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A Combinatorial Analog of the Poincaré Index Theorem

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The local configurations of arrows in directed graphs mapped on surfaces are related to one another by a relationship analogous to the Poincaré Index Theorem.

In recent work a mapping was proposed which captures certain structurally stable qualitative features of the continuous differential equations describing biochemical networks [1]. The mapping consists of an embedding of a directed graph on a surface. The strong correspondence between local configurations in the mapping, and the vector fields in the region of critical points of continuous differential equations, suggested the existence of a theorem for directed graphs analogous to the Poincaré Index Theorem [2, Appendix II].

Assume we have a directed graph composed of V vertices and E edges. Each edge terminates at two distinct vertices. For any graph, a compact surface, M, can be found so that the graph can be *embedded* on the surface so that no two edges intersect. This embedding partitions the surface into distinct regions, called faces, which are bounded by the edges of the graph. The embedding must be performed so that each face is simply connected. The number of faces is designated, F. For this construction Euler's theorem holds [3, Chapter 11],

$$F - E + V = \chi(M), \tag{1}$$

where $\chi(M)$ is the Euler-Poincaré characteristic of M. In Figure 1 we have embedded a directed graph on a torus. The Euler-Poincaré characteristic of the torus is 0, and the reader can confirm Euler's theorem.

The *degree of a vertex* is equal to the number of edges terminating on it. An *adjacent pair of edges* terminate on a common vertex and bound a common face. The number of *distinct* adjacent pairs of edges at a vertex is

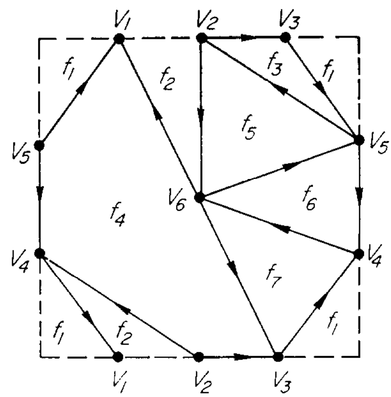

FIG. 1. A directed graph embedded on a torus. To reconstruct the torus opposite edges of the rectangle must be joined to form a cylinder, and the opposite ends of the cylinder joined to form a torus. Only the solid edges represent edges of the graph in the reconstructed figure. Both equation (1) and equation (3) can be confirmed.

equal to the degree of the vertex. By definition, for a vertex of degree one, there is a single adjacent pair of edges bounding an angle of 2π of a face of the embedded graph, and, for a vertex of degree two, there are two distinct adjacent pairs of edges. The internal region of each face of the embedded graph is simply connected. Starting from any vertex on the border of a face, the face can be traversed in a clockwise or counterclockwise direction until the starting vertex is reached again after the border of the disk has been traversed once. The number of edges traversed in this process is called the *circumference* of the face, where any edge which is traversed twice as the face is traversed once (for example, an isthmus and an edge terminating on a vertex of degree one) is counted twice in determining the circumference. The number of *distinct* adjacent pairs of edges bounding a face is equal to the circumference of the face.

For each vertex and each face of a directed graph embedded on M, we may compute a quantity which we call the *reversals*, designated R, at the vertex or face. Consider, in turn, each adjacent pair of edges terminating at a vertex. If the directions of both edges of the adjacent pair are the same (either both toward or both away from the vertex) they will contribute nothing to the reversals at the vertex. Count the number of adjacent pairs of edges at a vertex in which the directions of both edges are different. This number, which must be even, is called the reversals at the vertex. For example, in Figure 1, $R = 4$ for v_6 and $R = 0$ for v_1. In a similar fashion consider, in turn, each adjacent pair of edges bounding a face. A different convention is adopted to define the directions of adjacent edges of faces. Here the directions of both edges of the adjacent pair are called the same if they both are directed in a clockwise or counterclockwise direction. Count the number of adjacent pairs of edges bounding a face in which the directions of both edges are different. This number, which must again be even, is called the reversals of the face. For example, in Figure 1, $R = 6$ for f_4, and $R = 2$ for f_2.

We define the index of each vertex and face to be

$$I = 1 - (R/2), \tag{2}$$

where R is the number of reversals at the vertex or face.

THEOREM. *For directed graphs embedded on a compact surface, M,*

$$\sum_{F,V} I = \chi(M), \tag{3}$$

where the sum of the indices is taken over all faces and vertices of the figure.

