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Bifurcation analysis of modified Leslie-Gower predator-prey model with double Allee effect

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Abstract In the present article, a modified Leslie-Gower predator-prey model with double Allee effect, affecting the prey population, is proposed and analyzed. We have considered both strong and weak Allee effects separately. The equilibrium points of the system and their local stability have been studied. It is shown that the dynamics of the system are highly dependent upon the initial conditions. The local bifurcations (Hopf, saddle-node, Bogdanov-Takens) have been investigated by considering sufficient parameter(s) as the bifurcation parameter(s). The local existence of the limit cycle emerging through Hopf bifurcation and its stability is studied by means of the first Lyapunov coefficient. The numerical simulations have been done in support of the analytical findings. The result shows the emergence of homoclinic loop. The possible phase portraits and parametric diagrams have been depicted.

1. Introduction

The predator-prey interactions are the most challenging areas of the population ecology. It's universal existence and importance has attracted the Ecologists, Mathematicians and Biologists during the last few decades. A pioneer work Lotka-Volterra predator-prey model, proposed by Lotka [1] and Volterra [2] independently, is the first and simplest mathematical model. The Lotka-Volterra predator-prey model has neglected many real situations and complexities, so a number of changes in the model have been done by the researchers to improve the realism. Leslie and Gower [3] proposed a predator-prey model, the so-called Leslie-Gower predator-prey model, in which the predator growth function is different from the predator predation function. They assumed that the predator growth is described by a function of the ratio of predators and their prey. Hsu and Huang [4] studied this model and showed that the system has unique positive equilibrium which is globally asymptotically stable under all biologically admissible parameters. May [5] improved the Leslie-Gower predator-prey model by replacing the Holling type-I

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functional response by Holling type-II. This model has been studied extensively by many researchers [4,6–10]. Many authors have used this model to study the real world problems, for example Caughley [11] used this system to model the biological control of the prickly-pear cactus by the moth Cactoblastis cactorum, Wollkind and Logan [12] and Wollkind et al. [13] used this system to model the predator-prey mite outbreak interactions on fruit trees in Washington State. One of the main demerits of this model is that at low densities of prey population, predator population cannot switch to alternative prey since its growth will be limited by the fact that its most favorite food, the prey, is absent or is in short supply [14]. Aziz-Alaoui and Daher Okiye [15] has taken care of this situation and improved the model, known as modified Leslie-Gower predator-prey model.

Allee effect, a mechanism leading to a positive relationship between a component of individual fitness and the number or density of conspecifics [16,17], has been observed in different organisms such as vertebrates, invertebrates, and plants [18]. This effect has also been called a negative competition effect [19] in population dynamics or depensation [20–22] in fisheries sciences. The Allee effect may be one of the simple causes for the complex, richer and varied dynamics in predator-prey system. In [23,24] authors discussed the stabilizing or destabilizing effects and bifurcations on the predator-prey systems subject to Allee effect. A large variety of different biological phenomena may exhibit Allee effect dynamics [18, Table 1], [25, Table 2.1]. Two main types of Allee effects are well known, namely the strong Allee effect and the weak Allee effect. The main difference between the two is that the strong Allee effect includes a population threshold below which the population experiences extinction while the weak Allee effect does not have a threshold [20,26]. A number of mathematical forms have been introduced to model the Allee effect [18], and most of them are topologically equivalent [27]. Recent ecological research suggests the possibility that two or more Allee effects generate mechanisms acting simultaneously on a single population [18, Table 2], especially in renewable resources [28]. The combined influence of some of these phenomena has been named as the multiple (double) Allee effects [18,25,29,30]. The double Allee effect affecting the species has been seen in wild life ecosystem [18] and in marine ecosystem as well [31].

Gonzlez-Olivares et al. [32] considered that the growth of prey is affected by double Allee effect in Lotka-Volterra predator-prey model [33]. They proved the existence of two limit cycles by means of the Lyapunov quantities whenever the Allee effect is either strong or weak. Huincahue-Arcos and Gonzlez-Olivares in [34] studied the modified Rosenzweig-MacArthur predation model [33] in which two Allee effects affect the prey population. The authors [34] determined certain parametric conditions for which the unique interior equilibrium point is locally asymptotically stable or the existence of at least one stable limit cycle generated through Hopf bifurcation. Flores and Gonzalez-Olivares [35] studied a ratio-dependent predator-prey model with double Allee effect on the prey, and discussed the stability and bifurcation analysis. Feng and Kang [36] studied the stability and bifurcation of the modified Leslie-Gower predator-prey model with Allee effects in both predator and prey species. They also showed that the double Allee effects greatly alter the outcome of the survival of both species. Pal and saha [37] studied the stability and bifurcation analysis of a ratio dependent predator-prey system with a double Allee effect in prey population growth.

The motive of this work is to investigate the dynamical behavior of the modified Leslie-Gower predator-prey model with double Allee effect in growth of prey population. It is assumed that the extent to which the environment provides protection to both the predator and prey is the same. The rest of the paper is organized as follows: in Section 2, the mathematical model is formulated, boundedness of solutions and existence of a positively invariant and attracting set are shown. In Section 3, the conditions to the existence of possible equilibria of the system and their local stability are established. In Section 4, Hopf, saddle-node and Bogdanov-Takens bifurcations are discussed. In Section 5, numerical simulations and phase portrait diagrams are given to validate our analytical findings. Finally, a brief discussion is given in Section 6.

2. Model equations

We consider the following bidimensional predator-prey system

$$\begin{cases} \frac{dN}{dT} = rN\left(1-\frac{N}{K}\right) - \frac{\alpha NP}{a_1+N}, \\ \frac{dP}{dT} = sP\left(1-\frac{bP}{a_2+N}\right), \end{cases} \tag{2.1}$$

with the initial conditions $N(0) > 0, P(0) > 0$, where $N(T)$ and $P(T)$ are respectively, prey and predator density at time T. r, K, α, s, b are positive parameters, which represent intrinsic growth rate of prey, carrying capacity of prey in the absence of predator, maximal predator per capita consumption rate, intrinsic growth rate of predator, measure of the food quality that the prey provides for conversion into predator birth respectively, and a_1 and a_2 measure the extent to which the environment provides protection to prey and predator respectively. The system (2.1) is proposed by Aziz-Alaoui and Daher Okiye [15] and studied in [38–41].

We consider the following multiple Allee effect in the prey species.

$$\frac{dN}{dT} = rN\left(1-\frac{N}{K}\right)\left(1-\frac{m+n}{N+n}\right), \tag{2.2}$$

where m is the Allee threshold and $n > 0$ is the auxiliary parameter with $m > -n$. The above equation can be written as

$$\frac{dN}{dT} = \frac{rN}{N+n}\left(1-\frac{N}{K}\right)(N-m) \tag{2.3}$$

In Eq. (2.3), the intrinsic growth rate of the species is affected by two Allee effects; the factor $m(N) = N - m$ [42–44] and the other is the hyperbolic function $r(N) = \frac{rN}{N+n}$, which can be interpreted as an approximation of a population dynamics where the differences between fertile and non-fertile are not explicitly modeled. It is assumed in [45] that this factor indicates the impact of the Allee effect due to the non-fertile population n. The Allee effect in the above equation is strong if $m > 0$ and weak if $m < 0$. Moreover, the auxiliary parameter n affects the overall shape of the per-capita growth curve of the prey (see Fig. 1).

With the assumption that the extent to which the environment provides protection to both predator and prey is the same, that is, $a_1 = a_2 = a$ [40,41,46] and using Eq. (2.3), the model (2.1), reduces to

$$\begin{cases} \frac{dN}{dT} = \frac{rN}{N+n}\left(1-\frac{N}{K}\right)(N-m) - \frac{\alpha NP}{a+N}, \\ \frac{dP}{dT} = sP\left(1-\frac{bP}{a+N}\right), \end{cases} \tag{2.4}$$

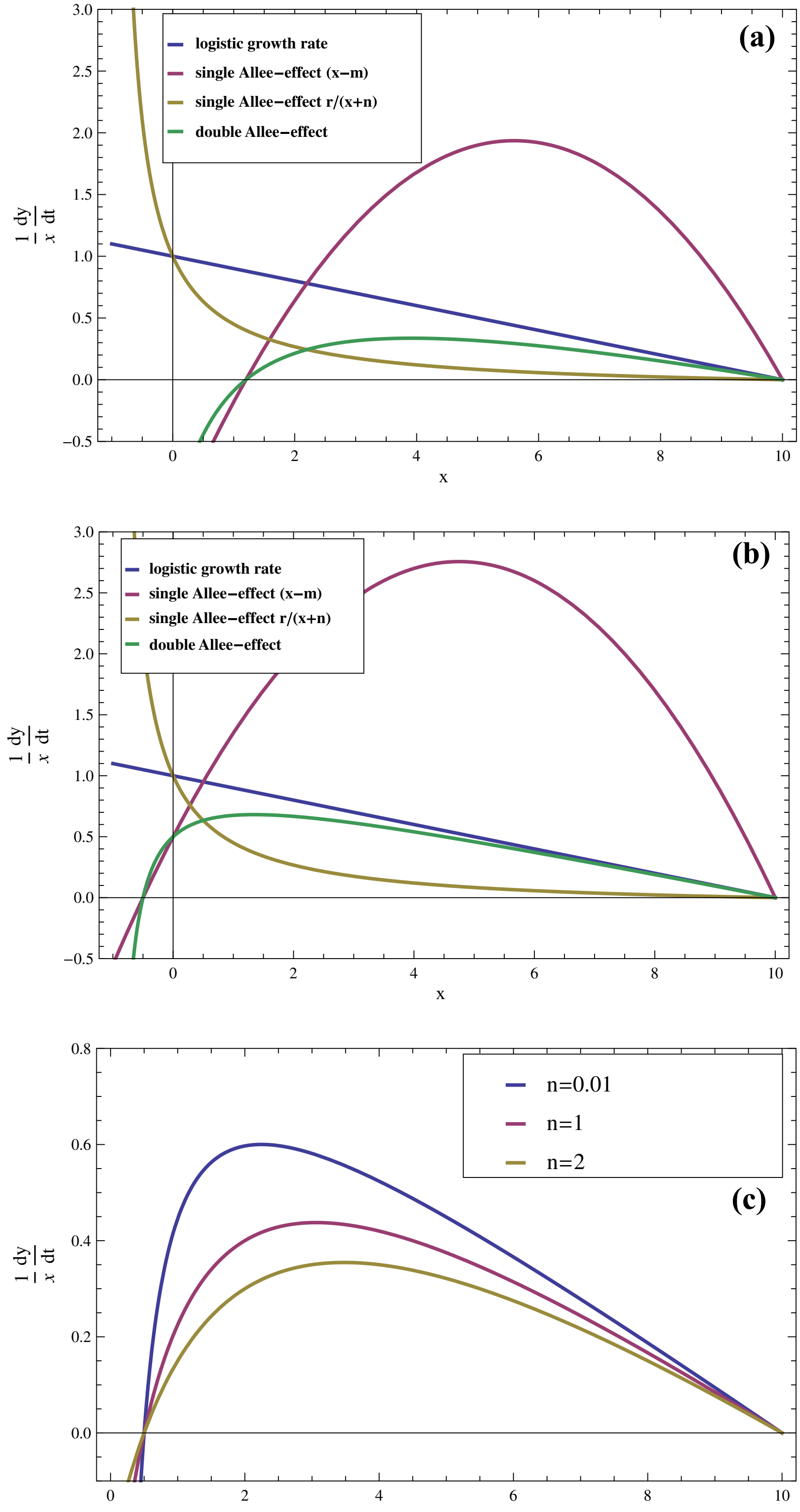

Figure 1 Per-capita growth rate of population of a single species. (a) Strong Allee effect ($m > 0$). (b) Weak Allee effect ($m < 0$). (c) Affect of n on the shape in double Allee effect.

