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Rivlin's Theorem on Walsh Equiconvergence

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1. INTRODUCTION

Recently Rivlin [2] has given a very interesting extension of Walsh's theorem on equiconvergence. Let C denote the complex plane, and let $\mathscr{A}(D(\rho))$, $1<\rho<\infty$, be the class of functions f that are analytic on the disc $D(\rho)=\{z\in C\colon |z|<\rho\}$ and have a singularity on the circle $\{z\in C\colon |z|=\rho\}$. If $f(z)=\sum_0^\infty a_j z^j$, we denote by $S_n(f; z)$ the partial sum $\sum_0^n a_j z^j$. For a positive integer $m=nq+c$, where q, c are fixed integers, let $\omega=e^{2\pi i/m}$. If π_n denotes the family of all polynomials of degree $\leqslant n$ and if $p_{n,m}(f; z)$ denotes such a polynomial minimizing

$$\sum_{k=0}^{m-1} |f(\omega^k)-q_n(\omega^k)|^2 \tag{1.1}$$

over all polynomials $q_n\in \Pi_n$, then Rivlin proved

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THEOREM A. *Let $f \in \mathscr{A}(D(\rho))$ and let q be a fixed positive integer. Then*

$$\lim_{n \to \infty} (p_{n,m}(f; z) - S_n(f; z)) = 0, \tag{1.2}$$

for all $z \in D(\rho^{1+q})$, the convergence being uniform and geometric in $|z| \leqslant \tau < \rho^{1+q}$, where $m = nq + c$, c a fixed integer. Moreover, the result is best possible in the sense that (1.2) fails for every z satisfying $|z| = \rho^{1+q}$ for some $f \in \mathscr{A}(D(\rho))$.

When $m = n + 1$, Theorem A reduces to a well-known theorem of J. L. Walsh [5, 4].

Rivlin gave another extension of Walsh's theorem for functions analytic in the ellipse $\mathscr{E}(\rho)$ in C which is the image of the disc $D(\rho)$ under the mapping $z = \frac{1}{2}(w + w^{-1})$. Let $\mathscr{A}(\mathscr{E}(\rho))$ denote the class of functions f that are analytic on $\mathscr{E}(\rho)$ but not on any region containing the closure of $\mathscr{E}(\rho)$. Let

$$f(z) = \sum_{0}^{\infty}{}' A_k T_k(z), \tag{1.3}$$

where $T_k(z)$ is the Chebyshev polynomial of degree k and where the prime means that the first term in Eq. (1.3) is to be halved. Let $\xi_j^{(m)}$ ($j = 1, ..., m$) be the zeros of $T_m(x)$ (i.e., $\xi_j^{(m)} = \cos[(2j-1)\pi/2m]$, $j = 1, ..., m$), and let $u_{n,m}(f; z)$ denote the algebraic polynomial which minimizes

$$\sum_{j=1}^{m} |f(\xi_j^{(m)}) - p_n(\xi_j^{(m)})|^2 \tag{1.4}$$

over all polynomials $p_n \in \Pi_n$. If $S_n(f; z) = \sum'^{n}_{k=0} A_k T_k(z)$, then Rivlin proved

THEOREM B [2]. *If $f \in \mathscr{A}(\mathscr{E}(\rho))$ and q is any integer > 1, then*

$$\lim_{n \to \infty} (u_{n,m}(f; z) - S_n(f; z)) = 0, \qquad m = nq + c \tag{1.5}$$

for all z in $\mathscr{E}(\rho^{2q-1})$, the convergence being uniform and geometric on $\mathscr{E}(\tau)$ for $\tau < \rho^{2q-1}$.

In addition, Rivlin also showed that Theorem B is also true if we replace $u_{n,m}(f; z)$ by the polynomial $t_{n,m}(f; z)$ which minimizes

$$\sum_{k=1}^{m} |f(\eta_k^{(m)}) - p_n(\eta_k^{(m)})|^2, \tag{1.6}$$

where $\eta_k^{(m)}$ ($k = 1, ..., m$) are the extrema of $T_n(x)$ on $[-1, 1]$.

The method of Rivlin is based on the properties of Chebyshev polynomials and their zeros. This makes a further extension of his results difficult. Our purpose here is to propose a mixed problem of interpolation and l_2-approximation and to extend Rivlin's result in two directions. As a special case we obtain "help" functions which give larger regions of equiconvergence as in [1].