Substituting equation (2) into (3), we find

$$F + V - \tfrac{1}{2} \sum_{F,V} R = \chi(M).$$

The theorem will be proved if we demonstrate that the total number of reversals of the figure equals $2E$.

Proof. Since each edge terminates at two vertices,

$$\sum_{V} \text{degree of vertex} = 2E. \tag{4}$$

When counting the total reversals of a figure, each adjacent pair of edges is considered twice, once when counting the reversals of the vertex at which they terminate, and once when counting the reversals of the face they bound. The rules for determining the reversals of faces and vertices are complementary, so that each adjacent pair of edges must make a contribution of $+1$ to either the reversals of their common vertex or common face, but not to both. Consequently, the total number of reversals of a figure is equal to the total number of distinct adjacent pairs of edges of the figure, or $2E$ (Eq. 4).

The index assigned using equation (2) is analogous to the Poincaré Index for closed curves for continuous differential systems. For both cases, the index is equal to the degree of the mapping of the circle into itself, where the mapping is defined by the field along the closed path for continuous systems [2, p. 365] or the configurations of arrows around the faces and vertices for directed graphs. To determine the degree of the mapping for a vertex of a directed graph, we first draw a circle around the vertex. If there are n edges terminating at the vertex, there will be n points of intersection of the edges with the circumference of the circle. These points are labeled consecutively $1, 2, \ldots, n$, where the directed angle between edges $i, i+1$ is designated θ_i $(n+1=1)$. For each point i on the circumference, the mapping generates a point i', where i' is located in the same position on the circumference as i if the arrow leaves the vertex, and i' is diametrically opposite to i if the arrow is toward the vertex (Fig. 2). The arc $i, i+1$ is thus mapped into the arc $i', (i+1)'$, where the

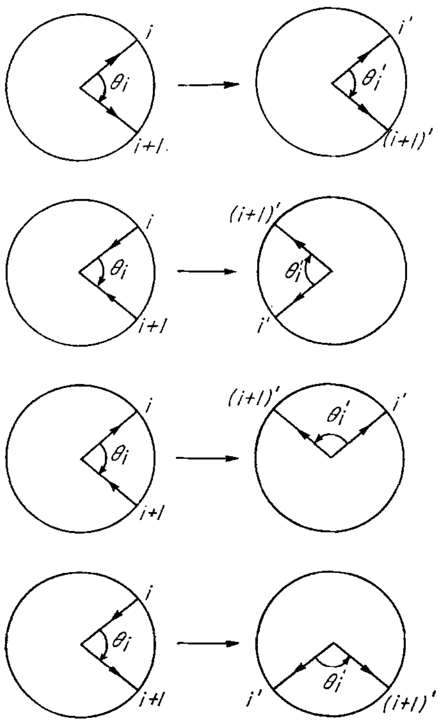

FIG. 2. The mapping, in the neighborhood of a vertex, defined by the directed graph. Arc $i, i+1$ is mapped into $i', (i+1)'$, where the directed angle, θ_i', subtended by $i', (i+1)'$ is given in equation (5).

directed angle between i' and $(i+1)'$ is designated θ_i'. We define a quantity R_i, which is 0 if the edges defining θ_i are in the same direction and 1 if the edges defining θ_i are in opposite directions. Then

$$\theta_i' = -\pi R_i + \theta_i \tag{5}$$

(see Fig. 2). The degree of the mapping is [4, p. 71]:

$$\text{degree of mapping} = \frac{1}{2\pi} \sum_{i=1}^{n} \theta_i'. \tag{6}$$

Substituting equation (5) into (6) we find the degree of the mapping is equal to the index defined in equation (2). If a continuous vector field is superposed at the vertex so that it is tangent to the directed edges, equation (5) specifies that the vector field in a sector is a *fan* for $R_i = 0$, or a *hyperbolic sector* for $R_i = 1$. This leads again to an immediate correspondence between the index of the vertex and the index of a critical point using a formula given by Bendixson [2, pp. 219–222]. A similar development can be given for the index of the faces. For systems which allow a graphic representation of the dynamics, vertices and faces with non-zero index play an important role in determining the qualitative dynamics of the flow, just as the critical points do in continuous dynamic systems.

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