On introducing the non-dimensional variables: $N = Kx,\ P = \frac{Ky}{b},\ T = \frac{1}{r}t$, in system (2.4), we obtain

$$\begin{cases} \frac{dx}{dt} = \frac{x(1-x)(x-\beta)}{(x+\theta)} - \frac{\xi xy}{\gamma+x}, \\ \frac{dy}{dt} = \rho y\left(1 - \frac{y}{\gamma+x}\right), \end{cases} \tag{2.5}$$

with the initial conditions: $x(0) > 0, y(0) > 0$, where $\beta = \frac{m}{K}, \theta = \frac{n}{K}, \xi = \frac{\alpha}{br}, \rho = \frac{s}{r}$, and $\gamma = \frac{a}{K}$.

The positivity and the boundedness of the solutions of the system (2.5) starting from an interior point of the first quadrant are proved below.

On integrating the first equation of the system (2.5), we get

$$x(t) = x(0)\exp\left[\int_0^t \left(\frac{(1-x(s))(x(s)-\beta)}{(x(s)+\theta)} - \frac{\xi y(s)}{\gamma + x(s)}\right)ds\right], \tag{2.6}$$

which is always nonnegative as $x(0) > 0$. Similarly from second equation of the system (2.5), we get

$$y(t) = y(0)\exp\left[\rho\int_0^t \left(1 - \frac{y(s)}{\gamma + x(s)}\right)ds\right], \tag{2.7}$$

which is always nonnegative as $y(0) > 0$. Therefore, all the solutions of the system (2.5) starting from an interior point of the first quadrant will remain in the first quadrant for all future time. Moreover, the solution trajectories starting from a point on the positive $x(y)$-axis will remain within the positive $x(y)$-axis for all future times. Hence, the set $R_+^2 = \{(x,y) : x, y \geqslant 0\}$ is an invariant set.

Now, we shall prove the boundedness of solutions of the system (2.5). We consider $(x(t), y(t))$ be any positive solution of the system (2.5), satisfies the initial conditions. There arise the following two cases;

Case I. We suppose $x(0) \leqslant 1$ and we claim $x(t) \leqslant 1$ for all $t \geqslant 1$. Otherwise, there are two positive real numbers t_1 and t_2 such that $t_2 > t_1, x(t_1) = 1$ and $x(t) > 1 \forall t \in (t_1, t_2)$. Then for all $t \in (t_1, t_2)$ the Eq. (2.6), can be written as

$$\begin{aligned} x(t) &= x(0)\exp\left[\int_0^{t_1} \left(\frac{(1-x(s))(x(s)-\beta)}{(x(s)+\theta)} - \frac{\xi y(s)}{\gamma + x(s)}\right)ds\right] \\ &\quad \times \exp\left[\int_{t_1}^{t_2} \left(\frac{(1-x(s))(x(s)-\beta)}{(x(s)+\theta)} - \frac{\xi y(s)}{\gamma + x(s)}\right)ds\right] \\ &= x(t_1)\exp\left[\int_{t_1}^{t_2} \left(\frac{(1-x(s))(x(s)-\beta)}{(x(s)+\theta)} - \frac{\xi y(s)}{\gamma + x(s)}\right)ds\right] \\ &< x(t_1), \end{aligned}$$

because $\left(\frac{(1-x(s))(x(s)-\beta)}{(x(s)+\theta)} - \frac{\xi y(s)}{\gamma + x(s)}\right) < 0$ for all $t \in (t_1, t_2)$ which contradicts our hypothesis. Thus $x(t) \leqslant 1$ for all $t \geqslant 0$.

Case II. Next, we suppose $x(0) > 1$, then as long as $x(t) \geqslant 1$

$$x(t) = x(0)\exp\left[\int_0^t \left(\frac{(1-x(s))(x(s)-\beta)}{(x(s)+\theta)} - \frac{\xi y(s)}{\gamma + x(s)}\right)ds\right] < x(0),$$

because $\left(\frac{(1-x(s))(x(s)-\beta)}{(x(s)+\theta)} - \frac{\xi y(s)}{\gamma + x(s)}\right) < 0$ for $x(t) \geqslant 1$. Hence, from the cases *I* and *II*, every positive solution holds $x(t) \leqslant \max\{x(0), 1\} \equiv M_1$ for all $t \geqslant 0$.

From second Eq. of the system (2.5)

$$\frac{dy}{dt} \leqslant \rho y\left(1 - \frac{y}{\gamma + M_1}\right),$$

then

$$y(t) \leqslant \max\{y(0), \gamma + M_1\} \quad \forall t \geqslant 0.$$

Thus, the above discussion can be concluded as follows:

Lemma 1.

(a) *All the solutions of the system (2.5) with its initial conditions are defined on* $[0, \infty)$ *and remain positive for all* $t \geqslant 0$.

(b) *All the solutions of the system (2.5) with its initial conditions are bounded for all* $t \geqslant 0$.

3. Equilibrium points and qualitative analysis

The equilibrium points of the system (2.5) are the points of intersection of the prey zero growth isocline ($\frac{dx}{dt} = 0$) and predator zero growth isocline ($\frac{dy}{dt} = 0$) which lie in first quadrant, that is, positive solutions of the following system

$$\frac{dx}{dt} = \frac{dy}{dt} = 0. \tag{3.1}$$

3.1. Strong Allee effect

The equilibrium points of the system (2.5) in case of strong Allee effect ($\beta > 0$) are

(a) $E_0 = (0, 0)$;
(b) $E_1 = (1, 0)$;
(c) $E_\beta = (\beta, 0)$;
(d) $E_\gamma = (0, \gamma)$;
(e) If $1 + \beta > \xi$, the system (2.5), has two positive interior equilibrium points $E_1^* = (x_1^*, y_1^*)$ and $E_2^* = (x_2^*, y_2^*)$ whenever $\left(\frac{1+\beta-\xi}{2}\right)^2 > \beta + \xi\theta$; a double multiple positive interior equilibrium point $E^* = (x^*, y^*) = \left(\frac{1+\beta-\xi}{2}, \gamma + \frac{1+\beta-\xi}{2}\right)$ whenever $\left(\frac{1+\beta-\xi}{2}\right)^2 = \beta + \xi\theta$, where $x_1^* = \frac{1+\beta-\xi+\sqrt{(1+\beta-\xi)^2 - 4(\beta+\xi\theta)}}{2}$, $x_2^* = \frac{1+\beta-\xi-\sqrt{(1+\beta-\xi)^2 - 4(\beta+\xi\theta)}}{2}$, $y_1^* = \gamma + x_1^*$ and $y_2^* = \gamma + x_2^*$.

Thus, the number and location of equilibrium points of system (2.5) can be described by the following lemma.

Lemma 2. *If* $1 + \beta > \xi$, *the system (2.5), has*

(a) *Four equilibrium points* E_0, E_1, E_β *and* E_γ *whenever* $\left(\frac{1+\beta-\xi}{2}\right)^2 < \beta + \xi\theta$.

(b) *Five equilibrium points* $E_0, E_1, E_\beta, E_\gamma$ *and* E^* *whenever* $\left(\frac{1+\beta-\xi}{2}\right)^2 = \beta + \xi\theta$.

(c) *Six equilibrium points* $E_0, E_1, E_\beta, E_\gamma, E_1^*$ *and* E_2^* *whenever* $\left(\frac{1+\beta-\xi}{2}\right)^2 > \beta + \xi\theta$.

Next, we discuss the dynamics of system (2.5) in the neighborhood of each feasible equilibria.

Theorem 1.

(a) *The equilibrium point* E_0 *is always a saddle point.*
(b) *The equilibrium point* E_1 *is always a saddle point.*
(c) *The equilibrium point* E_β *is always an unstable point.*
(d) *The equilibrium point* E_γ *is always a stable point.*

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(e) *The equilibrium point* E_1^*, *if exists, it is a stable point if* $x_1^*\left(\frac{1+\beta-\xi-2x_1^*}{x_1^*+\theta}+\frac{\xi}{\gamma+x_1^*}\right)-\rho<0$.
(f) *The equilibrium point* E_2^*, *if exists, is always a saddle point*.
(g) *The equilibrium point* E^*, *if exists, is a degenerate singularity*.

Proof 1.

(a) The Jacobian matrix of the system (2.5) at the equilibrium point E_0 is

$$J_{E_0}=\begin{bmatrix}-\frac{\beta}{\theta} & 0\\ 0 & \rho\end{bmatrix},$$

which confirms that the equilibrium point E_0 is a saddle point as $0<\beta<1$.
(b) The Jacobian matrix of the system (2.5) at the equilibrium point E_1 is

$$J_{E_1}=\begin{bmatrix}-\frac{1-\beta}{1+\theta} & -\frac{\xi}{1+\gamma}\\ 0 & \rho\end{bmatrix},$$

which confirms that the equilibrium point E_1 is a saddle point.
(c) The Jacobian matrix of the system (2.5) at the equilibrium point E_β is

$$J_{E_\beta}=\begin{bmatrix}\beta\frac{1-\beta}{\beta+\theta} & -\frac{\beta\xi}{\gamma+\beta}\\ 0 & \rho\end{bmatrix},$$

which confirms that the equilibrium point E_β is an unstable point.
(d) The Jacobian matrix of the system (2.5) at the equilibrium point E_γ is

$$J_{E_\gamma}=\begin{bmatrix}\frac{-\beta}{\theta}-\xi & 0\\ \rho & -\rho\end{bmatrix},$$

which confirms that the equilibrium point E_γ is a stable point.
(e) The Jacobian matrix of the system (2.5) at an interior equilibrium point $E(x,y)$ (say) is

$$J_E=\begin{bmatrix}x\left(\frac{1+\beta-\xi-2x}{x+\theta}+\frac{\xi}{\gamma+x}\right) & -\frac{\xi x}{\gamma+x}\\ \rho & -\rho\end{bmatrix}. \tag{3.2}$$

$\det(J_E)=-\rho x\frac{1+\beta-\xi-2x}{x+\theta}$ and $tr(J_E)=x\left(\frac{1+\beta-\xi-2x}{x+\theta}+\frac{\xi}{\gamma+x}\right)-\rho$. We have, $det(J_{E_1^*})>0$. Thus equilibrium point E_1^* is stable, if $x_1^*\left(\frac{1+\beta-\xi-2x_1^*}{x_1^*+\theta}+\frac{\xi}{\gamma+x_1^*}\right)-\rho<0$.
(f) From (3.2), $det(J_{E_2^*})<0$ which confirms that the equilibrium point E_2^* is a saddle.
(g) From (3.2), $det(J_{E^*})=0$, so the equilibrium point E^* is a degenerate singularity. □

It is proved in Theorem 1(g) that the interior equilibrium point E^* is a degenerate singularity, and so, the system may have complicated properties in the neighborhood of the point E^*. In the following, the dynamics of the system (2.5) in the neighborhood of the equilibrium point E^* have been discussed.

