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Periodic solutions for a kind of neutral functional differential systems

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Abstract

In this paper, we analyze some properties of the linear difference operator $A : C_T \to C_T$, $[Ax](t) = x(t) - V(t)x(t - \tau)$, and then, by using the coincidence degree theory of Mawhin, a kind of neutral differential systems with non-constant matrix is studied. Some new results on the existence of periodicity are obtained. It is worth noting that $V(t)$ is no longer a constant matrix, which is different from the corresponding ones of past work.

Keywords: periodic solutions; neutral; Mawhin's continuation theorem

1 Introduction

The field of neutral functional equations (in short NFDEs) is making significant breakthroughs in its practice; it is no longer only a specialist's field. In many practical systems, models of systems are described by NFDEs in which the models depend on the delays of state and state derivatives. Practical examples for neutral systems include population ecology, heat exchanges, mechanics, and economics; see [1–4]. In particular, qualitative analysis such as periodicity and stability of solutions of NFDEs has been studied extensively by many authors. We refer to [5–12] for some recent work on the subject of periodicity and stability of neutral equations.

In the last few years, the stability of neutral systems of various classes with time delays has received an ever-growing interest from many authors. Many sufficient conditions have been proposed to guarantee the asymptotic stability for neutral time delay systems. We only mention the work of some authors [13–15]. It is well known that the existence of periodic solutions of neutral equations and neutral systems is a very basic and important problem, which plays a role similar to stability. Thus, it is reasonable to seek conditions under which the resulting periodic neutral system would have a periodic solution. Much progress has been seen in this direction and many criteria are established based on different approaches. However, there is no paper for investigating the existence of periodic solutions of neutral system with non-constant matrix. In addition, to the best of our knowledge, most of the existing results deal with scalar NFEDs or neutral systems with a constant matrix. For example, in papers [16–20], based on Mawhin's continuation theorem, several types of scalar neutral equations have been studied:

$$\begin{aligned}
&\frac{d^2}{dt^2}\big(u(t)-ku(t-\tau)\big)=f\big(u(t)\big)u'(t)+\alpha(t)g\big(u(t)\big)+\sum_{j=1}^{n}\beta_j(t)g\big(u\big(t-\gamma_j(t)\big)\big)+p(t),\\
&\frac{dN}{dt}=N(t)\Bigg[\alpha(t)-\beta(t)N(t)-\sum_{j=1}^{n}b_j(t)N\big(t-\sigma_j(t)\big)-\sum_{i=1}^{m}c_i(t)N'\big(t-\tau_i(t)\big)\Bigg],\\
&\frac{dN}{dt}=N(t)\Bigg[r(t)-\sum_{j=1}^{n}\alpha_j(t)\ln N\big(t-\sigma_j(t)\big)-\sum_{i=1}^{m}b_i(t)\frac{d}{dt}\ln\big(t-\tau_i(t)\big)\Bigg],\\
&x'(t)+\alpha x'(t-\tau)=f\big(t,x(t)\big),\\
&\big(u(t)+Bu(t-\tau)\big)'=g_1\big(t,u(t)\big)-g_2\big(t,u(t-\tau_1)\big)+p(t).
\end{aligned}$$

For a neutral system, we note that Lu and Ge [21] studied the following system:

$$\frac{d^2}{dt^2}\big(x(t)-Cx(t-\widetilde{\tau})\big)+\frac{d}{dt}\operatorname{grad}F\big(x(t)\big)+\operatorname{grad}G\big(x\big(t-\tau(t)\big)\big)=p(t).$$

But C is a constant symmetric matrix. The purpose of this paper is to investigate the existence of periodic solutions to the nonlinear neutral system with non-constant matrix of the form

$$\frac{d^2}{dt^2}\big(x(t)-C(t)x(t-\tau)\big)+\frac{d}{dt}\operatorname{grad}F\big(x(t)\big)+\operatorname{grad}G\big(x\big(t-\gamma(t)\big)\big)=p(t), \tag{1.1}$$

where $x\in\mathbb{R}^n$, $C(t)=\operatorname{diag}(c_1(t),c_2(t),\dots,c_n(t))$, $C(t+T)=C(t)$; $F(x)\in C^2(\mathbb{R}^n,\mathbb{R})$, $G(x)\in C^1(\mathbb{R}^n,\mathbb{R})$; $p\in(\mathbb{R},\mathbb{R}^n)$, $p(t+T)=p(t)$; $\gamma\in C(\mathbb{R},\mathbb{R})$, $\gamma(t+T)=\gamma(t)$; T, and τ are given constants with $T>0$.

