 iterations the control \( v^n_k \approx 0.5 \) with discrete max error \( 1.7 \cdot 10^{-5} \), the controls \( u^n_k = -1 \), \( w^n_k = 1 \) exactly, the optimal state with discrete max error \( 2.142 \cdot 10^{-5} \), the approximate cost \( G^n(v^n_k, u^n_k, w^n_k) = -0.99999999914300 \), and \( e_k = -6.122 \cdot 10^{-6} \).

Finally, the progressively refining version of each method was also applied to the above problems, with successive step sizes \( 1/64 \), \( 1/128 \), \( 1/256 \) in three equal periods, and required here about half the computing time, but yielded results of practically similar accuracy. This is due to the fact that finer discretizations become progressively more essential as the control iterate of the optimization method gets closer to the extremal control, while relatively coarser discretizations in the early iterations have not much influence on the final result.

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References