Theorem 2. *The interior equilibrium point* E^*, *if exists, it is*

(a) *a saddle node whenever* $\frac{\xi x^*}{\gamma+x^*}\neq\rho$ *holds*.
(b) *a cusp of codimension 2 whenever* $\frac{\xi x^*}{\gamma+x^*}=\rho$ *and* $\frac{\xi\gamma}{(x^*+\gamma)^2}-2\frac{x^*}{x^*+\theta}\neq 0$ *hold*.

Proof 2.

(a) First, we use the transformation $\hat{x}=x-x^*$, $\hat{y}=y-y^*$ to shift the equilibrium point E^* of the system (2.5) to the origin and then expand the right-hand side of system as a Taylor series, the system (2.5) can be rewritten as

$$\begin{cases}\frac{d\hat{x}}{dt}=\frac{\xi x^*}{\gamma+x^*}\hat{x}-\frac{\xi x^*}{\gamma+x^*}\hat{y}+\alpha_{20}\hat{x}^2+\alpha_{11}\hat{x}\hat{y}+o|(x,y)^3|,\\ \frac{d\hat{y}}{dt}=\rho\hat{x}-\rho\hat{y}-\frac{\rho}{\gamma+x^*}\hat{x}^2+\frac{2\rho}{\gamma+x^*}\hat{x}\hat{y}-\frac{\rho}{\gamma+x^*}\hat{y}^2+o|(x,y)^3|,\end{cases} \tag{3.3}$$

where $\alpha_{20}=\frac{\xi\gamma}{(\gamma+x^*)^2}-\frac{x^*}{x^*+\theta}$, $\alpha_{11}=-\frac{\xi\gamma}{(x^*+\gamma)^2}$.
If $\frac{\xi x^*}{\gamma+x^*}\neq\rho$, the $tr(J_{E^*})\neq 0$ while $\det(J_{E^*})=0$. Hence, the equilibrium point E^* is a saddle node.
(b) Now, we consider $\frac{\xi x^*}{\gamma+x^*}=\rho$, then the system (3.3) reduces to

$$\begin{cases}\frac{d\hat{x}}{dt}=\rho\hat{x}-\rho\hat{y}+\alpha_{20}\hat{x}^2+\alpha_{11}\hat{x}\hat{y}+o|(x,y)^3|,\\ \frac{d\hat{y}}{dt}=\rho\hat{x}-\rho\hat{y}-\frac{\rho}{\gamma+x^*}\hat{x}^2+\frac{2\rho}{\gamma+x^*}\hat{x}\hat{y}-\frac{\rho}{\gamma+x^*}\hat{y}^2+o|(x,y)^3|.\end{cases} \tag{3.4}$$

On introducing the variable $\tau=\rho t$, the system (3.4) reduces to the following system

$$\begin{cases}\frac{d\hat{x}}{d\tau}=\hat{x}-\hat{y}+\hat{\alpha_{20}}\hat{x}^2+\hat{\alpha_{11}}\hat{x}\hat{y}+o|(x,y)^3|,\\ \frac{d\hat{y}}{d\tau}=\hat{x}-\hat{y}-\frac{1}{\gamma+x^*}\hat{x}^2+\frac{2}{\gamma+x^*}\hat{x}\hat{y}-\frac{1}{\gamma+x^*}\hat{y}^2+o|(x,y)^3|,\end{cases} \tag{3.5}$$

where $\hat{\alpha_{20}}=\frac{1}{\rho}\alpha_{20}$ and $\hat{\alpha_{11}}=\frac{1}{\rho}\alpha_{11}$.
Now, on using the transformation $x_1=\hat{x}$, $x_2=\hat{x}-\hat{y}$, the system (3.5) reduces to the following system

$$\begin{cases}\frac{dx_1}{d\tau}=x_2+\overline{\alpha_{20}}x_1^2-\hat{\alpha_{11}}x_1x_2+o|(y_1,y_2)^3|,\\ \frac{dx_2}{d\tau}=\overline{\alpha_{20}}x_1^2-\hat{\alpha_{11}}x_1x_2+\frac{1}{x^*+\gamma}x_2^2+o|(y_1,y_2)^3|,\end{cases} \tag{3.6}$$

where $\overline{\alpha_{20}}=\hat{\alpha_{20}}+\hat{\alpha_{11}}$.
On using the transformation $y_1=x_1$, $y_2=x_2-\frac{1}{x^*+\gamma}x_1x_2$, the system (3.6) reduces to

$$\begin{cases}\frac{dy_1}{d\tau}=y_2+\overline{\alpha_{20}}y_1^2+\overline{\alpha_{11}}y_1y_2+o|(y_1,y_2)^3|,\\ \frac{dy_2}{d\tau}=\overline{\alpha_{20}}y_1^2-\hat{\alpha_{11}}y_1y_2+o|(y_1,y_2)^3|,\end{cases} \tag{3.7}$$

where $\overline{\alpha_{11}}=\left(\frac{1}{x^*+\gamma}-\hat{\alpha_{11}}\right)$.
Finally, using the transformation $z_1=y_1-\frac{1}{2}\overline{\alpha_{11}}y_1^2$, $z_2=y_2+\overline{\alpha_{20}}y_1^2+o|(z_1,z_2)^3|$, the system (3.7) reduces to

$$\begin{cases}\frac{dz_1}{d\tau}=z_2,\\ \frac{dz_2}{d\tau}=\overline{\alpha_{20}}y_1^2+(2\overline{\alpha_{20}}-\hat{\alpha_{11}})y_1y_2+o|(z_1,z_2)^3|.\end{cases} \tag{3.8}$$

Since $\overline{\alpha_{20}} = -\frac{x^*}{\rho(x^*+\theta)} \neq 0$ and if $2\overline{\alpha_{20}} - \hat{\alpha_{11}} = \frac{1}{\rho}\left(\frac{\xi\gamma}{(x^*+\gamma)^2} - 2\frac{x^*}{x^*+\theta}\right) \neq 0$, the origin in z_1z_2 plane is a cusp of codimension 2, that is, E^* in xy-plane is a cusp of codimension 2. □

3.2. Weak Allee effect

The equilibrium points of the system (2.5) in case of weak Allee effect ($\beta < 0$) are

(a) $e_0 = (0,0)$;
(b) $e_1 = (1,0)$;
(c) $e_\gamma = (0,\gamma)$;
(d) If $\beta + \xi < 1$, the system (2.5), has two positive interior equilibrium points $e_1^* = (\overline{x_1}, \overline{y_1})$ and $e_2^* = (\overline{x_2}, \overline{y_2})$ whenever $\left(\frac{1-\beta-\xi}{2}\right)^2 + \beta > \xi\theta > \beta$; a double multiple positive interior equilibrium point $e^* = (x_*, y_*) = \left(\frac{1-\beta-\xi}{2}, \gamma + \frac{1-\beta-\xi}{2}\right)$ whenever $\left(\frac{1-\beta-\xi}{2}\right)^2 + \beta = \xi\theta > \beta$, has a unique positive interior equilibrium point $e_* = (\overline{x_3}, \overline{y_3})$ whenever $\beta > \xi\theta$, has a unique positive interior equilibrium point $e = (\overline{x_4}, \overline{y_4}) = (1-\beta-\xi, 1+\gamma-\beta-\xi)$ whenever $\beta = \xi\theta$, where $\overline{x_1} = \frac{1-\beta-\xi+\sqrt{(1-\beta-\xi)^2-4(\xi\theta-\beta)}}{2}$, $\overline{x_2} = \frac{1-\beta-\xi-\sqrt{(1-\beta-\xi)^2-4(\xi\theta-\beta)}}{2}$, $\overline{x_3} = \frac{1-\beta-\xi+\sqrt{(1-\beta-\xi)^2-4(\xi\theta-\beta)}}{2}$, $\overline{y_1} = \gamma + \overline{x_1}$, $\overline{y_2} = \gamma + \overline{x_2}$ and $\overline{y_3} = \gamma + \overline{x_3}$.

Thus, the number and location of equilibrium points of system (2.5) can be described by the following lemma.

Lemma 3. *If $\beta + \xi < 1$, the system (2.5), has*

(a) *Three equilibrium points e_0, e_1 and e_γ whenever $\left(\frac{1-\beta-\xi}{2}\right)^2 < \xi\theta - \beta$ and $\xi\theta > \beta$.*
(b) *Four equilibrium points e_0, e_1, e_γ and e^* whenever $\left(\frac{1-\beta-\xi}{2}\right)^2 + \beta = \xi\theta > \beta$.*
(c) *Four equilibrium points e_0, e_1, e_γ and e_* whenever $\beta > \xi\theta$.*
(d) *Four equilibrium points e_0, e_1, e_γ and e whenever $\beta = \xi\theta$.*
(e) *Five equilibrium points $e_0, e_1, e_\gamma, e_1^*$ and e_2^* whenever $\left(\frac{1-\beta-\xi}{2}\right)^2 + \beta > \xi\theta > \beta$.*

The dynamics of system (2.5) in the neighborhood of each feasible equilibria are concluded in the following.

Theorem 3.

(a) *The equilibrium point e_0 is always an unstable point.*
(b) *The equilibrium point e_1 is always a saddle point.*
(c) *The equilibrium point e_γ is asymptotically stable whenever $\beta < \xi\theta$ and a saddle whenever $\beta > \xi\theta$.*
(d) *The equilibrium point e_1^*, if exists, is a stable point if $\overline{x_1}\left(\frac{1-\beta-\xi-2\overline{x_1}}{\overline{x_1}+\theta} + \frac{\xi}{\gamma+\overline{x_1}}\right) - \rho < 0$.*
(e) *The equilibrium point e_2^*, if exists, is always a saddle point.*
(f) *The equilibrium point e_*, if exists, is a stable point if $\overline{x_3}\left(\frac{1-\beta-\xi-2\overline{x_3}}{\overline{x_3}+\theta} + \frac{\xi}{\gamma+\overline{x_3}}\right) - \rho < 0$.*
(g) *The equilibrium point e, if exists, is a stable point if $\overline{x_4}\left(\frac{1-\beta-\xi-2\overline{x_4}}{\overline{x_4}+\theta} + \frac{\xi}{\gamma+\overline{x_4}}\right) - \rho < 0$.*
(h) *The equilibrium point e^*, if exists, is a degenerate singularity. Moreover, it is*
   (1) *a saddle node whenever $\frac{\xi x_*}{\gamma+x_*} \neq \rho$ holds.*
   (2) *a cusp of codimension 2 whenever $\frac{\xi x_*}{\gamma+x_*} = \rho$ and $\frac{\xi\gamma}{(x_*+\gamma)^2} - 2\frac{x_*}{x_*+\theta} \neq 0$ hold.*

The proof of Theorem 3 is similar to Theorems 1 and 2.