In Section 2 we state the problem and the main results in Theorems 1 and 2. Section 3 deals with the proof of Theorem 1, and the proof of Theorem 2 is given in Section 4.

2. Preliminaries and Main Result

Let $A(\rho)$ denote the ring $\{z \in C: \rho^{-1} < |z| < \rho\}$, $\rho > 1$, and let $\mathscr{A}(A(\rho))$ denote the class of functions f which are analytic on $A(\rho)$ but not on any region containing the closure of $A(\rho)$. Let us set

$$f(z) = \sum_{-\infty}^{\infty} a_j z^j, \qquad z \in A(\rho). \tag{2.1}$$

We shall consider the following two problems:

Problem A. For given $f \in \mathscr{A}(A(\rho))$, find the polynomial P_{rm+n} defined by

$$P_{rm+n}(z) = P_{rm+n}(f; z) = \sum_{-rm-n}^{rm+n} c_\nu z^\nu \tag{2.2}$$

which satisfies

$$[P^{(\nu)}_{rm+n}(\omega^k) - f^{(\nu)}(\omega^k)] = 0 \qquad (\nu = 0, 1, ..., r-1, k = 0, 1, ..., 2m-1), \tag{2.3}$$

where $\omega^{2m} = 1$, and which minimizes

$$\sum_{k=0}^{2m-1} |P^{(r)}_{rm+n}(\omega^k) - f^{(r)}(\omega_k)|^2, \tag{2.4}$$

over all polynomials of the form (2.2) which satisfy (2.3).

Problem B. Find the region where the difference

$$P_{rm+n}(f; z) - S_{rm+n}(f; z) \tag{2.5}$$

tends to zero as $n \to \infty$, when $m = nq + c$, where c and q are positive integer constants, and where

$$S_{rm+n}(f; z) = \sum_{-rm-n}^{rm+n} a_j z^j \tag{2.6}$$

is a section of the Laurent series (2.1) of f.

The solution to Problem A is given by

THEOREM 1. *The polynomial $P_{rm+n}(f; z)$ of the form* (2.2) *which satisfies* (2.3) *and minimizes* (2.4) *is given by*

$$P_{rm+n}(f; z) = \frac{1}{2\pi i} \int_{C_R} f(t)\, t^{rm+n} K_1(t, z)\, dt, \tag{2.7}$$

where

$$z^{rm+n} K_1(t, z)(t-z) = 1 - \left(\frac{z^{2m}-1}{t^{2m}-1}\right)^r + \frac{(z^{2m}-1)^r}{(t^{2m}-1)^{r+1}}\, t^{2m-2n-1}(t^{2n+1} - z^{2n+1}), \tag{2.8}$$

and C_R is the oriented boundary of the ring $A(R)$.

We postpone the proof of Theorem 1 to Section 3 and proceed to state our main result.

THEOREM 2. *If $f \in \mathscr{A}(A(\rho))$, $f(z) = f(1/z)$ for all $z \in A(\rho)$ and $P_{rm+n}(f; z)$ is the solution to Problem* A, *and if $m = nq + c$, where n, q, and c are positive integers, then*

$$\lim_{n \to \infty} [P_{rm+n}(f; z) - S_{rm+n}(f; z)] = 0, \tag{2.9}$$

for all $z \in A(\tau(\rho))$, where

$$\begin{aligned} \tau(\rho) &= \rho^{2q-1}, && \text{when} \quad r = 0 \\ &= \min\{\rho^{1+(2q-2)/(qr+1)}, \rho^{1+2/(qr-1)}\}, && \text{when} \quad r \geqslant 1. \end{aligned} \tag{2.10}$$

Moreover, the convergence is uniform and geometric in any compact subset of the above ring. Also the result is best possible in the sense that (2.9) *fails for every z on the boundary of $A(\tau(\rho))$ for some $f \in \mathscr{A}(A(\rho))$.*

Remark. Problems A and B can also be formulated and solved in a similar way if instead of considering the minimization problem (2.4) on the zeros of $z^{2m} = 1$, we consider the same problem on the zeros of $z^{2m} = -1$. In this case, ω^k in (2.4) is replaced by $\omega^{k-1/2}$ and the corresponding polynomial $\tilde{P}_{rm+n}(f; z)$, which satisfies (2.3) and (2.4) on the zeros of $z^{2m} = -1$, is given by

$$\frac{1}{2\pi i} \int_{C_R} f(t)\, t^{rm+n} \tilde{K}_1(t, z)\, dt.$$

Here $\tilde{K}_1(t, z)$ is obtained from (2.8) by replacing $z^{2m}-1$ and $t^{2m}-1$ in (2.8) by $z^{2m}+1$ and $t^{2m}+1$, respectively. Also, Theorem 2 holds when $\tilde{P}_{rm+n}(f; z)$ replaces $P_{rm+n}(f; z)$.

