Rivlin's Theorem on Walsh Equiconvergence

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1. Introduction

Recently Rivlin [2] has given a very interesting extension of Walsh's theorem on equiconvergence. Let \( C \) denote the complex plane, and let \( \mathcal{A}(D(\rho)), 1 < \rho < \infty \), be the class of functions \( f \) that are analytic on the disc \( D(\rho) = \{ z \in C : |z| < \rho \} \) and have a singularity on the circle \( \{ z \in C : |z| = \rho \} \). If \( f(z) = \sum_{0}^{\infty} a_{j}z^{j} \), we denote by \( S_{n}(f; z) \) the partial sum \( \sum_{0}^{n} a_{j}z^{j} \). For a positive integer \( m = nq + c \), where \( q, c \) are fixed integers, let \( \omega = e^{2\pi i/m} \). If \( \pi_{n} \) denotes the family of all polynomials of degree \( \leq n \) and if \( p_{n,m}(f; z) \) denotes such a polynomial minimizing

\[
\sum_{k=0}^{m-1} |f(\omega^{k}) - q_{n}(\omega^{k})|^{2}
\]

over all polynomials \( q_{n} \in \Pi_{n} \), then Rivlin proved

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Theorem A. Let $f \in \mathcal{A}(D(\rho))$ and let $q$ be a fixed positive integer. Then

$$
\lim_{n \to \infty} (p_{n,m}(f; z) - S_n(f; z)) = 0,
$$

(1.2)

for all $z \in D(\rho^{1+q})$, the convergence being uniform and geometric in $|z| \leq \tau < \rho^{1+q}$, where $m = nq + c$, $c$ a fixed integer. Moreover, the result is best possible in the sense that (1.2) fails for every $z$ satisfying $|z| = \rho^{1+q}$ for some $f \in \mathcal{A}(D(\rho))$.

When $m = n + 1$, Theorem A reduces to a well-known theorem of J. L. Walsh [5, 4].

Rivlin gave another extension of Walsh's theorem for functions analytic in the ellipse $\mathcal{E}(\rho)$ in $C$ which is the image of the disc $D(\rho)$ under the mapping $z = \frac{1}{2}(w + w^{-1})$. Let $\mathcal{A}(\mathcal{E}(\rho))$ denote the class of functions $f$ that are analytic on $\mathcal{E}(\rho)$ but not on any region containing the closure of $\mathcal{E}(\rho)$. Let

$$
f(z) = \sum_{k=0}^{\infty} A_k T_k(z),
$$

(1.3)

where $T_k(z)$ is the Chebyshev polynomial of degree $k$ and where the prime means that the first term in Eq. (1.3) is to be halved. Let $\xi_j^{(m)}$ ($j = 1, ..., m$) be the zeros of $T_m(x)$ (i.e., $\xi_j^{(m)} = \cos [(2j - 1) \pi/2m]$, $j = 1, ..., m$), and let $u_{n,m}(f; z)$ denote the algebraic polynomial which minimizes

$$
\sum_{j=1}^{m} |f(\xi_j^{(m)}) - p_n(\xi_j^{(m)})|^2
$$

(1.4)

over all polynomials $p_n \in \Pi_n$. If $S_n(f; z) = \sum_{k=0}^{n} A_k T_k(z)$, then Rivlin proved

Theorem B [2]. If $f \in \mathcal{A}(\mathcal{E}(\rho))$ and $q$ is any integer $>1$, then

$$
\lim_{n \to \infty} (u_{n,m}(f; z) - S_n(f; z)) = 0, \quad m = nq + c
$$

(1.5)

for all $z$ in $\mathcal{E}(\rho^{2q-1})$, the convergence being uniform and geometric on $\mathcal{E}(\tau)$ for $\tau < \rho^{2q-1}$.

In addition, Rivlin also showed that Theorem B is also true if we replace $u_{n,m}(f; z)$ by the polynomial $t_{n,m}(f; z)$ which minimizes

$$
\sum_{k=1}^{m} |f(\eta_k^{(m)}) - p_n(\eta_k^{(m)})|^2,
$$

(1.6)

where $\eta_k^{(m)}$ ($k = 1, ..., m$) are the extrema of $T_m(x)$ on $[-1, 1]$.
The method of Rivlin is based on the properties of Chebyshev polynomials and their zeros. This makes a further extension of his results difficult. Our purpose here is to propose a mixed problem of interpolation and $l_2$-approximation and to extend Rivlin's result in two directions. As a special case we obtain "help" functions which give larger regions of equiconvergence as in [1].