4. Bifurcation analysis

This section concerns with the bifurcation analysis, occurring in system (2.5). It has been shown that for certain parametric conditions some of the equilibrium points may be hyperbolic or degenerate singularities, and hence, system may undergoes to some bifurcations.

4.1. Strong Allee effect

4.1.1. Hopf bifurcation

In Theorem 1, it is proved that the interior equilibrium point E_2^*, if exists, is always a saddle point while E_1^*, if exists, is stable whenever $x_1^*\left(\frac{1+\beta-\xi-2x_1^*}{x_1^*+\theta} + \frac{\xi}{\gamma+x_1^*}\right) < \rho$. If $x_1^*\left(\frac{1+\beta-\xi-2x_1^*}{x_1^*+\theta} + \frac{\xi}{\gamma+x_1^*}\right) = \rho$, the trace of the Jacobian matrix $J_{E_1^*}$ is zero and determinant is positive which confirms that the eigenvalues of the Jacobian matrix $J_{E_1^*}$ are purely imaginary, that is, the equilibrium point E_1^* is either a weak focus or a center. Now, we show that system (2.5) undergoes to a Hopf bifurcation. Consider ρ be the Hopf bifurcation parameter, then the threshold magnitude $\rho = \rho^{[hf]} = x_1^*\left(\frac{1+\beta-\xi-2x_1^*}{x_1^*+\theta} + \frac{\xi}{\gamma+x_1^*}\right)$ exists, which satisfies $\det(J_{E_1^*}) > 0$ and $tr(J_{E_1^*}) = 0$. Also at $\rho = \rho^{[hf]}$, we have

$$\frac{\partial}{\partial\rho}(trE_1^*) = -1 \neq 0.$$

Thus the transversality condition of Hopf bifurcation holds, which ensures that the system (2.5) undergoes to a Hopf bifurcation at the equilibrium point E_1^*.

Now, in order to discuss the stability of limit cycle, the first Lyapunov number σ at interior equilibrium point $E_1^*(x_1^*, y_1^*)$ of the system (2.5) is computed by using the procedure as given in [47]. Let $x = u - x_1^*,\ y = v - y_1^*$, the system (2.5), in the vicinity of origin, can be written as

$$\frac{du}{dt} = a_{10}u + a_{01}v + a_{20}u^2 + a_{11}uv + a_{02}v^2 + a_{30}u^3 + a_{21}u^2v + a_{12}uv^2 + a_{03}v^3 + P(u,v),$$

$$\frac{dv}{dt} = b_{10}u + b_{01}v + b_{20}u^2 + b_{11}uv + b_{02}v^2 + b_{30}u^3 + b_{21}u^2v + b_{12}uv^2 + b_{03}v^3 + Q(u,v),$$

where $a_{10} = x_1^*\left(\frac{1+\beta-\xi-2x_1^*}{x_1^*+\theta} + \frac{\xi}{\gamma+x_1^*}\right)$, $a_{01} = -\frac{\xi x_1^*}{\gamma+x_1^*}$, $a_{20} = \frac{\theta(1+\beta-\xi-2x_1^*)}{(\theta+x_1^*)^2} - \frac{x_1^*}{x_1^*+\theta} + \frac{\xi\gamma}{(x_1^*+\gamma)^2}$, $a_{11} = -\frac{\xi\gamma}{(x_1^*+\gamma)^2}$, $a_{02} = 0$, $a_{30} = \frac{\theta(-1-\beta+\xi+2x_1^*)}{(x_1^*+\theta)^3} - \frac{\theta}{(x_1^*+\theta)^2} - \frac{\xi\gamma}{(x_1^*+\gamma)^3}$, $a_{21} = \frac{\xi\gamma}{(x_1^*+\gamma)^3}$, $a_{12} = 0$,

$a_{03}=0,\quad b_{10}=\rho,\quad b_{01}=-\rho,\quad b_{20}=-\frac{\rho}{\gamma+x_1^*},\quad b_{11}=\frac{2\rho}{\gamma+x_1^*},$ $b_{02}=-\frac{\rho}{\gamma+x_1^*}, b_{30}=\frac{\rho}{(\gamma+x_1^*)^2},\quad b_{21}=-\frac{2\rho}{(\gamma+x_1^*)^2},\quad b_{12}=\frac{\rho}{(\gamma+x_1^*)^2},\quad b_{03}=0,$ $P(u,v)=\sum_{i+j=4}^{\infty}a_{ij}u^iv^j$ and $Q(u,v)=\sum_{i+j=4}^{\infty}b_{ij}u^iv^j$.

Hence the first Lyapunov number σ for the planer system is

$$\begin{aligned}\sigma=-\frac{3\pi}{2a_{01}\Delta^{3/2}}\{&[a_{10}b_{10}(a_{11}^2+a_{11}b_{02}+a_{02}b_{11})\\&+a_{10}a_{01}(b_{11}^2+a_{20}b_{11}+a_{11}b_{02})+b_{10}^2(a_{11}a_{02}+2a_{02}b_{02})\\&-2a_{10}b_{10}(b_{02}^2-a_{20}a_{02})-2a_{10}a_{01}(a_{20}^2-b_{20}b_{02})\\&-a_{01}^2(2a_{20}b_{20}+b_{11}b_{20})+(a_{01}b_{10}-2a_{10}^2)(b_{11}b_{02}-a_{11}a_{20})]\\&-(a_{10}^2+a_{01}b_{10})[3(b_{10}b_{03}-a_{01}a_{30})+2a_{10}(a_{21}+b_{12})\\&+(b_{10}a_{12}-a_{01}b_{21})]\},\end{aligned}$$

where $\Delta=\frac{\rho x_1^*}{(\theta+x_1^*)}\sqrt{(1+\beta-\xi)^2-4(\beta+\xi\theta)}$. Therefore, the subcritical Hopf bifurcation exists if $\sigma>0$ and supercritical Hopf bifurcation exists if $\sigma<0$.

From the above discussion, we conclude that

Theorem 4. *The system (2.5) undergoes a Hopf bifurcation with respect to bifurcation parameter ρ around the point E_1^*, if exist, whenever $x_1^*\left(\frac{1+\beta-\xi-2x_1^*}{x_1^*+\theta}+\frac{\xi}{\gamma+x_1^*}\right)=\rho$ and an unstable (stable) limit cycle arises around the point E_1^* if $\sigma>0$ ($\sigma<0$).*

4.1.2. Saddle-node bifurcation

In Section 3, it is shown that if $1+\beta>\xi$, the system (2.5) has two positive interior equilibrium points E_1^* and E_2^* whenever $\left(\frac{1+\beta-\xi}{2}\right)^2>\beta+\xi\theta$ and these two interior equilibrium points coincide with each other and a unique interior equilibrium point E^* is obtained whenever $\left(\frac{1+\beta-\xi}{2}\right)^2=\beta+\xi\theta$. Also the system (2.5) has no positive interior equilibrium points whenever $\left(\frac{1+\beta-\xi}{2}\right)^2<\beta+\xi\theta$. The annihilation of positive interior equilibrium points are may be due to the occurrence of saddle-node bifurcation at the interior equilibrium point, whenever the parameter θ crosses the critical value $\theta=\theta^{[SN]}=\frac{1}{\xi}\left(\left(\frac{1+\beta-\xi}{2}\right)^2-\beta\right)$. In Theorem 2 it is shown that the unique interior equilibrium point E^* is a saddle node whenever $\frac{\xi x^*}{\gamma+x^*}\neq\rho$. To ensure that the system (2.5) undergoes to a saddle-node bifurcation we use Sotomayor's theorem [47]. The parameter θ is taken as the bifurcation parameter.

Since $det(J_{E^*})=0$, therefore one eigenvalue of the Jacobian matrix J_{E^*} is zero. If $tr(J_{E^*})<0$, the other eigenvalue has negative real part. Suppose v and w be the eigenvectors corresponding to zero eigenvalue of the matrix J_{E^*} and $J_{E^*}^T$ respectively, then

$$V=\begin{bmatrix}1\\1\end{bmatrix};\quad W=\begin{bmatrix}1\\-\frac{\xi x^*}{x^*+\gamma}\end{bmatrix}.$$

Now we have,

$$W^TF_\theta(E^*,\theta^{[SN]})=-\frac{x^*(1-\beta+\xi)(1+\beta-\xi)}{2(x^*+\theta^{[SN]})^2}\neq 0,$$
$$\text{as } 1+\beta-\xi>0,\quad 1-\beta+\xi>0,$$

$$W^T[D^2F(E^*,\theta^{[SN]})(V,V)]=-\frac{2x^*}{x^*+\theta^{[SN]}}\neq 0.$$

where

$$F_\theta(E^*,\theta^{[SN]})=\begin{bmatrix}-\frac{x^*(1-x^*)(x^*-\beta)}{(x^*+\theta^{[SN]})^2}\\0\end{bmatrix};\quad D^2F(E^*,\theta^{[SN]})=\begin{bmatrix}-\frac{2x^*}{x^*+\theta^{[SN]}}\\0\end{bmatrix}.$$

Thus the transversality condition for saddle-node bifurcation are satisfied. The above discussion can be summarized as

Theorem 5. *The system (2.5) undergoes a saddle-node bifurcation with respect to the bifurcation parameter θ around the equilibrium point E^* whenever $1+\beta>\xi, \theta=\frac{1}{\xi}\left(\left(\frac{1+\beta-\xi}{2}\right)^2-\beta\right)$ and $\frac{\xi x^*}{\gamma+x^*}\neq\rho$.*

4.1.3. Bogdanov-Takens bifurcation

In Theorem 5, it is proved that the system (2.5) undergoes a saddle-node bifurcation at the equilibrium point E^*, if exist, whenever $\frac{\xi x^*}{\gamma+x^*}\neq\rho$, that is, $tr(J_{E^*})\neq 0$. Now, we consider $tr(J_{E^*})=0$. In this case the Jacobian matrix J_{E^*} has double zero eigenvalues but the Jacobian matrix J_{E^*} is not a zero matrix. So, here is a chance of co-dimension 2 bifurcation (Bogdanov-Takens bifurcation). In Theorem 2, it is shown that the equilibrium point E^* is a cusp of co-dimension 2 whenever $\frac{\xi x^*}{\gamma+x^*}=\rho$, and $\frac{\xi\gamma}{(x^*+\gamma)^2}-2\frac{x^*}{x^*+\theta}\neq 0$. Now, choose ξ and ρ as the bifurcation parameter as they are important from the ecological point of view. The Bogdanov-Taken point (in brief, BT-point) in the parameter space is the intersection point of the saddle-node bifurcation curve and the Hopf-bifurcation curve. We use the algorithm given in [48] to prove the non-degeneracy conditions of Bogdanov-Takens bifurcation.

