

ACADEMIC
PRESSAvailable at
WWW.MATHEMATICSWEB.ORG
POWERED BY SCIENCE @ DIRECT®

Journal of Number Theory 103 (2003) 71–76

JOURNAL OF
Number
Theory<http://www.elsevier.com/locate/jnt>

Some explicit badly approximable pairs

Keith Briggs

BT Exact, Adastral Park, Polaris 134, Suffolk IP5 3RE, UK

Received 13 November 2002; revised 1 May 2003

Communicated by D. Goss

Abstract

I consider the Diophantine approximation problem of sup-norm simultaneous rational approximation with common denominator of a pair of irrational numbers, and compute explicitly some pairs with large approximation constant. One of these pairs is the most badly approximable pair yet computed.

© 2003 Elsevier Inc. All rights reserved.

The theory of approximation of a single irrational number by rationals is well known, and for our purposes the relevant facts may be summarized as follows. We measure the goodness of approximation of the rational number p/q to α by $c(\alpha, p, q) \equiv q|q\alpha - p|$. For each irrational α (without loss of generality, we may assume $0 < \alpha < 1$) we know by Dirichlet's theorem that there are infinitely many rationals p/q such that $|\alpha - p/q| < 1/q^2$, or $c(\alpha, p, q) < 1$. It is therefore of interest to ask how small one may make γ in $c(\alpha, p, q) < \gamma$ before this property fails to hold. The *approximation constant* of α is thus defined as $c(\alpha) \equiv \liminf_{q \rightarrow \infty} c(\alpha, p, q)$. Here, of course, for each q we choose the p which minimizes $c(\alpha, p, q)$. Numbers α with a large $c(\alpha)$ are hard to approximate by rationals. The *one-dimensional Diophantine approximation constant*, defined as $c_1 = \sup_{\alpha \in \mathbb{R}} c(\alpha)$, has the value $1/\sqrt{5}$, attained at $\alpha = (\sqrt{5} - 1)/2$.

Otherwise expressed, this means that c_1 is the unique number such that for each $\varepsilon > 0$, the inequality $c(\alpha, p, q) < c_1 + \varepsilon$ has infinitely many rational solutions p/q for all α , whereas there is at least one α such that $c(\alpha, p, q) < c_1 - \varepsilon$ has only finitely many rational solutions.

These results completely solve the problem of rational approximation in one dimension, but by contrast the situation in two or more dimensions is much more

E-mail address: keith.briggs@bt.com.

complex and in fact the value of the analogous constant c_n for $n \geq 2$ is unknown [12,13].

We wish to simultaneously approximate a pair of irrationals by a pair of rationals with common denominator and to measure the closeness of approximation by the maximum error in the two components, so we make the definitions: for $p = (p_1, p_2) \in \mathbb{Z}^2$, $q \in \mathbb{Z}$, $\alpha = (\alpha_1, \alpha_2) \in \mathbb{R}^2$, let

$$c(\alpha, p, q) = q \max(|q\alpha_1 - p_1|^2, |q\alpha_2 - p_2|^2)$$

and

$$c(\alpha) = \liminf_{q \rightarrow \infty} \{c(\alpha, p, q), p \in \mathbb{Z}^2, q \in \mathbb{Z}\}.$$

The two-dimensional (sup-norm) simultaneous Diophantine approximation constant is then

$$c_2 = \sup_{\alpha \in \mathbb{R}^2} c(\alpha).$$

Despite much work over the last few decades [1,2,4,6–9,12,13], the value of c_2 is unknown, though folklore suggests that its value is $\frac{2}{7}$. Adams [1] has shown that this is the correct value if we restrict the pair (α_1, α_2) to cubic number fields, but his result does not give us a constructive procedure to identify pairs with large $c(\alpha)$.