Throughout this paper, we use some notation:

(1) $I_n=\{1,2,\dots,n\}$; $\forall a=(a_1,a_2,\dots,a_n)^T\in\mathbb{R}^n$, $|a|=(\sum_{i=1}^n|a_i|^2)^{\frac{1}{2}}$;

(2) $C_T=\{x:x\in C(\mathbb{R},\mathbb{R}^n),x(t+T)=x(t),\forall t\in\mathbb{R}\}$ with the norm

$$|\varphi|_0=\max_{t\in[0,T]}\big|\varphi(t)\big|,\quad \forall\varphi\in C_T;$$

(3) $C_T^1=\{x:x\in C^1(\mathbb{R},\mathbb{R}^n),x(t+T)=x(t),\forall t\in\mathbb{R}\}$ with the norm

$$\|\varphi\|=\max_{t\in[0,T]}\big\{|\varphi|_0,\big|\varphi'\big|_0\big\},\quad \forall\varphi\in C_T^1.$$

Clearly, C_T and C_T^1 are Banach spaces.

2 Main lemmas

Lemma 2.1 [22] *If $|c(t)|\neq 1$, then operator A_1 has a continuous inverse A_1^{-1} on C_T, satisfying*

$$(1)\quad \big[A_1^{-1}f\big](t)=\begin{cases} f(t)+\sum_{j=1}^{\infty}\prod_{i=1}^{j}c(t-(i-1)\tau)f(t-j\tau), & c_0<1,\forall f\in C_T,\\ -\frac{f(t+\tau)}{c(t+\tau)}-\sum_{j=1}^{\infty}\prod_{i=1}^{j+1}\frac{1}{c(t+i\tau)}f(t+j\tau+\tau), & \sigma>1,\forall f\in C_T,\end{cases}$$

(2) $$\int_0^T \left|\left[A_1^{-1}f\right](t)\right|\,dt \le \begin{cases} \frac{1}{1-c_0}\int_0^T |f(t)|\,dt, & c_0 < 1, \forall f \in C_T, \\ \frac{1}{\sigma-1}\int_0^T |f(t)|\,dt, & \sigma > 1, \forall f \in C_T, \end{cases}$$

(3) $$\left\|A_1^{-1}f\right\|_0 \le \begin{cases} \frac{|f|_0}{1-c_0}, & c_0 < 1, \forall f \in C_T, \\ \frac{|f|_0}{\sigma-1}, & \sigma > 1, \forall f \in C_T. \end{cases}$$

Here

$$c_0 = \max_{t\in[0,T]}\left|c(t)\right|, \qquad \sigma = \min_{t\in[0,T]}\left|c(t)\right|.$$

Let

$$\mathcal{A} : C_T \longrightarrow C_T, \qquad [\mathcal{A}](t) = x(t) - V(t)x(t-\tau),$$

where $\forall t \in \mathbb{R}$, $V(t) \in C_T^1$ *is a real symmetric matrix.*

We will give some properties of $\mathcal{A}$.

Lemma 2.2 *Suppose that* $\lambda_1(t), \lambda_2(t), \ldots, \lambda_n(t)$ *are eigenvalues of* $V(t)$. *Then the operator* $\mathcal{A}$ *has continuous inverse* $\mathcal{A}^{-1}$ *on* C_T, *satisfying*

(1) $$\int_0^T \left|\left[\mathcal{A}^{-1}f\right](t)\right|\,dt \le \begin{cases} \left(\sum_{i=1}^n \frac{1}{(1-\lambda_{i,L})^2}\right)^{\frac{1}{2}}\int_0^T |f(t)|\,dt, & \lambda_{i,L} < 1, \forall f \in C_T, \\ \left(\sum_{i=1}^n \frac{1}{(1-\lambda_{i,l})^2}\right)^{\frac{1}{2}}\int_0^T |f(t)|\,dt, & \lambda_{i,l} > 1, \forall f \in C_T, \end{cases}$$

(2) $$\left|\left[\mathcal{A}^{-1}f\right]\right|_0 \le \begin{cases} \left(\sum_{i=1}^n \frac{1}{(1-\lambda_{i,L})^2}\right)^{\frac{1}{2}}|f|_0, & \lambda_{i,L} < 1, \forall f \in C_T, \\ \left(\sum_{i=1}^n \frac{1}{(1-\lambda_{i,l})^2}\right)^{\frac{1}{2}}|f|_0, & \lambda_{i,l} > 1, \forall f \in C_T, \end{cases}$$

where

$$\lambda_{i,L} = \max_{t\in[0,T]}\left|\lambda_i(t)\right|, \qquad \lambda_{i,l} = \min_{t\in[0,T]}\left|\lambda_i(t)\right|, \quad i \in I_n.$$

Proof (1) Since $V(t)$ is a real symmetric matrix, there exists an orthogonal matrix $U(t)$ such that

$$U(t)V(t)U^T(t) = E_\lambda(t) = \operatorname{diag}\bigl(\lambda_1(t), \lambda_2(t), \ldots, \lambda_n(t)\bigr).$$