Suppose the bifurcation parameters ξ and ρ vary in a small domain of BT-point (ξ_0,ρ_0), and let $(\xi_0+\lambda_1,\rho_0+\lambda_2)$ be a point in the neighborhood of the BT-point (ξ_0,ρ_0) where λ_1,λ_2 are small. Thus, the system (2.5) reduces to

$$\begin{cases}\frac{dx}{dt}=\frac{x(1-x)(x-\beta)}{x+\theta}-\frac{(\xi+\lambda_1)xy}{x+\gamma},\\ \frac{dy}{dt}=(\rho+\lambda_2)y\left(1-\frac{y}{x+\gamma}\right).\end{cases}\tag{4.1}$$

The system (4.1) is C^∞ smooth with respect to the variables x,y in a small neighborhood of (ξ_0,ρ_0).

Define $x_1=x-x^*, x_2=y-y^*$, then the system (4.1) reduces to

$$\begin{cases}\frac{dx_1}{dt}=a_{00}+a_{10}x_1+a_{01}x_2+a_{20}x_1^2+a_{11}x_1x_2+R_1(x_1,x_2),\\ \frac{dx_2}{dt}=b_{10}x_1+b_{01}x_2+b_{20}x_1^2+b_{11}x_1x_2+b_{02}x_2^2+R_2(x_1,x_2),\end{cases}\tag{4.2}$$

where $a_{00}=-\lambda_1x^*,\ a_{10}=\frac{x^*(\xi+\lambda_1)}{x^*+\gamma}-\lambda_1,\ a_{01}=-\frac{(\xi+\lambda_1)x^*}{x^*+\gamma},\ a_{20}=-\frac{x^*}{x^*+\theta}+\frac{(\xi+\lambda_1)\gamma}{(x^*+\gamma)^2},\ a_{11}=-\frac{(\xi+\lambda_1)\gamma}{(x^*+\gamma)^2},\ b_{10}=\rho+\lambda_2,\ b_{01}=-(\rho+\lambda_2),$ $b_{20}=-\frac{\rho+\lambda_2}{x^*+\gamma}, b_{11}=\frac{2(\rho+\lambda_2)}{x^*+\gamma}, b_{02}=-\frac{\rho+\lambda_2}{x^*+\gamma}$ and R_1,R_2 are the power series in (x_1,x_2) with powers $x_1^ix_2^j$ satisfying $i+j\geqslant 3$.

Let us introduce the affine transformation $y_1=x_1,\ y_2=ax_1+bx_2\left(a=\frac{(\xi+\lambda_1)x^*}{x^*+\gamma}, b=-\frac{(\xi+\lambda_1)x^*}{x^*+\gamma}\right)$ in the system (4.2), we get

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$$\begin{cases} \frac{dy_1}{dt} = y_2 + \xi_{00}(\lambda) + \xi_{10}(\lambda)y_1 + \xi_{01}(\lambda)y_2 + \xi_{20}(\lambda)y_1^2 \\ \quad +\xi_{11}(\lambda)y_1y_2 + \overline{R}_1(y_1, y_2), \\ \frac{dy_2}{dt} = \eta_{00}(\lambda) + \eta_{10}(\lambda)y_1 + \eta_{01}(\lambda)y_2 + \eta_{20}(\lambda)y_1^2 + \eta_{11}(\lambda)y_1y_2 \\ \quad +\eta_{02}y_2^2 + \overline{R}_2(y_1, y_2), \end{cases} \tag{4.3}$$

where $\xi_{00}(\lambda) = -\lambda_1 x^*$, $\xi_{10}(\lambda) = -\lambda_1$, $\xi_{01} = \frac{\lambda_1}{\xi}$, $\xi_{20}(\lambda) = -\frac{x^*}{x^*+\theta}$, $\xi_{11}(\lambda) = \frac{(\xi+\lambda_1)\gamma}{(x^*+\gamma)\xi x^*}$, $\eta_{00}(\lambda) = -a\lambda_1 x^*$, $\eta_{10}(\lambda) = -a\lambda_1$, $\eta_{01}(\lambda) = -\left(\frac{\lambda_1 x^*}{x^*+\theta} + \lambda_2 + \rho\right)$, $\eta_{20}(\lambda) = -\frac{\xi(x^*)^2}{(x^*+\gamma)(x^*+\theta)}$, $\eta_{11}(\lambda) = \frac{(\xi+\lambda_1)\gamma}{(x^*+\gamma)^2}$, $\eta_{02}(\lambda) = \frac{\rho+\lambda_2}{\xi x^*}$ and $\overline{R_1}, \overline{R_2}$ are the power series in (y_1, y_2) with powers $y_1^i y_2^j$ satisfying $i + j \geqslant 3$.

The non-degeneracy conditions of Bogodanov-Takens bifurcation [48] are

$$\begin{bmatrix} \frac{(\xi+\lambda_1)x^*}{x^*+\gamma} & -\frac{(\xi+\lambda_1)x^*}{x^*+\gamma} \\ \rho & -\rho \end{bmatrix} \neq \theta_{2\times 2},$$

$$2\xi_{20}(0) + \eta_{11}(0) \neq 0,$$

$$\eta_{20}(0) \neq 0.$$

We have

$$2\xi_{20}(0) + \eta_{11}(0) = \frac{\xi\gamma}{(x^* + \gamma)^2} - 2\frac{x^*}{x^* + \theta},$$

$$\eta_{20}(0) = -\frac{\xi(x^*)^2}{(x^* + \gamma)(x^* + \theta)} \neq 0.$$

Thus the non-degeneracy condition of the Bogdanov-Takens bifurcation satisfy whenever $\frac{\xi\gamma}{(x^*+\gamma)^2} - 2\frac{x^*}{x^*+\theta} \neq 0$. We can summarized as

Theorem 6. *The system (2.5) undergoes a Bogdanov-Takens bifurcation with respect to the bifurcation parameter* ξ *and* ρ *around the equilibrium point* E^* *whenever* $1 + \beta > \xi, \theta = \frac{1}{\xi}\left(\left(\frac{1+\beta-\xi}{2}\right)^2 - \beta\right), \frac{\xi x^*}{\gamma+x^*} = \rho$ *and* $\frac{\xi\gamma}{(x^*+\gamma)^2} - 2\frac{x^*}{x^*+\theta} \neq 0$.

4.2. Weak Allee effect

As discussed in case of strong Allee effect the system (2.5) exhibits Hopf, saddle-node and Bogdanov-Taken bifurcations in case of weak Allee effect with respect to the corresponding parameter(s) which are concluded in the following theorem. The proof of these theorems are omitted for the sake of brevity.

Theorem 7. *The system (2.5) undergoes*

(a) *a hopf bifurcation with respect to bifurcation parameter* ρ *around the point*

(1) e_1^* *if* $\overline{x_1}\left(\frac{1-\beta-\xi-2\overline{x_1}}{\overline{x_1}+\theta} + \frac{\xi}{\gamma+\overline{x_1}}\right) = \rho$ *and unstable (stable) limit cycle arises around the point* e_1^* *if* $\sigma > 0 (\sigma < 0)$.

(2) e_* *if* $\overline{x_3}\left(\frac{1-\beta-\xi-2\overline{x_3}}{\overline{x_3}+\theta} + \frac{\xi}{\gamma+\overline{x_3}}\right) = \rho$ *and unstable (stable) limit cycle arises around the point* e_* *if* $\sigma > 0 (\sigma < 0)$.

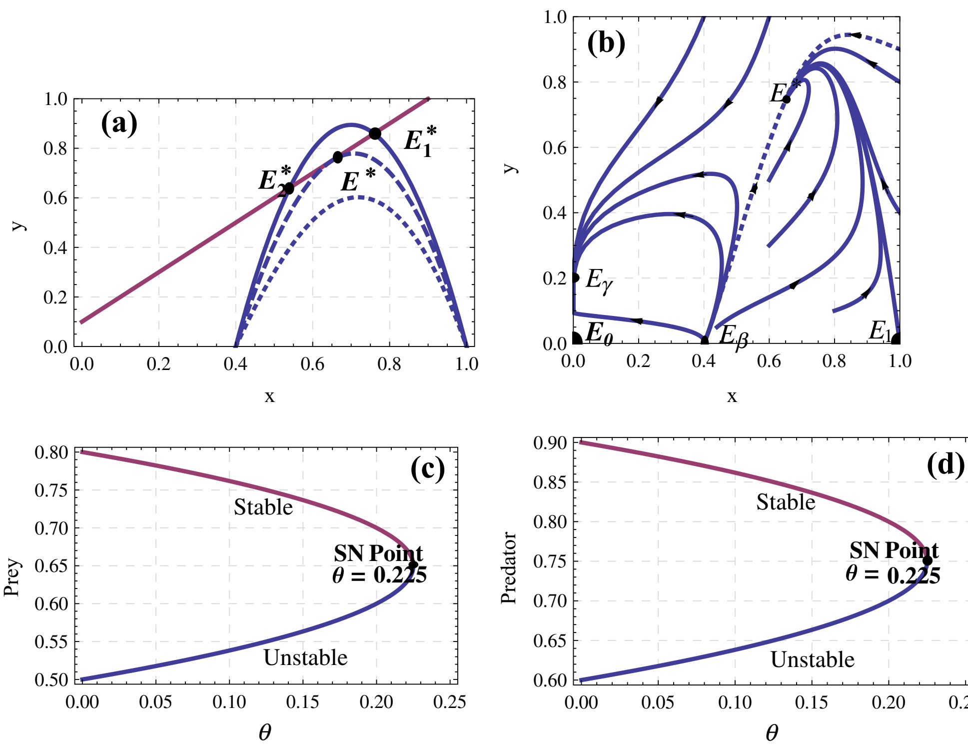

Figure 2 Strong Allee effect: $\beta = 0.4, \gamma = 0.1, \xi = 0.1, \rho = 0.2$. (a) This diagram shows how the number of interior equilibrium points changes with θ. All parabola are the prey nullcline for different values of θ and line is predator nullcline. For solid parabola $\theta = 0.05$ for dashed parabola $\theta = 0.225$ and for dotted parabola $\theta = 0.5$. (b) Phase portrait diagram of system (2.5) for $\theta = 0.225$. The dotted trajectories are the separatrix. (c) and (d) are the bifurcation diagram of the system (2.5). The upper curve stands for the stable equilibrium and the lower curve stands for unstable equilibrium.

