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Articulation sets in linear perfect matrices I: forbidden configurations and star cutsets*

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Abstract

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A (0, 1) matrix is linear if it does not contain a 2×2 submatrix of all ones. In these two papers we deal with perfect graphs whose clique-node incidence matrix is linear. We first study properties of some subgraphs that contain odd holes. We then prove that a graph whose clique-node incidence matrix is linear but not totally unimodular contains a node v such that the removal of v and all its neighbors disconnects the graph. These results yield a proof of the strong perfect graph conjecture for this class of graphs.

1. Introduction

Given a simple, undirected graph, let α be the maximum size of a stable set and θ the minimum number of cliques which cover its set of nodes. A graph is perfect if the equality $\alpha = \theta$ holds for all of its induced subgraphs. Berge [1] formulated the following two conjectures.

Conjecture 1.1. A graph is perfect if and only if its complement is perfect.

Conjecture 1.2. A graph is perfect if and only if it does not contain an odd hole or an antihole.

A hole is a chordless cycle of length $\geqslant 5$ and an antihole is its complement.

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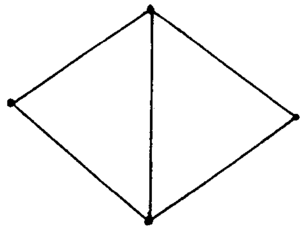

Fig. 1.

Lovasz [16] has proven the first conjecture. The second conjecture (the strong conjecture), which is stronger than Conjecture 1.1, is still unsolved but has been shown to hold for several classes of graphs, such as planar graphs [20], claw-free graphs [19], circular arc graphs [21], to mention a few. An exhaustive collection of papers dealing with perfect graphs can be found in [3].

Many attempts to prove Conjecture 1.2 for special classes of graphs have relied on the hypothesis that all perfect graphs or matrices can be built by repeated compositions of elementary graphs or matrices with operations that preserve the perfection property. Hence, for this approach to successfully yield proofs of Conjecture 1.2 for particular classes of graphs or eventually in its entirety, it is important to study articulation sets of perfect graphs or matrices and composition operations that preserve the perfection property. Results on this subject have been recently obtained by, among others, Burlet and Fonlupt [5] on Meyniel graphs, Chvatal and Sbihi [8] on claw-free graphs, Bixby [4], Cornuejols and Cunningham [14] and Chvatal [7].

In this paper we study the decomposition of graphs not containing the graph K_4-e of Fig. 1 as an induced subgraph. Using a theorem of Chvatal [7] on critically imperfect graphs, we show in Section 4, the validity of Conjecture 1.2 for (K_4-e) free graphs. In the second paper this result is used to find articulation sets of perfect (K_4-e)-free graphs. Our findings are based on the study of the structure of the $(0, 1)$ matrix whose set of rows contains the incidence vectors of the maximal cliques of the graph. Whenever the graph is (K_4-e)-free, such a matrix is linear, i.e., it does not contain a 2×2 submatrix of all ones.

2. Definitions, notation and the main result

In this section we discuss the main results contained in the two papers [10, 11].

2.1. Bipartite representation of a (0, 1) matrix

The bipartite representation of a $(0,1)$ matrix A is the bipartite graph $G(V^+, V^-; E)$ where V^+, V^- are the two sets of nodes representing the columns and rows of A. For each entry $a_{ij}=1$ there exists an edge (i, j) of E whose end nodes are the two nodes representing row i and column j. A *path* of $G(V^+, V^-; E)$ is a sequence of distinct nodes $v_1, v_2, \ldots, v_n$ such that $(v_i, v_{i+1}) \in E$, for all $1 \leq i \leq n-1$. The edges (v_i, v_{i+1}) are the edges of the path and an edge (v_i, v_{i+l}), $l \geq 2$ is a *chord* of the path. A path with nodes v_1, v_n as end nodes is said to be a v_1v_n-path. If v_i and v_l are two nodes of a path P, the path $v_i, v_{i+1}, \ldots, v_l$ is said to be the v_iv_l-subpath of P. The *length* of a path is the number of edges in the path. If both end nodes v_1, v_n of P belong to either V^+ or V^-, the length of P is congruent to 0 or $2 \bmod 4$. If only one end node of P belongs to V^+, the length of P is congruent to 1 or $3 \bmod 4$. For sake of brevity, the word 'congruent' will be often omitted.