Consider the system

$$x(t) - V(t)x(t-\tau) = f(t),$$

where we have equivalence to

$$y(t) - E_\lambda(t)y(t-\tau) = \widetilde{f}(t), \tag{2.1}$$

where $\widetilde{f}(t) = U(t)f(t)$, $y(t) = U(t)x(t)$. On the other hand, a component of the vector in system (2.1) is

$$y_i(t) - \lambda_i(t)y_i(t-\tau) = \widetilde{f}_i(t), \quad i \in I_n.$$

From Lemma 2.1, we have

$$y_i(t) = \begin{cases} \widetilde{f}_i(t) + \sum_{j=1}^{\infty} \prod_{k=1}^{j} \lambda_i(t-(k-1)\tau)\widetilde{f}_i(t-j\tau), & \lambda_{i,L} < 1, \\ -\frac{\widetilde{f}_i(t+\tau)}{\lambda_i(t+\tau)} - \sum_{j=1}^{\infty} \prod_{k=1}^{j+1} \frac{1}{\lambda_i(t+k\tau)}\widetilde{f}_i(t+j\tau+\tau), & \lambda_{i,l} > 1. \end{cases} \tag{2.2}$$

Thus, $\mathcal{A}^{-1}$ exists and

$$\mathcal{A}^{-1}: C_T \to C_T, \qquad \mathcal{A}^{-1}f(t) = x(t) = U^T(t)y(t), \quad t \in [0,T]. \tag{2.3}$$

When $\lambda_{i,L} < 1$, by (2.2) we get

$$|y_i(t)| \le \frac{\max_{t\in[0,T]} |\widetilde{f}_i(t)|}{1-\lambda_{i,L}}, \quad i \in I_i,$$

i.e.,

$$\max_{t\in[0,T]} |y_i(t)| \le \frac{\max_{t\in[0,T]} |\widetilde{f}_i(t)|}{1-\lambda_{i,L}}, \quad i \in I_i.$$

Thus, by (2.3) we have

$$\begin{aligned} |\mathcal{A}^{-1}f|_0 &= \max_{t\in[0,T]} |U^T(t)y(t)| = \max_{t\in[0,T]} |y(t)| = \max_{t\in[0,T]} \left(\sum_{i=1}^n y_i^2(t)\right)^{\frac{1}{2}} \\ &\le \left(\sum_{i=1}^n \max_{t\in[0,T]} y_i^2(t)\right)^{\frac{1}{2}} \le \left(\sum_{i=1}^n \frac{\max_{t\in[0,T]} |\widetilde{f}_i(t)|^2}{(1-\lambda_{i,L})^2}\right)^{\frac{1}{2}} \\ &\le \left(\sum_{i=1}^n \frac{1}{(1-\lambda_{i,L})^2}\right)^{\frac{1}{2}} |\widetilde{f}|_0 = \left(\sum_{i=1}^n \frac{1}{(1-\lambda_{i,L})^2}\right)^{\frac{1}{2}} |Uf|_0 \\ &= \left(\sum_{i=1}^n \frac{1}{(1-\lambda_{i,L})^2}\right)^{\frac{1}{2}} |f|_0. \end{aligned}$$

Obviously,

$$\int_0^T |\mathcal{A}^{-1}f(t)|\,dt \le \left(\sum_{i=1}^n \frac{1}{(1-\lambda_{i,L})^2}\right)^{\frac{1}{2}} \int_0^T |f(t)|\,dt.$$

(2) Similar to the above proof, when $\lambda_{i,l} > 1$, we get

$$|\mathcal{A}^{-1}f|_0 \le \left(\sum_{i=1}^n \frac{1}{(1-\lambda_{i,l})^2}\right)^{\frac{1}{2}} |f|_0,$$

$$\int_0^T |\mathcal{A}^{-1}f(t)|\,dt \le \left(\sum_{i=1}^n \frac{1}{(1-\lambda_{i,l})^2}\right)^{\frac{1}{2}} \int_0^T |f(t)|\,dt.$$

□

Let X and Y be two Banach spaces and let $L: D(L) \subset X \to Y$ be a linear operator, a Fredholm operator with index zero (meaning that $\operatorname{Im} L$ is closed in Y and $\dim \operatorname{Ker} L =$

codim Im $L < +\infty$). If L is a Fredholm operator with index zero, then there exist continuous projectors $P : X \to X$, $Q : Y \to Y$ such that $\operatorname{Im} P = \operatorname{Ker} L$, $\operatorname{Im} L = \operatorname{Ker} Q = \operatorname{Im}(I - Q)$, and $L_{D(L)\cap \operatorname{Ker} P} : (I - P)X \to \operatorname{Im} L$ is invertible. Denote by K_p the inverse of L_P.

Let Ω be an open bounded subset of X, a map $N : \overline{\Omega} \to Y$ is said to be L-compact in $\overline{\Omega}$ if $QN(\overline{\Omega})$ is bounded and the operator $K_p(I - Q)N(\overline{\Omega})$ is relatively compact. We first give the famous Mawhin continuation theorem.