(3) *e if* $\overline{x_4}\left(\frac{1-\beta-\xi-2\overline{x_4}}{\overline{x_4}+\theta}+\frac{\xi}{\gamma+\overline{x_4}}\right)=\rho$ *and unstable (stable) limit cycle arises around the point e if* $\sigma>0(\sigma<0)$.

(b) *a saddle-node bifurcation with respect to the bifurcation parameter* θ *around the equilibrium point* e^* *whenever* $\beta+\xi<1,\theta=\frac{1}{\xi}\left(\left(\frac{1-\beta-\xi}{2}\right)^2+\beta\right)$ *and* $\frac{\xi x_*}{\gamma+x_*}\neq\rho$.

(c) *a Bogdanov-Takens bifurcation with respect to the bifurcation parameter* ξ *and* ρ *around the equilibrium point* e^* *whenever* $\beta+\xi<1,\theta=\frac{1}{\xi}\left(\left(\frac{1-\beta-\xi}{2}\right)^2+\beta\right),\frac{\xi x_*}{\gamma+x_*}=\rho$ *and* $\frac{\xi\gamma}{(x_*+\gamma)^2}-2\frac{x_*}{x_*+\theta}\neq 0$.

5. Numerical simulation

In this section numerical simulations are carried out to support the analytical results obtained above. The MATHEMATICA 7.0 software has been used to plot phase portrait diagrams.

(1) $\beta=0.4,\ \gamma=0.1,\ \xi=0.1,\ \rho=0.2,\ \theta=0.05$. The system (2.5) has two positive interior equilibrium points; $E_1^*(x_1^*,y_1^*)=(0.782288,0.882288)$ and $E_2^*(x_2^*,y_2^*)=(0.517712,0.617712)$. If $\theta=\theta^{[SN]}=0.225$, the two interior equilibrium points coincide and the system (2.5) has only one interior equilibrium point $E^*(x^*,y^*)=(0.65,0.75)$. If $\theta=0.5$, the system (2.5) has no interior equilibrium point (see Fig. 2a). The phase portrait diagram for $\theta=\theta^{[SN]}=0.225$ is depicted in Fig. 2b in which the equilibrium point E^* is stable for the region lie right to the separatrix (dashed trajectories) while unstable for the region lie left to separatrix. The saddle-node bifurcation diagram is depicted in Fig. 2c, d.

(2) $\beta=0.4,\ \gamma=0.1,\ \xi=0.1,\ \rho=0.2,\ \theta=0.2$. The system (2.5) has two interior equilibrium points; $E_1^*(x_1^*,y_1^*)=(0.7,0.8)$ and $E_2^*(x_2^*,y_2^*)=(0.6,0.7)$. The equilibrium point E_2^* is always a saddle point and the equilibrium point E_1^* is stable (see Fig. 3a). If $\rho=\rho^{[hf]}=0.009722222$, the system (2.5) undergoes to a Hopf bifurcation at the point E_1^* and since the first Lyapunov number $\sigma=2804.28\pi>0$, an unstable limit cycle arises around the point E_1^* (see Fig. 3b). If $\rho=0.0166067444209$, a homoclinic loop is created around E_1^* (see Fig. 3c). If $\rho=0.008$ the equilibrium point E_1^* is unstable (see Fig. 3d).

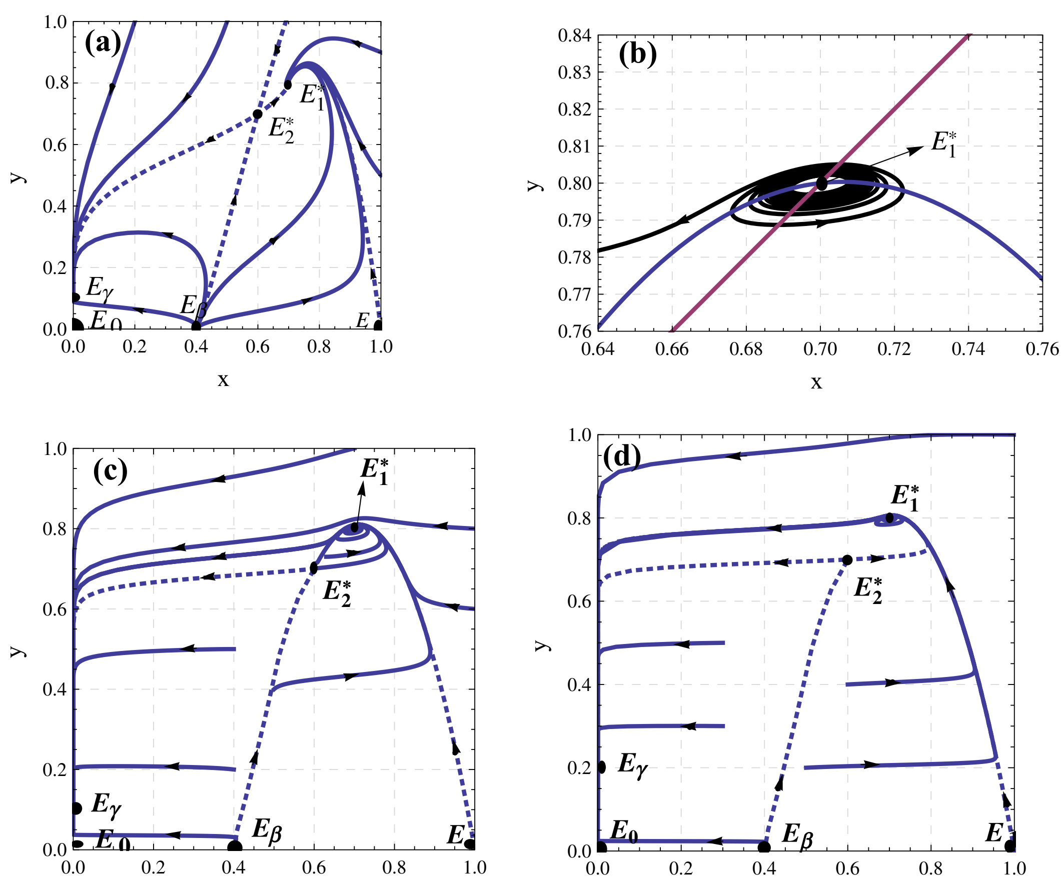

Figure 3 Strong Allee effect: $\beta=0.4,\gamma=0.1,\xi=0.1,\theta=0.2$. (a) $\rho=0.2$ two interior equilibrium points exist. E_1^* is asymptotically stable and E_2^* is saddle. (b) $\rho=0.009722$ an unstable limit cycle bifurcates through Hopf - bifurcation around E_1^* (c) $\rho=0.0166067444209415$ The diagram shows that the limit cycle collides with the saddle point E_2^* to give a homoclinic loop. (d) $\rho=0.008$ E_1^* is unstable point. The Dotted trajectories are the stable and unstable manifolds.

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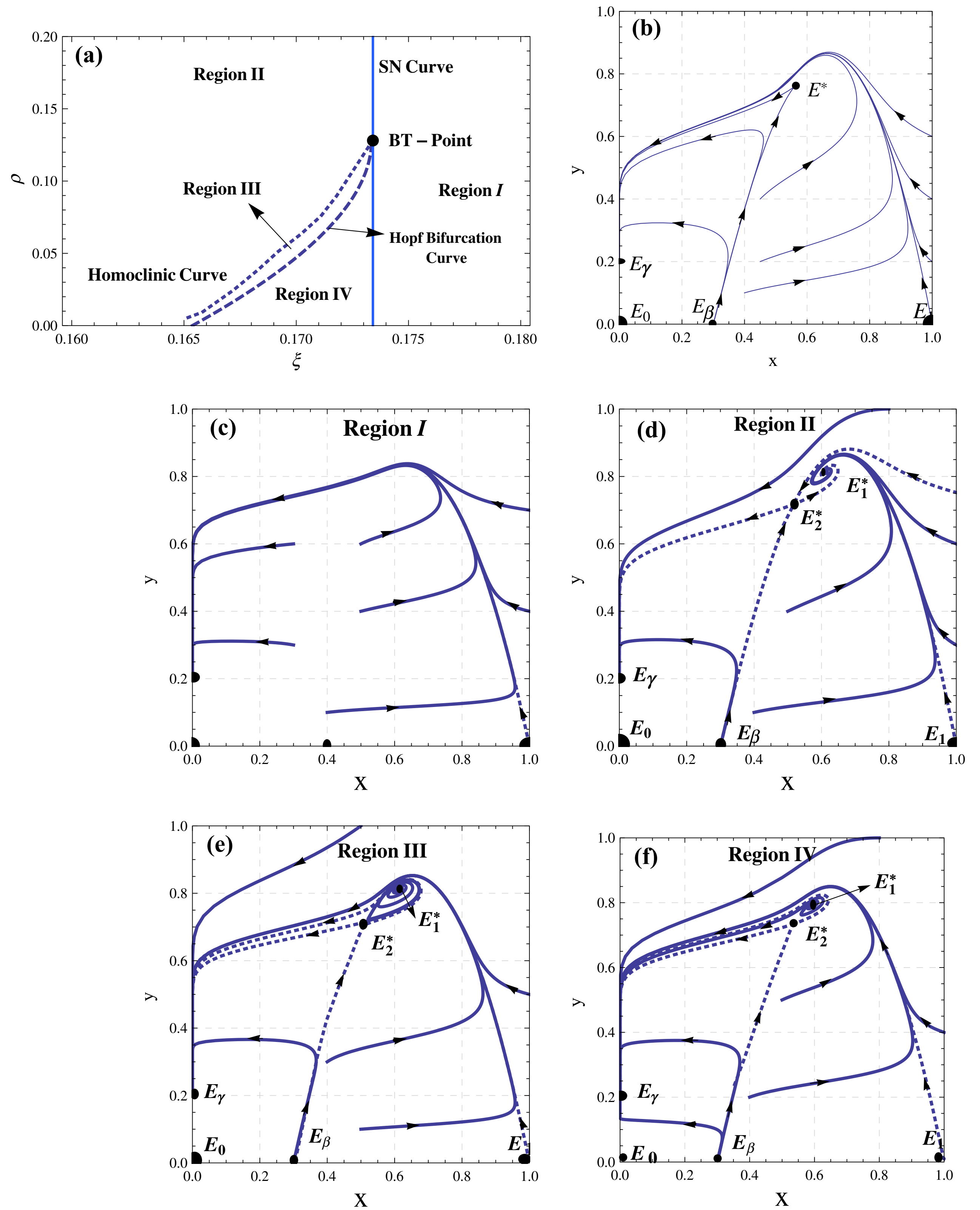

Figure 4 Strong Allee effect: $\beta = 0.3, \theta = 0.1, \gamma = 0.2$. (a) Bifurcation diagram of system (2.5) in $\xi\rho$-space (b) $\xi = 0.17335, \rho = 0.12793$ phase portrait diagram of the system (2.5). (c) $\xi = 0.175, \rho = 0.05$ lies in region *I*. No interior equilibrium point exist. The equilibrium point E_γ is globally stable. (d) $\xi = 0.170, \rho = 0.1$ lies in region *II*. Two interior equilibrium points exist. (e) $\xi = 0.169, \rho = 0.0634$ lies in region *III*. Two interior equilibrium points exist. (f) $\xi = 0.172, \rho = 0.05$ lies in region *IV*. Two interior equilibrium points exist.