When $r=0$, Theorem A gives the polynomials $t_{n,m}(f; z)$ and $u_{n,m}(f; z)$ of Rivlin [2] according as we use the zeros of $z^{2m}+1$ or of $z^{2m}-1$ respectively in (2.3) and (2.4).

3. Proof of Theorem 1

Since $P_{rm+n}(f; z)$ is of the form (2.2) and satisfies (2.3), we have

$$z^{rm+n} P_{rm+n}(f; z) = Q_{2rm-1}(z) + (z^{2m}-1)^r R_{2n}(z), \tag{3.1}$$

where $R_{2n}(z) \in \Pi_{2n}$. From (2.3), we require that

$$[Q_{2rm-1}(z)\, z^{-rm-n}]^{(v)}_{z=\omega^k} = f^{(v)}(\omega_k) \quad (v=0, 1, ..., r-1, k=0, 1, ..., 2m-1).$$

Equivalently, we require

$$Q^{(v)}_{2mr-1}(\omega^k) = [z^{rm+n} f(z)]^{(v)}_{z=\omega^k}.$$

From a known formula [1], we have

$$Q_{2mr-1}(z) = \frac{1}{2\pi i} \int_{C_R} \frac{f(t)\, t^{rm+n}}{t-z} \left\{1 - \left(\frac{z^{2m}-1}{t^{2m}-1}\right)^r\right\} dt. \tag{3.2}$$

In order to find $P^{(r)}_{rm+n}(\omega^k)$, we need to evaluate

$$A_1 := [Q_{2rm-1}(z)\, z^{-rm-n}]^{(v)}_{\omega^k} \quad \text{and} \quad A_2 := [(z^{2m}-1)^r R_{2n}(z)\, z^{-rm-n}]^{(v)}_{\omega^k}.$$

Since

$$\left[\frac{d^r}{dz^r}(z^{2m}-1)^r\right]_{\omega^k} = r!(2m)^r\, \omega^{-kr},$$

it is easy to see that

$$A_2 = r!(2m)^r\, \omega^{-kr} R_{2n}(\omega^k)\, \omega^{-k(rm+n)} \qquad (k=0, 1, ..., 2m-1). \tag{3.3}$$

Also from (3.1) and (3.2), we have

$$\begin{aligned} A_1 &= \left[z^{-rm-n} \frac{1}{2\pi i} \int_{C_R} \frac{f(t)\, t^{rm+n}}{t-z}\, dt\right]^{(r)}_{\omega^k} \\ &\quad - \frac{1}{2\pi i} \int_{C_R} \frac{f(t)\, t^{rm+n}}{(t^{2m}-1)^r} \left[\frac{(z^{2m}-1)^r\, z^{-rm-n}}{t-z}\right]^{(r)}_{z=\omega^k} dt \\ &= f^{(r)}(\omega^k) - \frac{r!(2m)^r}{2\pi i}\, \omega^{-k(rm+n+r)} \int_{C_R} \frac{f(t)\, t^{rm+n}}{(t^{2m}-1)^r (t-\omega^k)}\, dt. \end{aligned} \tag{3.4}$$

From (3.3) and (3.4), the problem of minimizing (2.4) reduces to minimizing

$$\sum_{k=0}^{2m-1} |R_{2n}(\omega^k) - g(\omega^k)|^2 \tag{3.5}$$

over all polynomials $R_{2n} \in \pi_{2n}$, where

$$g(z) = \frac{1}{2\pi i} \int_{C_R} \frac{f(t)\, t^{rm+n}}{(t^{2m}-1)^r (t-z)}\, dt.$$

In order to minimize (3.5), we replace $g(z)$ by its Lagrange interpolant on the $2m$ roots of unity and use a result of Rivlin [2]. Accordingly, the Lagrange interpolant of $g(z)$ is

$$L_{2m-1}(z; g) = \frac{1}{2\pi i} \int_{C_R} \frac{f(t)\, t^{rm+n}(t^{2m} - z^{2m})}{(t^{2m}-1)^{r+1}(t-z)}\, dt.$$

If $s_{2n}(z; L_{2m-1})$ denotes the Taylor polynomial of degree $2n$ for $L_{2m-1}(z; g)$, the result of Rivlin yields

$$\begin{aligned} R_{2n}(z) &= s_{2n}(z; L_{2m-1}(z; g)) \\ &= \frac{1}{2\pi i} \int_{C_R} \frac{f(t)\, t^{mr+2m-n-1}(t^{2n+1} - z^{2n+1})}{(t^{2m}-1)^{r+1}(t-z)}\, dt. \end{aligned} \tag{3.6}$$

The formula (2.7) is obtained now on using (3.1), (3.2), and (3.6).