Here, however, I use a theorem of Cusick together with high-precision numerical computation to explicitly compute examples of such pairs. These have potential applications to numerical simulation studies of dynamical systems on the 2-torus, where (α_1, α_2) represent the winding number of periodic orbits.

Cusick's construction makes use of the cubic number field $\mathbb{Q}(\theta)$, where $\theta = 2 \cos(2\pi/7)$, of smallest positive discriminant, namely 49. For details on cubic fields and their integral bases, I refer to [5].

The results of Cusick [7,8] state that for any integral basis $\{1, \alpha, \beta\}$ of $\mathbb{Q}(\theta)$, we have $c^* < \frac{2}{7}$, where c^* is the infimum of those c such that

$$|x + \alpha y + \beta z| \max(y^2, z^2) < c$$

(with y and z not both zero) has infinitely many solutions in integers x, y, z . Additionally, for any $\varepsilon > 0$ there is an integral basis $\{1, \alpha, \beta\}$ such that

$$\frac{2}{7} - c^*(\alpha, \beta) < \varepsilon$$

iff

1. The continued fraction of θ has patterns $[\dots, n_1, 1, 1, n_2, \dots]$ with n_1, n_2 arbitrarily large; or,
2. The continued fraction of θ has patterns $[\dots, n_1, 2, n_2, \dots]$ with n_1, n_2 arbitrarily large.

It is not known whether either of the last two conditions are satisfied. Note that this theorem relates to the dual problem to simultaneous Diophantine approximation, namely approximation to zero by linear forms. Hence, it is not immediately apparent that the upper bound of $\frac{2}{7}$ that it gives for c^* that it defines is relevant to the problem of determining c_2 . However, from other results of Cusick (Corollary 1 on p. 187 of [6], along with Theorem 3 on p. 127 of [7]), we have that for the particular field $\mathbb{Q}(\theta)$, $c^*(\alpha, \beta) = c(\alpha, \beta)$ for all integral bases. Also, by a theorem of Davenport [10], we have $\sup c^*(\alpha, \beta) = \sup c(\alpha, \beta)$, where the sups are over all irrational pairs, not necessarily in a cubic field.

Thus, if the above patterns in the continued fraction of θ do in fact exist, Cusick’s theorem gives us a way of finding explicit pairs (which together with 1 form an integral basis of $\mathbb{Q}(\theta)$) with a value of c close to $\frac{2}{7}$. Even if n_1, n_2 do not become arbitrarily large, just the presence of some large values gives us potential candidates for very badly approximable pairs.

From results in [8], it follows that for an integral basis of the form $\{1, p\theta + q\theta^2, r\theta + s\theta^2\}$, $\begin{pmatrix} p & q \\ r & s \end{pmatrix} \in PSL(2, \mathbb{Z})$, where $-q/p$ and $-s/r$ are rational approximants to θ obtained by truncating the continued fraction at the points where condition 1 or 2 is satisfied, c^* is explicitly given by

$$\begin{aligned} c^* &= 1/\max\{|A + B + C|, |A - B + C|, |C - B^2/(4A)|, |A - B^2/(4C)|\} \\ &= 1/\max\{|A + B + C|, |A - B + C|, 49/|4A|, 49/|4C|\}, \end{aligned}$$

where

$$\begin{bmatrix} A \\ B \\ C \end{bmatrix} = \begin{bmatrix} s^2 & -rs & r^2 \\ -2qs & ps + qr & -2pr \\ q^2 & -pq & p^2 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

with

$$\begin{aligned} a &= (\theta_2^2 - \theta^2)(\theta^2 - \theta_1^2), \\ b &= (\theta_2^2 - \theta^2)(\theta_1 - \theta) + (\theta_2 - \theta)(\theta_1^2 - \theta^2), \\ c &= (\theta - \theta_2)(\theta_1 - \theta), \\ \theta &= 2 \cos(2\pi/7), \\ \theta_1 &= 2 \cos(4\pi/7), \\ \theta_2 &= 2 \cos(6\pi/7). \end{aligned}$$