In Section 2 we state the problem and the main results in Theorems 1 and 2. Section 3 deals with the proof of Theorem 1, and the proof of Theorem 2 is given in Section 4.

2. PRELIMINARIES AND MAIN RESULT

Let $A(\rho)$ denote the ring \( \{z \in \mathbb{C}: \rho^{-1} < |z| < \rho, \rho > 1, \) and let $\mathcal{A}(A(\rho))$ denote the class of functions $f$ which are analytic on $A(\rho)$ but not on any region containing the closure of $A(\rho)$. Let us set

\[
f(z) = \sum_{-\infty}^{\infty} a_j z^j, \quad z \in A(\rho). \tag{2.1}
\]

We shall consider the following two problems:

**Problem A.** For given $f \in \mathcal{A}(A(\rho))$, find the polynomial $P_{rm+n}$ defined by

\[
P_{rm+n}(z) = P_{rm+n}(f; z) = \sum_{-rm-n}^{rm+n} c_r z^r \tag{2.2}
\]

which satisfies

\[
[P_{rm+n}^{(v)}(\omega^k) - f^{(v)}(\omega^k)] = 0 \quad (v = 0, 1, ..., r - 1, k = 0, 1, ..., 2m - 1), \tag{2.3}
\]

where $\omega^{2m} = 1$, and which minimizes

\[
\sum_{k=0}^{2m-1} \left| P_{rm+n}^{(v)}(\omega^k) - f^{(v)}(\omega_k) \right|^2, \tag{2.4}
\]

over all polynomials of the form (2.2) which satisfy (2.3).

**Problem B.** Find the region where the difference

\[
P_{rm+n}(f; z) - S_{rm+n}(f; z) \tag{2.5}
\]

tends to zero as $n \to \infty$, when $m = nq + c$, where $c$ and $q$ are positive integer constants, and where

\[
S_{rm+n}(f; z) = \sum_{-rm-n}^{rm+n} a_j z^j \tag{2.6}
\]

is a section of the Laurent series (2.1) of $f$. 
The solution to Problem A is given by

**Theorem 1.** The polynomial $P_{rm+n}(f; z)$ of the form (2.2) which satisfies (2.3) and minimizes (2.4) is given by

$$P_{rm+n}(f; z) = \frac{1}{2\pi i} \int_{C_R} f(t) t^{rm+n}K_i(t, z) \, dt,$$

(2.7)

where

$$z^{rm+n}K_i(t, z)(t-z) = 1 - \left(\frac{z^{2m-1}}{t^{2m-1}}\right)^r + \frac{(z^{2m-1})^r}{(t^{2m-1})^r+1} t^{2m-2n-1}(t^{2n+1} - z^{2n+1}),$$

(2.8)

and $C_R$ is the oriented boundary of the ring $A(R)$.

We postpone the proof of Theorem 1 to Section 3 and proceed to state our main result.

**Theorem 2.** If $f \in \mathcal{A}(A(\rho))$, $f(z) = f(1/z)$ for all $z \in A(\rho)$ and $P_{rm+n}(f; z)$ is the solution to Problem A, and if $m = nq + c$, where $n$, $q$, and $c$ are positive integers, then

$$\lim_{n \to \infty} [P_{rm+n}(f; z) - S_{rm+n}(f; z)] = 0,$$

(2.9)

for all $z \in A(\tau(\rho))$, where

$$\tau(\rho) = \rho^{2q-1},$$

when $r = 0$

$$= \min\{\rho^{1+(2q-2)/(qr+1)}, \rho^{1+2/(qr-1)}\},$$

when $r \geq 1$.

(2.10)

Moreover, the convergence is uniform and geometric in any compact subset of the above ring. Also the result is best possible in the sense that (2.9) fails for every $z$ on the boundary of $A(\tau(\rho))$ for some $f \in \mathcal{A}(A(\rho))$.