COROLLARY 1. *If $f \in \mathscr{A}(A(\rho))$ and if moreover $f(z) = f(z^{-1})$ for all $z \in A(\rho)$, then*

$$\begin{aligned} z^{rm+n} P_{rm+n}(f; z) = \frac{1}{2\pi i} \int_{\Gamma} \frac{f(t)}{t} \Big\{ & t^{rm+n+1} K_1(t, z) \\ & - \left(\frac{1}{t}\right)^{rm+n+1} K_1\left(\frac{1}{t}, z\right) \Big\}\, dt, \end{aligned} \tag{3.7}$$

where Γ is the circle $|z| = R$, $1 < R < \rho$.

Proof. Since C_R is the union of the circles $|z| = R$ and $|z| = R^{-1}$, a change of variable in the integral on $|z| = R^{-1}$ gives the result after an elementary calculation, because $f(t) = f(t^{-1})$.

Remark. We remark that when $r = 0$, $P_n(f; z)$ is the polynomial $t_{n,m}(z; f)$ of Rivlin [2].

Also from (2.6), we see that if $f(t) = f(t^{-1})$, then

$$z^{rm+n} S_{rm+n}(f; z) = \frac{1}{2\pi i} \int_{\Gamma} \frac{f(t)}{t} K_0(t, z)\, dt, \tag{3.8}$$

where

$$K_0(t, z) = \left(\frac{z}{t}\right)^{rm+n} \frac{z^{rm+n+1} - t^{rm+n+1}}{z-t} + \frac{1}{t} \frac{z^{rm+n} - (1/t)^{rm+n}}{z-(1/t)}. \quad (3.9)$$

This follows also from the representation of $f(z)$, viz.,

$$f(z) = \frac{1}{2\pi i} \int_\Gamma \frac{f(t)}{t} \left[\frac{t}{t-z} - \frac{t^{-1}}{t^{-1}-z}\right] dt, \quad (3.10)$$

when $f(z) = f(z^{-1})$.

COROLLARY 2. *If* $f \in \mathscr{A}(A(\rho))$ *and if moreover* $f(z) = f(z^{-1})$ *for all* $z \in A(\rho)$, *then*

$$P_{rm+n}(f; z) = P_{rm+n}(f; z^{-1}). \quad (3.11)$$

Proof. From (2.7) and (2.8), we have

$$z^{rm+n} P_{rm+n}(f; z) = \frac{1}{2\pi i} \int_{C_R} \frac{f(t)\, t^{rm+n}}{t-z} \times \left[1 + \frac{(z^{2m}-1)(1-t^{2m-2n-1} z^{2n+1})}{(t^{2m}-1)^{r+1}}\right] dt$$

and

$$\left(\frac{1}{z}\right)^{rm+n} P_{rm+n}(f; z^{-1}) = \frac{1}{2\pi i} \int_{C_R} \frac{f(t)\, t^{rm+n}}{t-z^{-1}} \times \left[1 + \frac{(1-z^{2m})^r z^{-2mr}}{(t^{2m}-1)^{r+1}} \left(1 - \frac{t^{2m-2n-1}}{z^{2n+1}}\right)\right] dt.$$

Changing t into t^{-1} in the above and simplifying, we have

$$P_{rm+n}(f; z^{-1}) = \frac{1}{2\pi i} \int_{C_R} \left(\frac{z}{t}\right)^{rm+n+1} \times \left[1 - \frac{(z^{2m}-1)^r z^{-2m(r+1)}}{(t^{2m}-1)^{r+1} z^{2mr}} \left(1 - \frac{z^{-2n-1}}{t^{2m-2n-1}}\right)\right] dt.$$

From these we obtain after simplifying that

$$P_{rm+n}(f; z) - P_{rm+n}(f; z^{-1}) = \frac{1}{2\pi i} \int_{C_R} \frac{f(t)}{t-z} \left[\left(\frac{t}{z}\right)^{rm+n} - \left(\frac{z}{t}\right)^{rm+n}\right] dt = 0,$$

because the integrand is single-valued analytic in the annulus C_R.