Lemma 2.3 [23] *Suppose that X and Y are two Banach spaces and $L : D(L) \subset X \to Y$ is a Fredholm operator with index zero. Furthermore, $\Omega \subset X$ is an open bounded set and $N : \overline{\Omega} \to Y$ is L-compact on $\overline{\Omega}$. If all the following conditions hold*:

(1) $Lx \neq \lambda Nx$, $\forall x \in \partial\Omega \cap D(L)$, $\forall \lambda \in (0, 1)$,

(2) $Nx \notin \operatorname{Im} L$, $\forall x \in \partial\Omega \cap \operatorname{Ker} L$,

(3) $\deg\{QN, \Omega \cap \operatorname{Ker} L, 0\} \neq 0$,

then the equation $Lx = Nx$ has a solution on $\overline{\Omega} \cap D(L)$.

3 Main results

Theorem 3.1 *Suppose that $\int_0^T p(t)\,dt = 0$, $\varphi(t)$ is a nonzero periodic solution of* (3.1) *and there exists a constant $M > 0$ such that*

(H_1) $\forall i \in I_n$, $\frac{\partial G}{\partial x_i}$ *is bounded in the set $\triangle_1$ (or $\triangle_2$), where*

$$\triangle_1 = \{x : x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n, x_i \in (-\infty, M], x_j \in \mathbb{R}, j \in I_n - \{i\}\},$$
$$\triangle_2 = \{x : x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n, x_i \in [-M, \infty), x_j \in \mathbb{R}, j \in I_n - \{i\}\}.$$

(H_2) $x_i \frac{\partial G}{\partial x_i} > 0$ *(or < 0), whenever $|x_i| > M$, $i \in I_n$.*

(H_3) *Suppose that $\mu_1, \mu_2, \ldots, \mu_n$ are eigenvalues of $\frac{\partial^2 F(v)}{\partial x^2}$, $v \in \mathbb{R}^n$, and there exists a constant $\lambda_F \geq 0$ such that*

$$\max\{|\mu_1|, |\mu_2|, \ldots, |\mu_n|\} \leq \lambda_F.$$

Then system (1.1) *has at least one T-periodic solution, if $\lambda_{0,i} < \frac{1}{2}$ (or $\sigma_{0,i} > 1$), $(\lambda_{2,i}Tn + n\lambda_{1,i}\sqrt{n})T + \lambda_{0,i} < 1$, and $\tau = mT$, $m \in \mathbb{Z}$, where*

$$\lambda_{0,i} = \max_{t\in[0,T]}\{|c_i(t)|, i \in I_n\}, \qquad \lambda_{1,i} = \max_{t\in[0,T]}\{|c_i'(t)|, i \in I_n\},$$
$$\lambda_{2,i} = \max_{t\in[0,T]}\{|c_i''(t)|, i \in I_n\}, \qquad \sigma_{0,i} = \min_{t\in[0,T]}\{|c_i(t)|, i \in I_n\}.$$

Proof Define

$$A : C_T \to C_T, \qquad [Ax](t) = x(t) - C(t)x(t - \tau), \quad \forall t \in \mathbb{R},$$
$$N : C_T^1 \to C_T, \qquad (Nx)(t) = -\frac{d}{dt}\operatorname{grad} F(x(t)) - \operatorname{grad} G(x(t - \gamma(t))) + p(t),$$
$$L : D(L) \subset C_T \to C_T^1, \qquad Lx = (Ax)'',$$

where $D(L) = \{x : x \in C_T^1, x'' \in C(\mathbb{R}, \mathbb{R}^n)\}$. Then system (1.1) obeys the operator equation $Lx = Nx$. We have $(x(t) - C(t)x(t-\tau))'' = 0$, $\forall x \in \operatorname{Ker} L$. Then

$$x(t) - C(t)x(t-\tau) = \widetilde{c}_1 t + \widetilde{c}_2,$$

where $\widetilde{c}_1, \widetilde{c}_2 \in \mathbb{R}^n$. Since $x(t) - C(t)x(t-\tau) \in C_T$, we have $\widetilde{c}_1 = 0$. Let $\varphi(t) \in C(\mathbb{R}, \mathbb{R}^n)$ be a nonzero periodic solution of

$$x(t) - C(t)x(t-\tau) = I, \tag{3.1}$$

then $|\varphi(t)|^2 > 0$ and $\int_0^T \varphi^2(t)\,dt \neq 0$, where I is an unit matrix. We get

$$\operatorname{Ker} L = \{a_0\varphi(t) : a_0 \in \mathbb{R}\}, \qquad \operatorname{Im} L = \left\{y : y \in C_T, \int_0^T y(s)\,ds = 0\right\}.$$