**Remark.** Problems A and B can also be formulated and solved in a similar way if instead of considering the minimization problem (2.4) on the zeros of $z^{2m} = 1$, we consider the same problem on the zeros of $z^{2m} = -1$. In this case, $\omega^k$ in (2.4) is replaced by $\omega^{k-1/2}$ and the corresponding polynomial $\tilde{P}_{rm+n}(f; z)$, which satisfies (2.3) and (2.4) on the zeros of $z^{2m} = -1$, is given by

$$\frac{1}{2\pi i} \int_{C_R} f(t) t^{rm+n}\tilde{K}_i(t, z) \, dt.$$
Here $\tilde{K}_1(t, z)$ is obtained from (2.8) by replacing $z^{2m} - 1$ and $t^{2m} - 1$ in (2.8) by $z^{2m} + 1$ and $t^{2m} + 1$, respectively. Also, Theorem 2 holds when $P_{rm + n}(f; z)$ replaces $P_{rm + n}(f; z)$.

When $r = 0$, Theorem A gives the polynomials $t_{n,m}(f; z)$ and $u_{n,m}(f; z)$ of Rivlin [2] according as we use the zeros of $z^{2m} + 1$ or of $z^{2m} - 1$ respectively in (2.3) and (2.4).

3. Proof of Theorem 1

Since $P_{rm + n}(f; z)$ is of the form (2.2) and satisfies (2.3), we have

$$z^{rm + n} P_{rm + n}(f; z) = Q_{2rm - 1}(z) + (z^{2m} - 1)^{r} R_{2n}(z),$$

where $R_{2n}(z) \in \Pi_{2n}$. From (2.3), we require that

$$[Q_{2rm - 1}(z) z^{-rm - n}]^{(v)}_{z = \omega^k} = f^{(v)}(\omega^k) \quad (v = 0, 1, ..., r - 1, k = 0, 1, ..., 2m - 1).$$

Equivalently, we require

$$Q^{(v)}_{2rm - 1}(\omega^k) = [z^{rm + n} f(z)]^{(v)}_{z = \omega^k}.$$

From a known formula [1], we have

$$Q_{2rm - 1}(z) = \frac{1}{2\pi i} \int_{C_R} \frac{f(t) t^{rm + n}}{t - z} \left\{ 1 - \left(\frac{z^{2m} - 1}{t^{2m} - 1}\right)^r \right\} dt. \quad (3.2)$$

In order to find $P^{(r)}_{rm + n}(\omega^k)$, we need to evaluate

$$A_1 := [Q_{2rm - 1}(z) z^{-rm - n}]^{(v)}_{\omega^k}, \quad \text{and} \quad A_2 := [(z^{2m} - 1)^{r} R_{2n}(z) z^{-rm - n}]^{(v)}_{\omega^k}.$$

Since

$$\left[ \frac{d^r}{dz^r} (z^{2m} - 1)^r \right]_{\omega^k} = r!(2m)^r \omega^{-kr},$$

it is easy to see that

$$A_2 = r!(2m)^r \omega^{-kr} R_{2n}(\omega^k) \omega^{-k(rm + n)} \quad (k = 0, 1, ..., 2m - 1). \quad (3.3)$$

Also from (3.1) and (3.2), we have

$$A_1 = \left[ z^{-rm - n} \frac{1}{2\pi i} \int_{C_R} \frac{f(t) t^{rm + n}}{t - z} dt \right]^{(r)}_{\omega^k}$$

$$- \frac{1}{2\pi i} \int_{C_R} \frac{f(t) t^{rm + n}}{(t^{2m} - 1)^r} \left[ \frac{(z^{2m} - 1)^r z^{-rm - n}}{t - z} \right]^{(r)}_{z = \omega^k} dt$$

$$= f^{(r)}(\omega^k) - \frac{r!(2m)^r}{2\pi i} \omega^{-k(rm + n + r)} \int_{C_R} \frac{f(t) t^{rm + n}}{(t^{2m} - 1)^r(t - \omega^k)} dt. \quad (3.4)$$
From (3.3) and (3.4), the problem of minimizing (2.4) reduces to minimizing

\[ \sum_{k=0}^{2m-1} |R_{2n}(\omega^k) - g(\omega^k)|^2 \]  

(3.5)

over all polynomials \( R_{2n} \in \pi_{2n} \), where

\[ g(z) = \frac{1}{2\pi i} \int_{C_R} \frac{f(t) t^{rm+n}}{(t^{2m}-1)^r+1(t-z)} \, dt. \]

In order to minimize (3.5), we replace \( g(z) \) by its Lagrange interpolant on the \( 2m \) roots of unity and use a result of Rivlin [2]. Accordingly, the Lagrange interpolant of \( g(z) \) is

\[ L_{2m-1}(z; g) = \frac{1}{2\pi i} \int_{C_R} \frac{f(t) t^{rm+n}(t^{2m}-z^{2m})}{(t^{2m}-1)^r+1(t-z)} \, dt. \]