4. Some Lemmas and Proof of Theorem 2

The proof of Theorem 2 will require a number of estimates and to this effect we prove

LEMMA 1. *If $f(z)$ satisfies the conditions of Theorem 2, then we have*

$$P_{rm+n}(f; z) - S_{rm+n}(f; z) = \frac{1}{2\pi i} \int_{\Gamma} \frac{f(t)}{t} A(t, z)\, dt, \tag{4.1}$$

where $A(t, z)$ is given by

$$A(t, z) = S_1(t, z) + S_2(t, z) + S_3(t, z) - S_2(t^{-1}, z) - S_3(t^{-1}, z) \tag{4.2}$$

and

$$\begin{aligned} S_1(t, z) &= -\frac{(t^{2mr+2n+1} - z^{2mr+2n+1})}{(t-z)(tz)^{rm+n}}, \\ S_2(t, z) &= \frac{t^{mr+n+1}\{(t^{2m}-1)^r - (z^{2m}-1)^r\}}{(t^{2m}-1)^r (t-z)\, z^{rm+n}}, \\ S_3(t, z) &= \frac{(z^{2m}-1)^r\, t^{mr+2m-n}(t^{2n+1} - z^{2n+1})}{(t^{2m}-1)^{r+1}(t-z)\, z^{rm+n}}. \end{aligned} \tag{4.3}$$

Proof. These formulae are obtained from (3.7) and (3.8) and on adding the integral

$$\frac{1}{2\pi i} \int_{\Gamma} \frac{f(t)}{t} \left\{ \frac{z^{rm+n} - t^{rm+n}}{t-z} + \frac{1}{t} \frac{z^{rm+n} - (1/t)^{rm+n}}{z - t^{-1}} \right\} dt$$

to the right side of (3.7), since it is easily seen to be zero when $f(t) = f(1/t)$.

LEMMA 2. *The following identity holds:*

$$\frac{(t^{2m}-1)^r - (z^{2m}-1)^r}{t-z} = \sum_{k=0}^{2mr-1} z^k t^{-k-1} A_k(t), \tag{4.4}$$

where $A_k(t)$ is a polynomial such that

$$\begin{aligned} A_{2m\nu}(t) &= A_{2m\nu+1}(t) = \cdots = A_{2m(\nu+1)-1}(t) \\ &= (t^{2m}-1)^r - \sum_{j=0}^{\nu} (-1)^{r-j} \binom{r}{j} t^{2mj} \\ &= \sum_{j=\nu+1}^{r} (-1)^{r-j} \binom{r}{j} t^{2mj} \qquad (\nu = 0, 1, \ldots, r-1). \end{aligned} \tag{4.5}$$

This is easily verified. When $r=0$, $A_k(t)$'s are all zero, and when $r=1$, $A_k(t)=t^{2m}$.

LEMMA 3. *If we set*

$$A(t,z)=\sum_{j=-mr-n}^{mr+n} \lambda_j(t)\, z^j, \tag{4.6}$$

then $\lambda_j(t)=\lambda_{-j}(t)$, $j=1, 2, ..., rm+n$, *and for* $|t|=R$ $(1<R<\rho)$, *we have*

$$\begin{aligned}\lambda_{|j|}(t)&=O(R^{-mr-n-1}), &&m(r-2\lambda-2)+n+1\leqslant |j| \leqslant m(r-2\lambda)-n-1\\ &=O(R^{-mr-2m+n}), &&\max(0, m(r-2\lambda)-n)\leqslant |j| \leqslant m(r-2\lambda)+n.\end{aligned} \tag{4.7}$$

The proof of this lemma depends on Lemma 2 and (4.3). The estimates (4.7) can be used to prove Theorem 2, but we provide here a simple proof.