Obviously, $\operatorname{Im} L$ is closed in C_T and $\dim \operatorname{Ker} L = \operatorname{codim} \operatorname{Im} L = n$. So L is a Fredholm operator with index zero. Define continuous projectors P, Q:

$$P : C_T \to \operatorname{Ker} L, \qquad (Px)(t) = \frac{\int_0^T x(t)\varphi(t)\,dt}{\int_0^T \varphi^2\,dt}\varphi(t)$$

and

$$Q : C_T \to C_T/\operatorname{Im} L, \qquad Qy = \frac{1}{T}\int_0^T y(s)\,ds.$$

Let

$$L_P = L|_{D(L)\cap \operatorname{Ker} P} : D(L) \cap \operatorname{Ker} P \to \operatorname{Im} L,$$

then

$$L_P^{-1} = K_P : \operatorname{Im} L \to D(L) \cap \operatorname{Ker} P.$$

Since $\operatorname{Im} L \subset C_T$ and $D(L) \cap \operatorname{Ker} P \subset C_T^1$, K_P is an embedding operator. Hence K_P is a completely continuous operator in $\operatorname{Im} L$. By the definitions of Q and N, one knows that $QN(\overline{\Omega})$ is bounded on $\overline{\Omega}$. Hence the nonlinear operator N is L-compact on $\overline{\Omega}$. We complete the proof by three steps.

Step 1. Let $\Omega_1 = \{x \in D(L) \subset C_T^1 : Lx = \lambda Nx, \lambda \in (0,1)\}$. We show that Ω_1 is a bounded set. We have $Lx = \lambda Nx$ $\forall x \in \Omega_1$, *i.e.*,

$$\frac{d^2}{dt^2}\bigl(x(t) - C(t)x(t-\tau)\bigr) + \lambda\frac{d}{dt}\operatorname{grad} F\bigl(x(t)\bigr) + \lambda \operatorname{grad} G\bigl(x\bigl(t-\gamma(t)\bigr)\bigr) = \lambda p(t). \tag{3.2}$$

Integrating both sides of (3.2) over $[0, T]$, we have

$$\int_0^T \operatorname{grad} G\bigl(x\bigl(t-\gamma(t)\bigr)\bigr)\,dt = 0,$$

i.e., $\forall i \in I_n$,

$$\int_0^T \frac{\partial G(x(t-\gamma(t)))}{\partial x_i}\,dt = 0. \tag{3.3}$$

Let $\frac{\partial G}{\partial x_i}$ be bounded in $\triangle_1$ and

$$x_i \frac{\partial G}{\partial x_i} > 0, \quad \text{whenever } |x_i| > M. \tag{3.4}$$

Let

$$E_1 = \bigl\{t : t \in [0,T], x\bigl(t-\gamma(t)\bigr) \le M\bigr\}, \qquad E_2 = \bigl\{t : t \in [0,T], x\bigl(t-\gamma(t)\bigr) > M\bigr\}.$$

By assumption (H_1), if $x_i \le M$, there exists a constant $M_1 > 0$ such that $|\frac{\partial G}{\partial x_i}| \le M_1$. From (3.3) and (3.4), we get

$$\int_{E_2} \left|\frac{\partial G(x(t-\gamma(t)))}{\partial x_i}\right| dt = \int_{E_2} \frac{\partial G(x(t-\gamma(t)))}{\partial x_i}\,dt \le \int_{E_1} \left|\frac{\partial G(x(t-\gamma(t)))}{\partial x_i}\right| dt \le M_1 T.$$

Thus

$$\int_0^T \left|\frac{\partial G(x(t-\gamma(t)))}{\partial x_i}\right| dt = \int_{E_1} \left|\frac{\partial G(x(t-\gamma(t)))}{\partial x_i}\right| dt + \int_{E_2} \left|\frac{\partial G(x(t-\gamma(t)))}{\partial x_i}\right| dt \le 2M_1 T,$$

i.e.,

$$\begin{aligned}
\int_0^T \bigl|\operatorname{grad} G\bigl(x\bigl(t-\gamma(t)\bigr)\bigr)\bigr|\,dt &= \int_0^T \left[\sum_{i=1}^n \left(\frac{\partial G(x(t-\gamma(t)))}{\partial x_i}\right)^2\right]^{\frac{1}{2}} dt \\
&\le \int_0^T \left[\sum_{i=1}^n \left|\frac{\partial G(x(t-\gamma(t)))}{\partial x_i}\right|\right] dt \\
&\le 2nM_1 T.
\end{aligned} \tag{3.5}$$

We claim that there exists a point $t_1 \in \mathbb{R}$ such that

$$\bigl|x_i(t_1)\bigr| \le M. \tag{3.6}$$

In fact, for $\forall t \in [0,T]$, we have $x_i(t-\gamma(t)) > M$, and by (3.4), we have $\int_0^T \frac{\partial G(x(t-\gamma(t)))}{\partial x_i}\,dt > 0$, which is a contradiction; see (3.3). So there must be a point $\xi \in [0,T]$ such that

$$x_i\bigl(\xi - \lambda(\xi)\bigr) \le M. \tag{3.7}$$

Similar to the above proof, there must be a point $\eta \in [0,T]$ such that

$$x_i\bigl(\eta - \gamma(\eta)\bigr) \ge -M. \tag{3.8}$$

(1) If $x_i(\xi - \gamma(\xi)) \geq -M$, by (3.7) we have

$$\left|x_i\left(\xi - \lambda(\xi)\right)\right| \leq M.$$

Let $t_1 = \xi - \gamma(\xi)$. This proves (3.6).