With this background, I can now state the main result of this paper: I have exactly computed over 6 million partial quotients of the continued fraction of θ (directly from the defining cubic $x^3 + x^2 - 2x - 1$), and the required patterns do indeed occur, though very infrequently. The largest values of c^* , with the corresponding fractional parts of $\alpha = p\theta + q\theta^2$ and $\beta = r\theta + s\theta^2$ occur at:

- (A) positions 57–60: [..., 60, 1, 1, 50, ...], $c^* \approx 0.2851877$:
 $\alpha \approx 0.4563286858107963651609830446124431560745665647128596153008802$,
 $\beta \approx 0.4781573193903170892895817415258772866671562381178937772663665$,
- (B) positions 2927–2930: [..., 22, 1, 1, 22, ...], $c^* \approx 0.2853154$:
 $\alpha \approx 0.1554011929520066325796747316744656830061413509133865038820677$,
 $\beta \approx 0.6003679362632065361061389158735863615694126556922931077332356$,
- (C) positions 3626–3629: [..., 272, 1, 1, 215, ...], $c^* \approx 0.2855726$:
 $\alpha \approx 0.6530646111210617321254054547968773238346090082060701183776580$,
 $\beta \approx 0.9410463762107594592302548739412493098027738320829952592216557$,
- (D) positions 33 877–33 880: [..., 81, 1, 1, 78, ...], $c^* \approx 0.2856261$:
 $\alpha \approx 0.9319638477108390366188499907354642637920661848031694636081724$,
 $\beta \approx 0.7032571495109702868148790086182835032528572663181375225766851$,
- (E) positions 215 987–215 990: [..., 124, 1, 1, 129, ...], $c^* \approx 0.2856678$:
 $\alpha \approx 0.4375520476578757564544576313180510209212270982522655674846137$,
 $\beta \approx 0.5646614639128419094417646922292433724548272488193131214134926$,
- (F) positions 957 740–957 743 [..., 460, 1, 1, 415, ...], $c^* \approx 0.28568046$:
 $\alpha \approx 0.6134980317071692745070006892224661159462079954445253478668675$,
 $\beta \approx 0.9411544329571988683307282702558980820407618535628393885417987$,
- (G) positions 1 650 050–1 650 053: [..., 648, 1, 1, 666, ...], $c^* \approx 0.2857082$:
 $\alpha \approx 0.4848739572889332951989678247806190621159456336657613155291560$,
 $\beta \approx 0.5404925035004667478257428539575752367424111926723566428410541$.

These calculations involve extremely large integer and floating-point numbers; in case (G) the absolute values of the integers p, q, r, s are of the order $2^{3 \times 10^6}$, and the calculation of c^* requires floating-point operations of about twice this precision. In fact, these examples all come from cases of Cusick's first condition, and c^* is given by $49/|4A|$ or $49/|4C|$. Of course, the approximate decimal values for α, β given above are insufficient to represent the true values, but these may be reconstructed if required from the continued fraction of θ .

An independent verification of these results may be obtained by giving the values α, β as input to a simultaneous Diophantine approximation algorithm. Such an algorithm finds all best simultaneous approximants up to a given denominator. For the computation of sup-norm best approximants, an algorithm has been given by