Proof of Theorem 2. Set $A:=S_1(t,z)+S_2(t,z)+S_3(t,z)$. Then from (4.3) we obtain

$$\begin{aligned}A&=\frac{z^{mr+n+1}}{(t-z)\,t^{mr+n}}-\frac{(1-z^{-2m})^r}{(1-t^{-2m})^{r+1}}\frac{z^{mr-n}}{t^{mr+2m-n-1}}\frac{(-1+z^{2n+1}t^{2m-2n-1})}{(t-z)}\\ &=\frac{1}{t-z}\left[\frac{z^{mr+n+1}}{t^{mr+n}}+\frac{z^{mr-n}(1-z^{-2m})^r}{t^{mr+2m-n-1}(1-t^{-2m})^{r+1}}\right.\\ &\qquad\left.-\frac{z^{mr+n+1}}{t^{mr+n}}\frac{(1-z^{-2m})^r}{(1-t^{-2m})^{r+1}}\right]\\ &=\frac{1}{t-z}\left[\frac{z^{mr+n+1}}{t^{mr+n}}\{O(z^{-2m})+O(t^{-2m})\}\right.\\ &\qquad\left.+\frac{z^{mr-n}}{t^{mr+2m-n-1}}\{1+O(z^{-2m})+O(t^{-2m})\}\right],\end{aligned}$$

where we may assume without loss of generality that $|z|>1$, $|t|>1$. Moreover, we observe that if we set $B:=S_1(1/t,z)+S_3(1/t,z)$, then using (4.3) again we see that

$$\begin{aligned}B&=\frac{t^{-mr-n-1}\{(t^{-2m}-1)^r-(z^{2m}-1)^r\}}{(t^{-2m}-1)^r(t^{-1}-z)\,z^{mr+n}}\\ &\qquad+\frac{(z^{2m}-1)^r}{(t^{-2m}-1)^{r+1}}\frac{t^{-mr-2m+n}(t^{-2n-1}-z^{2n+1})}{(t^{-1}-z)\,z^{mr+n}}.\end{aligned}$$

Some simplification yields

$$B=\frac{1}{1-tz}\left[\frac{1}{(tz)^{rm+n}}-\frac{(-1)^r z^{mr-n}}{t^{mr+n}}(1+O(z^{-2m})+O(t^{-2m}))\right.$$
$$\left.+\frac{(-1)^r z^{mr+n+1}}{t^{mr+2m-n-1}}(1+O(z^{-2m})+O(t^{-2m}))\right].$$

From the above estimates for A and B we see that as $n\to\infty$,

$$A=O\left(\frac{z^{mr+n}}{t^{mr+n+2m}}\right)+O\left(\frac{z^{mr-n}}{t^{mr+2m-n-1}}\right)$$

and

$$B=O\left(\frac{z^{mr-n}}{t^{mr+n}}\right)+O\left(\frac{z^{mr+n+1}}{t^{mr+2m-n-1}}\right).$$

Thus we have

$$A(t,z)=O\left(\frac{z^{mr-n}}{t^{mr+n}}\right)+O\left(\frac{z^{mr+n+1}}{t^{mr+2m-m-1}}\right), \tag{4.8}$$

which tends to zero if

$$|z|<\min\{\rho^{(mr+n)/(mr-n)},\ \rho^{(mr+2m-n-1)/(mr+n+1)}\}.$$

This gives the result when $n\to\infty$ and completes the proof for $r>1$.

For $r=0$, $S_2(t,z)$ and $S_2(1/t,z)$ do not occur and the estimate in Theorem 2 is easily obtained from (4.8), since in this case $A(t,z)=O(z^{n+1}/t^{2m-n-1})$.

Remark. For $r=0$, the polynomial $P_n(z)$ in Theorem 1 can be easily seen to be the polynomial $t_{n,m(n)}(z,f)$ of Rivlin [2]. In fact we can see from (2.7) and (2.8) that

$$P_n(z)=t_{n,m(n)}(z,f)=\sum_{j=0}^{n}{}' T_j(z)\frac{1}{2\pi i}\int_\Gamma f(t)\left(\frac{t^{2m-j-1}}{t^{2m}-1}+\frac{t^{j-1}}{t^{2m}-1}\right)dt.$$

If we set

$$s_{n,v}(z,f)=\sum_{j=0}^{n}{}'(A_{2vm+j}+A_{2vm-j})\,T_j(z) \qquad (v=1,2,3,...),$$

where $f(z)$ is given by (1.3), then

$$\lim_{n\to\infty} t_{n,m}(z;f)-s_n(z;f)-\sum_{v=1}^{l-1}s_{n,v}(z;f)=0 \tag{4.9}$$

for $|z|<\rho^{2lq-1}$.

Theorems 1 and 2 can be formulated for functions in $\mathscr{A}(A(\rho))$ and an analogue of (4.9) can also be obtained from the representation (4.1).

It would be interesting to obtain sharpness results analogous to those of Saff and Varga [3] and the analogue of Theorem 2 above when Hermite interpolation is replaced by lacunary interpolation as in [6].

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