(2) If $x_i(\xi - \gamma(\xi)) < -M$, from (3.8) and the fact that $x_i(t)$ is continuous on $\mathbb{R}$, there is a point t_1 between $\xi - \gamma(\xi)$ and $\eta - \gamma(\eta)$ such that $|x_i(t_1)| \leq M$. This also proves (3.6). Let $t_1 = k\pi + t_2$, $k \in \mathbb{Z}$, $t_2 \in [0, T]$. Then $|x_i(t_1)| = |x_i(t_2)| \leq M$. Hence we get

$$\left|x_i(t)\right| = \max_{t\in[0,T]}\left|x_i(t_2) + \int_{t_2}^{t} x_i'(s)\,ds\right| \leq \left|x_i(t_2)\right| + \int_0^T \left|x_i'(s)\right|ds \leq M + \int_0^T \left|x'(s)\right|ds,$$

$$|x|_0 \leq \sqrt{n}\left(M + \int_0^T \left|x'(s)\right|ds\right) \leq \sqrt{n}\left(M + T^{\frac{1}{2}}\left(\int_0^T \left|x'(s)\right|^2 ds\right)^{\frac{1}{2}}\right).$$

Multiplying the two sides of system (3.2) by $x^T(t)$ and integrating them over $[0, T]$, combining with $\tau = mT$, by (3.9) we have

$$\begin{aligned}
&-\int_0^T \left|x'(t)\right|^2 dt + \lambda_{2,i}\int_0^T \left|x(t)\right|^2 dt + n\lambda_{1,i}|x|_0\int_0^T \left|x'(t)\right|dt + \lambda_{0,i}\int_0^T \left|x'(t)\right|^2 dt \\
&\quad + \lambda\int_0^T x^T(t)\operatorname{grad} G(x\left(t-\gamma(t)\right)dt - \lambda\int_0^T x^T(t)p(t)\,dt \geq 0,
\end{aligned}$$

i.e.,

$$\begin{aligned}
(1-\lambda_{0,i})\int_0^T \left|x'(t)\right|^2 dt &\leq \lambda_{2,i}Tn\left(M + \int_0^T \left|x'(t)\right|dt\right)^2 \\
&\quad + n\lambda_{1,i}\sqrt{n}\left(M + \int_0^T \left|x'(t)\right|dt\right)\int_0^T \left|x'(t)\right|dt \\
&\quad + \left(|p|_0 + 2nM_1\right)T\left(M + \int_0^T \left|x'(t)\right|dt\right) \\
&= (\lambda_{2,i}Tn + n\lambda_{1,i}\sqrt{n})\left(\int_0^T \left|x'(t)\right|dt\right)^2 \\
&\quad + \left(2\lambda_{2,i}TnM + n\lambda_{1,i}\sqrt{n}M + |p|_0T + 2nM_1T\right)\int_0^T \left|x'(t)\right|dt \\
&\quad + \lambda_{2,i}TnM^2 + \left(|p|_0 + 2nM_1\right)TM \\
&\leq (2\lambda_{2,i}Tn + n\lambda_{1,i}\sqrt{n})T\int_0^T \left|x'(t)\right|^2 dt \\
&\quad + \left(2\lambda_{2,i}TnM + n\lambda_{1,i}\sqrt{n}M + |p|_0T + 2nM_1T\right)T^{\frac{1}{2}} \\
&\quad \times\left(\int_0^T \left|x'(t)\right|^2 dt\right)^{\frac{1}{2}} + \lambda_{2,i}TnM^2 + \left(|p|_0 + 2nM_1\right)TM. \qquad (3.9)
\end{aligned}$$

From (3.9) and $(\lambda_{2,i}Tn + n\lambda_{1,i}\sqrt{n})T + \lambda_{0,i} < 1$, there is a constant $M_2 > 0$ such that

$$\int_0^T \left|x'(t)\right|^2 dt \leq M_2. \qquad (3.10)$$

In view of (3.9) and (3.10), we get

$$|x|_0 \leq \sqrt{n}\big(M + T^{\frac{1}{2}} M_2^{\frac{1}{2}}\big) := M_3. \tag{3.11}$$