If \( s_{2n}(z; L_{2m-1}) \) denotes the Taylor polynomial of degree \( 2n \) for \( L_{2m-1}(z; g) \), the result of Rivlin yields

\[ R_{2n}(z) = s_{2n}(z; L_{2m-1}(z; g)) \]

\[ = \frac{1}{2\pi i} \int_{C_R} \frac{f(t) t^{rm+2m-n-1} t^{2n+1}(t-z^{2n+1})}{(t^{2m}-1)^r+1(t-z)} \, dt. \]  

(3.6)

The formula (2.7) is obtained now on using (3.1), (3.2), and (3.6).

**Corollary 1.** If \( f \in \mathcal{A}(A(\rho)) \) and moreover \( f(z) = f(z^{-1}) \) for all \( z \in A(\rho) \), then

\[ z^{rm+n}P_{rm+n}(f; z) = \frac{1}{2\pi i} \int_{\Gamma} \left\{ t^{rm+n+1}K_1(t, z) \right\} dt, \]  

(3.7)

where \( \Gamma \) is the circle \( |z| = R, 1 < R < \rho \).

**Proof.** Since \( C_R \) is the union of the circles \( |z| = R \) and \( |z| = R^{-1} \), a change of variable in the integral on \( |z| = R^{-1} \) gives the result after an elementary calculation, because \( f(t) = f(t^{-1}) \).

**Remark.** We remark that when \( r = 0 \), \( P_n(f; z) \) is the polynomial \( t_{n,m}(z; f) \) of Rivlin [2].

Also from (2.6), we see that if \( f(t) = f(t^{-1}) \), then

\[ z^{rm+n}S_{rm+n}(f; z) = \frac{1}{2\pi i} \int_{\Gamma} \frac{f(t)}{t} K_0(t, z) \, dt, \]  

(3.8)
where

\[ K_0(t, z) = \left( \frac{z}{t} \right)^{rn+n} \frac{z^{rn+n+1} - t^{rn+n+1}}{z - t} + \frac{1}{t} \frac{z^{rn+n} - (1/t)^{rn+n}}{z - (1/t)}. \quad (3.9) \]

This follows also from the representation of \( f(z) \), viz.,

\[ f(z) = \frac{1}{2\pi i} \int_{\partial R} \frac{f(t)}{t(z - t)} \left[ \frac{t}{t - z} - \frac{t^{-1}}{t^{-1} - z} \right] dt, \quad (3.10) \]

when \( f(z) = f(z^{-1}) \).

**Corollary 2.** If \( f \in \mathcal{A}(A(p)) \) and if moreover \( f(z) = f(z^{-1}) \) for all \( z \in A(p) \), then

\[ P_{rm+n}(f; z) = P_{rm+n}(f; z^{-1}). \quad (3.11) \]

**Proof.** From (2.7) and (2.8), we have

\[ z^{rm+n} P_{rm+n}(f; z) = \frac{1}{2\pi i} \int_{\partial R} \frac{f(t) t^{rm+n}}{t(z - t)} \left[ 1 + \frac{(z^{2m} - 1)(1 - t^{2m-2n-1}z^{2n+1})}{(t^{2m-1})^{r+1}} \right] dt \]

and

\[ \left( \frac{1}{z} \right)^{rm+n} P_{rm+n}(f; z^{-1}) = \frac{1}{2\pi i} \int_{\partial R} \frac{f(t) t^{rm+n}}{t(z^{-1} - t)} \left[ 1 + \frac{(1 - z^{2m})^{r} z^{-2mr}}{(t^{2m-1})^{r+1} \left( 1 - \frac{t^{2m-2n-1}}{z^{2n+1}} \right)} \right] dt. \]

Changing \( t \) into \( t^{-1} \) in the above and simplifying, we have

\[ P_{rm+n}(f; z^{-1}) = \frac{1}{2\pi i} \int_{\partial R} \left( \frac{z}{t} \right)^{rn+n+1} \left[ 1 - \frac{(z^{2m} - 1)^{r} z^{-2m(r+1)}}{(t^{2m-1})^{r+1} z^{2mr}} \left( 1 - \frac{z^{-2n-1}}{t^{2m-2n-1}} \right) \right] dt. \]