(3) $\beta = 0.3,\ \gamma = 0.2,\ \theta = 0.1$. The BT bifurcation point in the $\xi\rho$-space is $(\xi_0, \rho_0) = (0.17335, 0.12793)$, intersection point of the saddle-node bifurcation curve and the Hopf-bifurcation curve and $E^* = (0.563325, 0.763325)$. The bifurcation diagram in the vicinity of the BT point in the parameter space is shown in Fig. 4a. A third curve (dotted curve) coming out from the BT point is a curve of non-local bifurcation of a formation of a separatrix loop obtained numerically. Fig. 4b shows that the unique interior equilibrium point E^* is a cusp of codimension 2. If ξ and ρ lie in first region $((\xi_0, \rho_0) = (0.175, 0.05))$, the system (2.5) has no interior equilibrium point (see Fig. 4c). If ξ and ρ lie in second region $((\xi_0, \rho_0) = (0.170, 0.10))$, then the system (2.5) has two interior equilibrium points one is a saddle point and other is asymptotically stable. The stable manifold of the saddle equilibrium point serves as separatrix for the basin of attraction of the axial equilibrium point E_γ and the stable interior equilibrium (see Fig. 4d). If ξ and ρ lie in third region $((\xi_0, \rho_0) = (0.169, 0.0634))$, the system (2.5) has two interior equilibrium points one is a saddle and other is a stable point surrounded by an unstable limit cycle. The basin of attraction of the stable equilibrium point increases in this domain (see Fig. 4e). If ξ and ρ lie in fourth region $((\xi_0, \rho_0) = (0.172, 0.05))$, the system (2.5) has two interior equilibrium points one is a saddle and other is an unstable point (see Fig. 4f).

(4) $\beta = -0.05,\ \gamma = 0.3,\ \xi = 0.4,\ \rho = 0.3,\ \theta = 0.3$. The system (2.5) has two interior equilibrium points $e_1^* = (0.35, 0.65)$ and $e_2^* = (0.2, 0.5)$. The equilibrium point e_2^* is always a saddle point and the equilibrium point e_1^* is stable (see Fig. 5a). If $\rho = \rho^{[hf]} = 0.134615$, the system (2.5) undergoes to a Hopf bifurcation at the point e_1^* and since the first Lyapunov number $\sigma = 318.808\pi > 0$, an unstable limit cycle arises around

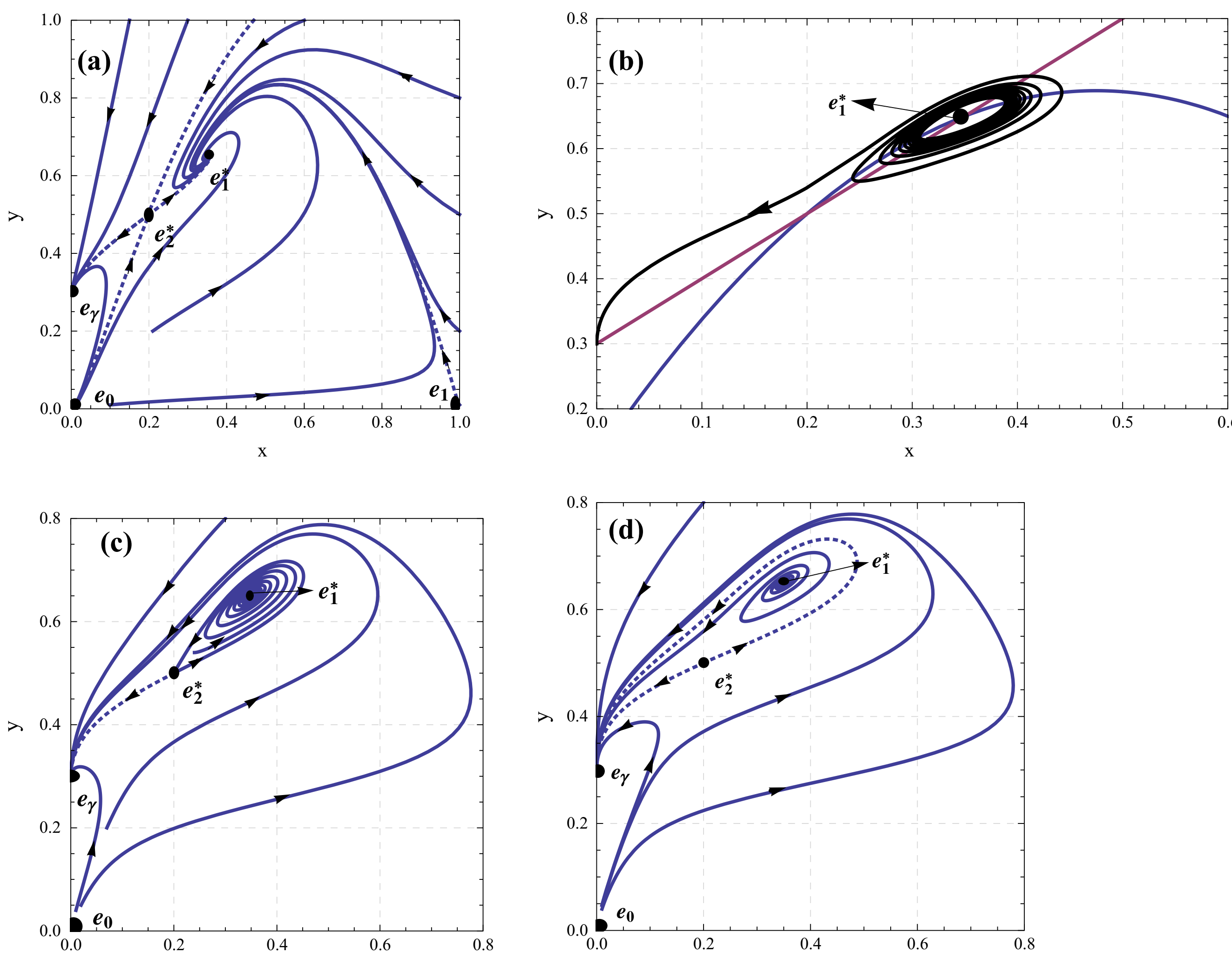

Figure 5 Weak Allee effect: $\beta = -0.05, \gamma = 0.3, \xi = 0.4, \theta = 0.3$. (a) $\rho = 0.3$ two interior equilibrium points exist. e_1^* is asymptotically stable and e_2^* is saddle. (b) $\rho = 0.134615$ an unstable limit cycle bifurcates through Hopf - bifurcation around e_1^* (c) $\rho = 0.14681$ The diagram shows that the limit cycle collides with the saddle point e_2^* to give a homoclinic loop. (d) $\rho = 0.12$ e_1^* is unstable point. The Dotted trajectories are the stable and unstable manifolds.

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the point e_1^* (see Fig. 5b). If $\rho = 0.14681$, a homoclinic loop is created around e_1^* (see Fig. 5c). If $\rho = 0.12$ the equilibrium point e_1^* is unstable (see Fig. 5d).

(5) $\beta = -0.25,\ \gamma = 0.3,\ \xi = 0.5,\ \rho = 0.3,\ \theta = 0.45.$ The system (2.5) has only one interior equilibrium point $e_* = (0.326556, 0.626556)$ which is always a stable point (see Fig. 6a). If $\rho = \rho^{[hf]} = 0.0910797$, the system (2.5) undergoes to a Hopf bifurcation at the point e_* and since the first Lyapunov number $\sigma = -173.22\pi < 0$, an stable limit cycle arises around the point e_1^* (see Fig. 6b).

(6) $\beta = -0.225,\ \gamma = 0.3,\ \xi = 0.5,\ \rho = 0.5,\ \theta = 0.45,$ then the system (2.5) has only one interior equilibrium point $e = (0.275, 0.575)$ which is always a stable point(see Fig. 7a). If $\rho = \rho^{[hf]} = 0.13482$, the system (2.5) undergoes to a Hopf bifurcation at the point e and since the first Lyapunov number $\sigma = -274.131\pi < 0$, an stable limit cycle arises around the point e (see Fig. 7b).

(7) $\beta = -0.2,\ \gamma = 0.3,\ \theta = 0.6$. The BT bifurcation point in the $\xi\rho$-space is $(\xi_0, \rho_0) = (0.4, 0.16)$ also $e^* = (0.2, 0.4)$. The bifurcation diagram in the vicinity of the BT point in the parameter space is shown in Fig. 8a. The blue dotted curve is the Hopf bifurcation curve and the red dotted curve is the non-local bifurcation curve. The Fig. 8b shows that the unique interior equilibrium point e^* is a cusp of codimension 2. If ξ and ρ lie in first region $((\xi_0, \rho_0) = (0.45, 0.10))$, then the system (2.5) has no interior equilibrium point (see Fig. 8c). If ξ and ρ lie in second region $((\xi_0, \rho_0) = (0.36, 0.20))$, then the system (2.5) has two interior equilibrium points one is a saddle

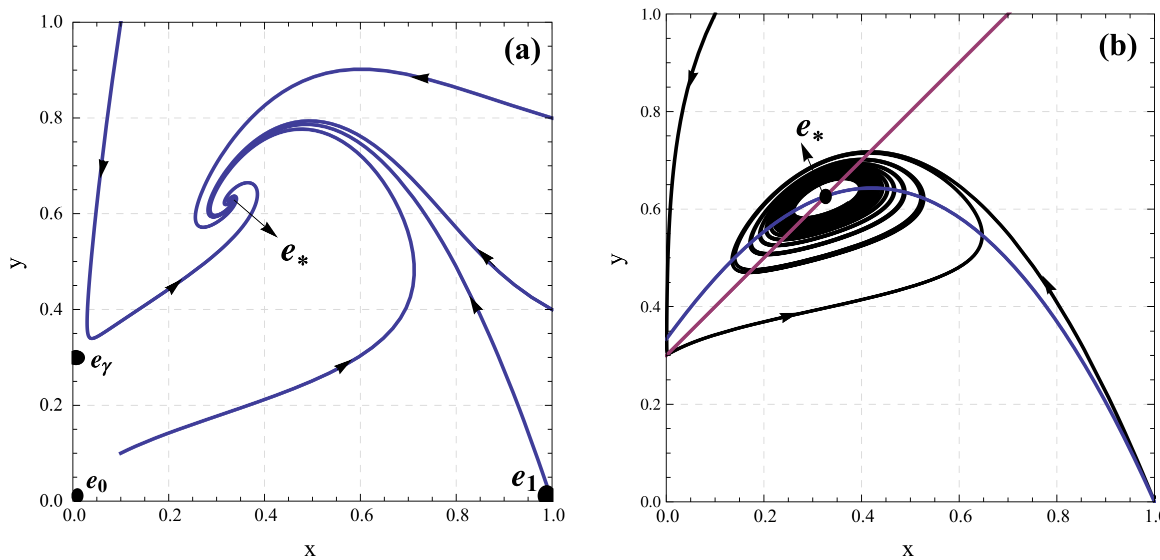

Figure 6 Weak Allee effect: $\beta = -0.25, \gamma = 0.3, \xi = 0.5, \theta = 0.45$. (a) $\rho = 0.3$ only one interior equilibrium point e_* exist which is asymptotically stable. (b) $\rho = 0.0910797$ a stable limit cycle bifurcates through Hopf - bifurcation around e_*.