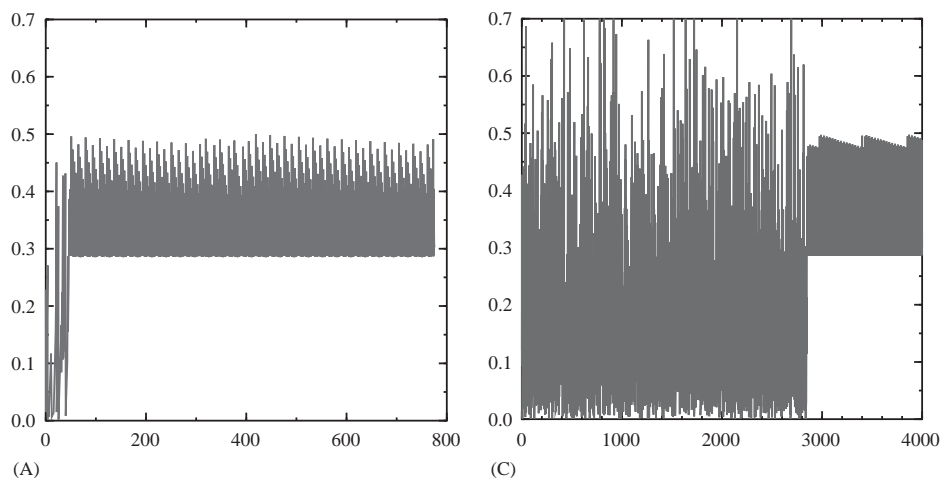


Fig. 1. $c(x, \beta, q)$ vs. $\log_{10}(q)$ at best approximants for two integral bases $(1, \alpha, \beta)$ of the field $\mathbb{Q}(\theta)$. Left: case (A), right: case (C).

Furtwängler [3,11]. Fig. 1 shows the behaviour of the Furtwängler algorithm applied to the pairs (A) and (C) above. The approximation constant estimated from the minimum c after ignoring the initial transient is about 0.2856, verifying the more precise value of c^* above. But the chief point to be noted is the extremely long initial transient. Until a sufficient large denominator q is reached, these pairs would in fact appear to be *not* badly approximable.

I have thus exhibited some explicit pairs which are very badly approximable by rationals. I believe that the value 0.2857082 above is the largest explicitly computed lower bound for the two-dimensional simultaneous Diophantine approximation constant c_2 .

The question remains open as to whether there are pairs (necessarily unrelated to the field $\mathbb{Q}(\theta)$) with approximation constant larger than $\frac{2}{7}$.

References

- [1] W.W. Adams, Simultaneous Diophantine approximations and cubic irrationals, *Pacific J. Math.* 30 (1969) 1–14.
- [2] W.W. Adams, The best two-dimensional Diophantine approximation constant for cubic irrationals, *Pacific J. Math.* 91 (1980) 29–30.
- [3] K.M. Briggs, On the Furtwängler algorithm for simultaneous rational approximation, preprint, to be submitted for publication, 2001.
- [4] J.W.S. Cassels, Simultaneous Diophantine approximation, *J. Lond. Math. Soc.* 30 (1955) 119–121.
- [5] H. Cohen, A Course in Computational Algebraic Number Theory, in: *Graduate Texts in Mathematics*, Vol. 138, Springer, Berlin, 1993.

- [6] T.W. Cusick, Formulas for some Diophantine approximation constants, *Math. Ann.* 197 (1972) 182–188.
- [7] T.W. Cusick, Formulas for some Diophantine approximation constants II, *Acta Arith.* 26 (1974) 117–128.
- [8] T.W. Cusick, The two-dimensional Diophantine approximation constant, *Monatsh. Math.* 78 (1974) 297–304.
- [9] T.W. Cusick, The two-dimensional Diophantine approximation constant. II, *Pacific J. Math.* 105 (1983) 53–67.
- [10] H. Davenport, Simultaneous Diophantine approximation, *Proc. London Math. Soc.* 2 (1952) 403–416.
- [11] Ph. Furtwängler, Über die simultane Approximation von Irrationalzahlen (Zweite Mitteilung), *Math. Ann.* 99 (1928) 71–83.
- [12] G. Szekeres, The N -dimensional approximation constant, *Bull. Austral. Math. Soc.* 29 (1984) 119–125.
- [13] G. Szekeres, Computer examination of the 2-dimensional simultaneous approximation constant, *Ars Combin.* 19A (1985) 237–243.