From these we obtain after simplifying that

\[ P_{rm+n}(f; z) - P_{rm+n}(f; z^{-1}) = \frac{1}{2\pi i} \int_{\partial R} \frac{f(t)}{t(z - t)} \left[ \left( \frac{z}{t} \right)^{rn+n} - \left( \frac{t}{z} \right)^{rn+n} \right] dt = 0, \]

because the integrand is single-valued analytic in the annulus \( C_R \).
4. SOME LEMMAS AND PROOF OF THEOREM 2

The proof of Theorem 2 will require a number of estimates and to this effect we prove

**Lemma 1.** If \( f(z) \) satisfies the conditions of Theorem 2, then we have

\[
P_{m+n}(f; z) - S_{m+n}(f; z) = \frac{1}{2\pi i} \int_{\Gamma} \frac{f(t)}{t} A(t, z) \, dt,
\]

where \( A(t, z) \) is given by

\[
A(t, z) = S_1(t, z) + S_2(t, z) + S_3(t, z) - S_2(t^{-1}, z) - S_3(t^{-1}, z)
\]

and

\[
S_1(t, z) = -\frac{(t^{2m} + z^{2m} + 1)}{(t - z)(tz)^{m+n}},
\]

\[
S_2(t, z) = \frac{t^{m+n+1} (t^2 + z^2 - 1)}{(t^2 - 1)'(t - z) z^{m+n}},
\]

\[
S_3(t, z) = \frac{(t^2 - 1)' t^{m+n} - (z^2 - 1)'}{(t^2 - 1)'(t - z) z^{m+n}}.
\]

**Proof.** These formulae are obtained from (3.7) and (3.8) and on adding the integral

\[
\frac{1}{2\pi i} \int_{\Gamma} \frac{f(t)}{t} \left\{ \frac{z^{m+n} - t^{m+n}}{t - z} + \frac{1}{t} \frac{z^{m+n} - (1/t)^{m+n}}{z - t^{-1}} \right\} dt
\]

to the right side of (3.7), since it is easily seen to be zero when \( f(t) = f(1/t) \).

**Lemma 2.** The following identity holds:

\[
\frac{(t^{2m} - 1)' - (z^{2m} - 1)'}{t - z} = \sum_{k=0}^{2m-1} z^k t^{-k-1} A_k(t),
\]

where \( A_k(t) \) is a polynomial such that

\[
A_{2mv}(t) = A_{2mv+1}(t) = \cdots = A_{2m(v+1) - 1}(t)
\]

\[
= (t^{2m} - 1)' - \sum_{j=0}^{v} (-1)^{r-j} \binom{r}{j} t^{2mj}
\]

\[
= \sum_{j=v+1}^{r} (-1)^{r-j} \binom{r}{j} t^{2mj} \quad (v = 0, 1, \ldots, r - 1).
\]
This is easily verified. When \( r = 0 \), \( A_k(t) \)'s are all zero, and when \( r = 1 \), \( A_k(t) = t^{2m} \).

**Lemma 3.** If we set

\[
A(t, z) = \sum_{j = 1}^{mr + n} \lambda_j(t) z^j,
\]

then \( \lambda_j(t) = \lambda_{-j}(t) \), \( j = 1, 2, \ldots, rm + n \), and for \( |t| = R \) \( (1 < R < \rho) \), we have

\[
\lambda_{1, j}(t) = O(R^{-mr - n - 1}), \quad m(r - 2\lambda - 2) + n + 1 \leq |j| \leq m(r - 2\lambda) - n - 1
\]

\[
= O(R^{-mr - 2m + n}), \quad \max(0, m(r - 2\lambda) - n) \leq |j| \leq m(r - 2\lambda) + n.
\]

(4.7)

The proof of this lemma depends on Lemma 2 and (4.3). The estimates (4.7) can be used to prove Theorem 2, but we provide here a simple proof.