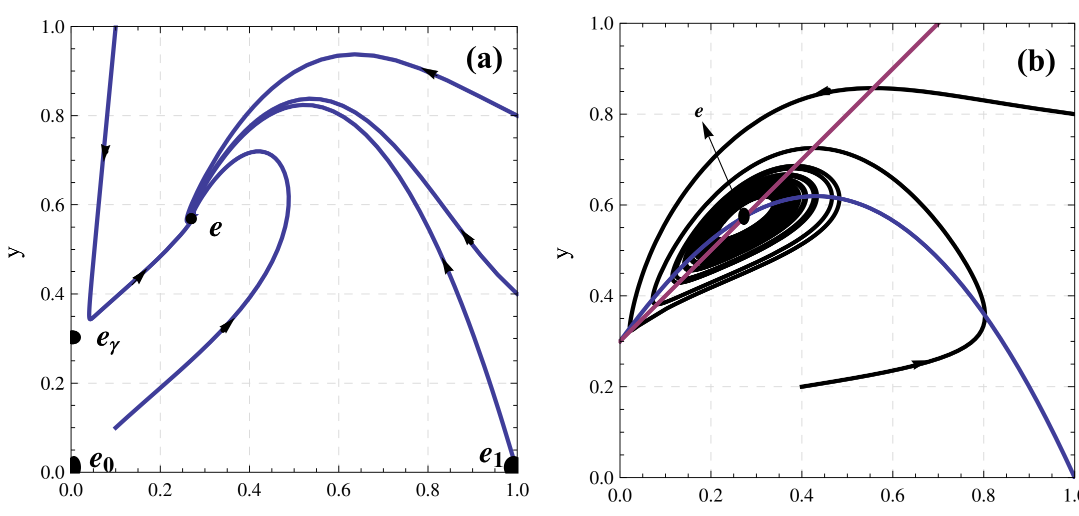

Figure 7 Weak Allee effect: $\beta = -0.225, \gamma = 0.3, \xi = 0.4, \theta = 0.45$. (a) $\rho = 0.5$ only one interior equilibrium point e exist which is asymptotically stable. (b) $\rho = 0.13482$ a stable limit cycle bifurcates through Hopf - bifurcation around e.

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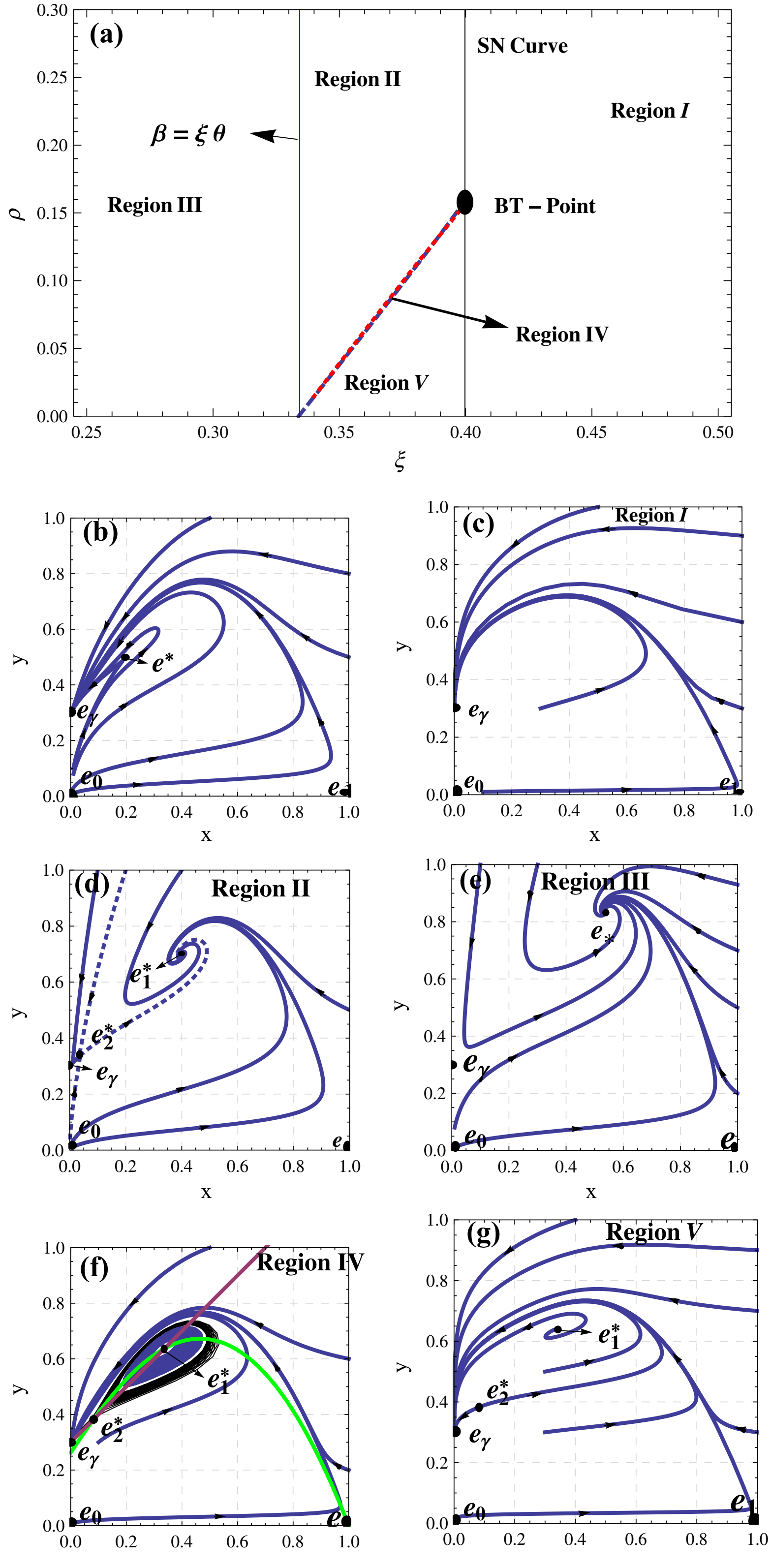

Figure 8 Weak Allee effect: $\beta = -0.2, \gamma = 0.3, \xi = 0.5, \theta = 0.45$. (a) Bifurcation diagram of system (2.5) (b) $\xi = 0.4, \rho = 0.16$ phase portrait diagram of the system (2.5). (c) $\xi = 0.45, \rho = 0.10$ lies in region I. No interior equilibrium point exist. The equilibrium point E_γ is globally stable. (d) $\xi = 0.36, \rho = 0.20$ lies in region II. Two interior equilibrium points exist. (e) $\xi = 0.3, \rho = 0.2$ lies in region III. Only one interior equilibrium points exist which is globally stable. (f) $\xi = 0.38, \rho = 0.110355$ lies in region IV (region between red and blue curve). Two interior equilibrium points exist. (g) $\xi = 0.38, \rho = 0.05$ lies in region V. Two interior equilibrium points exist.

point and other is asymptotically stable (see Fig. 8d). If ξ and ρ lie in third region ($(\xi_0, \rho_0) = (0.30, 0.20)$), the system (2.5) has only one interior equilibrium point which is globally stable (see Fig. 8e). If ξ and ρ lie in fourth region ($(\xi_0, \rho_0) = (0.38, 0.110355)$), the system (2.5) has two interior equilibrium points one is a saddle and other is an unstable point surrounded by an unstable limit cycle (see Fig. 8f). If ξ and ρ lie in fifth region ($(\xi_0, \rho_0) = (0.38, 0.05)$), the system (2.5) has two interior equilibrium points one is saddle and another is unstable (see Fig. 8g).

6. Result and discussion

In this article, we have analyzed a bidimensional modified Leslie-Gower predator-prey model in the presence of double Allee effect in the prey population, where the protection provided by the environment for both the prey and predator species is the same. From the ecological point of view, multiple (double) Allee effect has a great importance than single Allee effect whenever managing threatened or exploited populations as combined effect accelerates population decline and extinction risk and more theoretical work are necessary to promote co-existence of such diverse communities of threatened population [37].

The proposed model is shown biologically well-posed in the sense that any positive solution starts in the first quadrant remains both non-negative and bounded. The local stability of the system in different steady states has been discussed. Further, the system cannot collapse for any value of parameters as the origin is never stable. The existence of separatrix curves (stable manifold of the saddle interior equilibrium point) which separates the behavior of trajectories of the system is obtained, implying that dynamics of the system is very sensitive to the variation of the initial conditions. The solutions initiating from the domain lie to the left of the separatrix curve tend to the prey free axial equilibrium while the solutions initiating from the domain lie to the right of the separatrix curve tend to the positive interior equilibrium which indicates coexistence of both species.

The proposed system can have zero, one or two positive interior equilibrium points through saddle-node bifurcation as the bifurcation parameter $\theta = \frac{n}{K}$ crosses its critical value. The Sotomayor's theorem [47] is applied to ensure the existence of saddle-node bifurcation. Ecologically speaking, if the ratio of the non-fertile population of prey and the carrying capacity of prey is below the maximum threshold value, both the populations co-exist and above the maximum threshold the prey species suddenly collapse to extinction, and the system suddenly experiences a transition to a qualitatively exceptionally. It is found that if two interior equilibrium points exist, one of them being always a saddle point and other is stable, unstable or the system undergoes to a Hopf bifurcation around this point depending upon the parametric conditions. The emergence of homoclinic loop has been shown through numerical simulation when the limit cycle arising through Hopf bifurcation collides with a saddle point. The non-degeneracy conditions of the Bogdanov-Takens bifurcation are also proved. In both the strong and weak Allee effect the Bogdanov-Takens bifurcation demonstrates that there is a parametric region in which the predator and prey coexist in the form of a positive equilibrium or prey species can be driven to extinction, depends upon the initial values. Moreover, in strong Allee effect there exists other region in which the predator and prey coexist in the form of a positive equilibrium for all initial values lying inside the unstable limit cycle while in weak Allee effect there exists other region in which predator and prey coexist in form of a positive equilibrium for any initial value and also there exists another region in which the predator and prey coexist in the form of a periodic orbit for all initial values lying inside the unstable limit cycle.

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