From Lemma 2.2, $(Ax(t))'' = Ax''(t) - 2C'(t)x'(t-\tau) - C''(t)x(t-\tau)$ and (3.2), if $\lambda_{0,i} < \frac{1}{2}$, we have

$$\begin{aligned}
& x''(t) + A^{-1}\left[\lambda \frac{d}{dt} \operatorname{grad} F\big(x(t)\big) + \lambda \operatorname{grad} G\big(x\big(t - \gamma(t)\big)\big)\right] \\
&\quad = A^{-1}\big[2C'(t)x'(t-\tau) + C''(t)x(t-\tau) + A^{-1}\big(\lambda p(t)\big)\big], \\
& \int_0^T \big|x''(t)\big|\,dt \leq \left(\sum_{i=1}^n \frac{1}{(1-\lambda_{0,i})^2}\right)^{\frac{1}{2}} \\
&\qquad \times \left(\int_0^T \left|\frac{\partial^2 F(x(t)}{\partial x^2}\right| \big|x'(t)\big|\,dt + \int_0^T \big|\operatorname{grad} G\big(x\big(t-\gamma(t)\big)\big)\big|\,dt \right. \\
&\qquad \left. + 2T\lambda_{1,i} \int_0^T \big|x'(t)\big|\,dt + T\lambda_{2,i}|x|_0 + T|p|_0\right).
\end{aligned} \tag{3.12}$$

From assumption (H_3) and (3.10)-(3.12), we get

$$\begin{aligned}
& \int_0^T \big|x''(t)\big|\,dt \\
&\quad \leq \left(\sum_{i=1}^n \frac{1}{(1-\lambda_{0,i})^2}\right)^{\frac{1}{2}} \big(\lambda_F T^{\frac{1}{2}} M_2^{\frac{1}{2}} + 2nM_1 T + 2T\lambda_{1,i} T^{\frac{1}{2}} M_2^{\frac{1}{2}} + T\lambda_{2,i} M_3 + T|p|_0\big).
\end{aligned}$$

So there exists a constant $M_4 > 0$ such that

$$\int_0^T \big|x''(t)\big|\,dt \leq M_4. \tag{3.13}$$

Since $x(t) \in C_T^1$, $\int_0^T x'(t)\,dt = 0$, there is a constant vector $\alpha \in \mathbb{R}^n$ such that $x'(\alpha) = 0$; then by (3.13) we get

$$\big|x'(t)\big| \leq \int_0^T \big|x''(t)\big|\,dt \leq M_4.$$

Thus

$$\big|x'\big|_0 \leq M_4.$$

Step 2. Let $\Omega\{x \in \operatorname{Ker} L : QNx = 0\}$, we shall prove that Ω_2 is a bounded set. We have $x(t) = a_0\varphi(t)$, $a_0 \in \mathbb{R}$ $\forall x \in \Omega_2$; then

$$\int_0^T \operatorname{grad} G\big(a_0\varphi\big(t-\gamma(t)\big)\big)\,dt = \int_0^T \operatorname{grad} G\big(a_0\varphi(t)\big)\,dt = 0. \tag{3.14}$$

When $\lambda_{0,i} < \frac{1}{2}$, $i \in I_n$, we have

$$\begin{aligned} \varphi_i(t) &= A^{-1}(1) = 1 + \sum_{j=1}^{\infty} \prod_{k=1}^{j} c_i\bigl(t-(k-1)\tau\bigr) \\ &\geq 1 - \sum_{j=1}^{\infty} \prod_{k=1}^{j} \lambda_{0,i} = 1 - \frac{\lambda_{0,i}}{1-\lambda_{0,i}} \\ &= \frac{1-2\lambda_{0,i}}{1-\lambda_{0,i}} := \delta > 0. \end{aligned}$$

Then we have

$$\bigl|\varphi(t)\bigr| \geq \sqrt{n}\delta.$$

Thus

$$a_0 \leq \frac{M}{\sqrt{n}\delta}.$$

Otherwise, if, $\forall t \in [0, T]$, $|a_0\varphi(t)| > M$, then from assumption (H_2), we have

$$\frac{\partial G(a_0\varphi(t))}{\partial x_i} > 0 \quad (\text{or } < 0), \quad i \in I_n,$$

which is a contradiction; see (3.14). When $\sigma_{0,i} > 1$, $i \in I_n$, we have

$$\begin{aligned} \varphi_i(t) &= A^{-1}(1) = -\frac{1}{c_i(t+\tau)} - \sum_{j=1}^{\infty} \prod_{k=1}^{j} \frac{1}{c_i(t+k\tau)} \\ &\leq -\frac{1}{\lambda_{i,l}} - \sum_{j=1}^{\infty} \prod_{k=1}^{j+1} \frac{1}{\lambda_{0,i}} \\ &= -\frac{1}{\lambda_{i,l}-1} := \gamma < 0. \end{aligned}$$

Then we have

$$\bigl|\varphi(t)\bigr| \geq \sqrt{n}|\gamma|.$$

Thus

$$a_0 \leq \frac{M}{\sqrt{n}|\gamma|}.$$

Otherwise, if $\forall t \in [0, T]$, $|a_0\varphi(t)| > M$, then from assumption (H_2), we have

$$\frac{\partial G(a_0\varphi(t))}{\partial x_i} > 0 \quad (\text{or } < 0), \quad i \in I_n,$$

which is a contradiction; see (3.14). Hence Ω_2 is a bounded set.