**Proof of Theorem 2.** Set \( A := S_1(t, z) + S_2(t, z) + S_3(t, z) \). Then from (4.3) we obtain

\[
A = \frac{z^{mr + n + 1}}{(t - z) t^{mr + n}} \frac{(1 - z^{-2m})^r}{(1 - t^{-2m})^r} \frac{z^{mr - n}}{t^{mr + 2m - n - 1}} \frac{(-1 + z^{2n + 1} t^{2m - 2n - 1})}{(t - z)}
\]

\[
eq \frac{1}{t - z} \left[ \frac{z^{mr + n + 1}}{t^{mr + n}} + \frac{z^{mr - n}}{t^{mr + 2m - n - 1}} \frac{(1 - z^{-2m})^r}{(1 - t^{-2m})^r + 1} \right]
\]

\[
eq \frac{1}{t - z} \left[ \frac{z^{mr + n + 1}}{t^{mr + n}} \left\{ O(z^{-2m}) + O(t^{-2m}) \right\} + \frac{z^{mr - n}}{t^{mr + 2m - n - 1}} \left\{ 1 + O(z^{-2m}) + O(t^{-2m}) \right\} \right],
\]

where we may assume without loss of generality that \( |z| > 1, |t| > 1 \). Moreover, we observe that if we set \( B := S_1(1/t, z) + S_3(1/t, z) \), then using (4.3) again we see that

\[
B = \frac{t^{-mr - n - 1} \left\{ \left( t^{-2m - 1} \right)^r - \left( z^{2m - 1} \right)^r \right\}}{(t^{-2m - 1})^r (t^{-1} - z) z^{mr + n}}
\]

\[
+ \frac{(z^{2m - 1})^r}{(t^{-2m - 1})^r + 1} \frac{t^{-mr - 2m + n} (t^{-2n - 1} - z^{2n + 1})}{(t^{-1} - z) z^{mr + n}}.
\]
Some simplification yields

\[ B = \frac{1}{1-tz} \left[ \frac{1}{(tz)^{m+n}} - \frac{(-1)^r z^{mr-n}}{t^{mr+n}} (1 + O(z^{-2m}) + O(t^{-2m})) \right. \\
\left. + \frac{(-1)^r z^{mr+n+1}}{t^{mr+2m-n-1}} (1 + O(z^{-2m}) + O(t^{-2m})) \right]. \]

From the above estimates for \( A \) and \( B \) we see that as \( n \to \infty \),

\[ A = O \left( \frac{z^{mr+n}}{t^{mr+n+2m}} \right) + O \left( \frac{z^{mr-n}}{t^{mr+2m-n-1}} \right) \]

and

\[ B = O \left( \frac{z^{mr-n}}{t^{mr+n}} \right) + O \left( \frac{z^{mr+n+1}}{t^{mr+2m-n-1}} \right). \]

Thus we have

\[ A(t, z) = O \left( \frac{z^{mr-n}}{t^{mr+n}} \right) + O \left( \frac{z^{mr+n+1}}{t^{mr+2m-n-1}} \right), \quad (4.8) \]

which tends to zero if

\[ |z| < \min \{ \rho^{(mr+n)/(mr-n)}, \rho^{(mr+2m-n-1)/(mr+n+1)} \}. \]

This gives the result when \( n \to \infty \) and completes the proof for \( r > 1 \).

For \( r = 0 \), \( S_3(t, z) \) and \( S_3(1/t, z) \) do not occur and the estimate in Theorem 2 is easily obtained from (4.8), since in this case \( A(t, z) = O(z^{n+1}/t^{2m-n-1}) \).

Remark. For \( r = 0 \), the polynomial \( P_n(z) \) in Theorem 1 can be easily seen to be the polynomial \( t_{n,m(n)}(z, f) \) of Rivlin [2]. In fact we can see from (2.7) and (2.8) that

\[ P_n(z) = t_{n,m(n)}(z, f) = \sum_{j=0}^{n} T_j(z) \frac{1}{2\pi i} \int_{C} f(t) \left( \frac{t^{2m-j-1}}{t^{2m} - 1} + \frac{t^{j-1}}{t^{2m} - 1} \right) dt. \]

If we set

\[ s_{n,v}(z, f) = \sum_{j=0}^{n} (A_{2vm+j} + A_{2vm-j}) T_j(z) \quad (v = 1, 2, 3, \ldots), \]

where \( f(z) \) is given by (1.3), then

\[ \lim_{n \to \infty} t_{n,m}(z; f) - s_n(z; f) - \sum_{v=1}^{l-1} s_{n,v}(z; f) = 0 \quad (4.9) \]

for \( |z| < \rho^{2lq-1} \).
Theorems 1 and 2 can be formulated for functions in $A(\rho)$ and an analogue of (4.9) can also be obtained from the representation (4.1).

It would be interesting to obtain sharpness results analogous to those of Saff and Varga [3] and the analogue of Theorem 2 above when Hermite interpolation is replaced by lacunary interpolation as in [6].

REFERENCES