Step 3. Let $\Omega = \{x \in C_T^1 : |x|_0 < nM_2 + 1, |x'|_0 < nM_4 + 1\}$, then $\Omega_1 \cup \Omega_2 \subset \Omega$, $\forall (x,\lambda) \in \partial\Omega \times (0,1)$, and from the above proof, $Lx \neq \lambda Nx$ is satisfied. Obviously, condition (2) of Lemma 2.3 is also satisfied. Now we prove that condition (3) of Lemma 2.3 is satisfied. We have $|x^0| = |a_0\varphi|_0$, $a_0 \in \mathbb{R}$, $\forall x^0 \in \partial\Omega \cap \operatorname{Ker} L$. There at least exists a $i \in I_n$ such that $|x_i^0| > M$. When $x_i^0 > M$, take the homotopy

$$H(x,\mu) = \mu x + (1-\mu)QNx, \quad x \in \overline{\Omega} \cap \operatorname{Ker} L, \mu \in [0,1].$$

Then, by using assumption (H_2), we have $H(x,\mu) \neq 0$. When $x_i^0 < -M$, take the homotopy

$$H(x,\mu) = -\mu x - (1-\mu)QNx, \quad x \in \overline{\Omega} \cap \operatorname{Ker} L, \mu \in [0,1].$$

We also have $H(x,\mu) \neq 0$. Then by degree theory,

$$\begin{aligned} \deg\{QN, \Omega \cap \operatorname{Ker} L, 0\} &= \deg\bigl\{H(\cdot,0), \Omega \cap \ker L, 0\bigr\} \\ &= \deg\bigl\{H(\cdot,1), \Omega \cap \ker L, 0\bigr\} \\ &= \deg\{I, \Omega \cap \ker L, 0\} \neq 0. \end{aligned}$$

Applying Lemma 2.3, we reach the conclusion. □

Remark 3.1 When $\frac{1}{2} \leq \lambda_{0,i} < 1$ or $\sigma_{0,i} < 1$, there are no existence results for periodic solutions for system (1.1). We hope that there is interest in doing further research on this issue.

As an application, we consider the following system:

$$\frac{d^2}{dt^2}\bigl(x(t) - C(t)x(t-4\pi)\bigr) + \frac{d}{dt}\operatorname{grad} F\bigl(x(t)\bigr) + \operatorname{grad} G\bigl(x(t - 5\cos t)\bigr) = p(t), \tag{3.15}$$

where

$$\begin{aligned} &x(t) = \bigl(x_1(t), x_2(t)\bigr)^T, \qquad \tau = 4\pi, \qquad \gamma(t) = 5\cos t, \qquad C(t) = \operatorname{diag}\left(\frac{\sin t}{1{,}000}, \frac{\cos t}{1{,}000}\right), \\ &F(x) = \frac{1}{2\pi}\bigl(x_1^2 + 2x_1x_2 + x_2^2 + 2x_1 + 3x_2 + 1\bigr), \qquad G(x) = \frac{1}{\sqrt{2\pi}}(x_1 + x_2), \\ &p(t) = (\sin t, \cos t)^T. \end{aligned}$$

Clearly, system (3.15) is a particular case of system (1.1). Obviously,

$$\operatorname{grad} G(x) = \frac{1}{\sqrt{2\pi}}(x_1, x_2)^T, \qquad \frac{\partial^2 F(\nu)}{\partial x^2} = \begin{pmatrix} \frac{1}{\pi} & \frac{1}{\pi} \\ \frac{1}{\pi} & \frac{1}{\pi} \end{pmatrix}.$$

Here assumptions (H_1)-(H_2) are all satisfied. In addition,

$$\begin{aligned} &T = 2\pi, \qquad \lambda_{0,i} = \lambda_{1,i} = \lambda_{2,i} = \frac{1}{1{,}000}, \quad n = 2, \\ &(\lambda_{2,i}Tn + n\lambda_{1,i}\sqrt{n})T + \lambda_{0,i} \approx 0.0976 < 1. \end{aligned}$$

By using Theorem 3.1, when $\lambda_{0,i} < \frac{1}{2}$, we know that system (3.15) has at least one 2π-periodic solution.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
ZM performed all the steps of proof in this research and also wrote the paper. BD suggested many good ideas that made this paper possible and helped to improve the manuscript. All authors read and approved the final manuscript.

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