A *cycle* is a path $v_1, v_2, \ldots, v_{n+1}$ in which nodes v_1, v_{n+1} coincide. An edge (v_i, v_{i+1}) $1 \leq i \leq n$ is an edge of the cycle. An edge of G connecting two nonconsecutive nodes of a cycle is a chord of the cycle. A chordless cycle is a hole. A cycle is referred to as an odd cycle if its length is congruent to 2 modulo 4 and as an even cycle if its length is congruent to 0 modulo 4.

A node y is adjacent to (or is a neighbor of) node x if edge $(x, y) \in E$. Two nodes x, y are adjacent in a path or a cycle if edge (x, y) is an edge of the path on the cycle.

Let G' be a subgraph of G. A node x not belonging to G' is said to be *strongly adjacent* to G' if x is adjacent to at least two nodes of G'.

We say that a subset V of nodes of G is an articulation set if the subgraph of G induced by $(V^+ \cup V^-) \setminus V$ is disconnected.

The set $N(x)$ consists of node x and its neighbors. If $N(x)$ is an articulation set, it is referred to as a *star cutset.*

Given a set S, we indicate its cardinality as $|S|$. In this paper we focus on congruence relationships ($\equiv$) modulo 4.

It is straightforward to interpret the notions of linearity of a matrix in terms of the associated bipartite graph. A $(0, 1)$ matrix is linear if the corresponding bipartite graph does not contain cycles of length 4. We define a bipartite graph to be *linear* if the above condition is satisfied.

2.2. The main result

All the matrices introduced in this section have $(0, 1)$ entries. A matrix A is *perfect* if the following set packing polytope

$$\{x \geq 0,\ Ax \leq 1\}$$

has all integer valued extreme points (is integral). A matrix is *balanced* if the bipartite graph G does not contain a chordless cycle of length $2 \bmod 4$ (odd hole). A matrix A is *totally unimodular* (TU) if the polytope

$$\{Ax \leq b,\ x \geq 0\}$$

is integral for every integer vector b. A matrix is *without odd cycles* if the corresponding graph G has no cycle of length $2 \bmod 4$. We say that a bipartite graph G is without odd cycles, or TU, or balanced, or perfect if the corresponding matrix has the property. Characterizations of TU and the matrices or graphs have been given by Camion, [18], and Padberg [17] respectively. A survey of totally unimodular, balanced and perfect matrices can be found in [18]. In particular, a matrix without odd cycles is TU, a TU matrix is balanced, and a balanced matrix is perfect.

This paper contains the following result: *Let G (V^+, V^-, E) be a bipartite graph which is perfect but contains odd cycles; then there exists a node v of V^+ such that the removal of v, together with all the nodes of G connected to v by a path of length ≤ 2, disconnects the graph.* In fact, node v can be chosen as the end node of a chord of a minimal odd cycle of G. As discussed in Section 4, this yields a proof of the validity of Conjecture 1.2 for this class of graphs.

The main result contained in our second paper is the following: *Let G be a linear perfect bipartite graph which is not balanced; then G has a node in V^- which is a star cutset.*

Conforti and Rao [11] have proven the following: *Let G be a linear balanced bipartite graph containing odd cycles; then G has a star cutset $N(v)$.*

The class of bipartite graphs without odd cycles is well studied from a structural as well as an algorithmic point of view. In particular, Yannakakis [2] has given a composition procedure that constructs all bipartite graphs without odd cycles, starting with 'elementary' matrices which are easily recognizable. Based on his results, he has also given polynomial time procedures to recognize whether a bipartite graph is without odd cycles and to solve integer programs for these classes of constraint matrices. Conforti and Rao [9] have given a different procedure to test whether a bipartite graph is without odd cycles. In [13] polynomial algorithms to test membership in the classes of linear balanced and perfect matrices are given.

2.3. Bipartite graphs and intersection graphs

To any (0.1) matrix A one can associate an *intersection graph* I_A as follows: The node set of I_A corresponds to the columns of A and two nodes are adjacent if and only if the corresponding columns are not orthogonal. Conversely, given a graph I, the matrix which has as rows the incidence vectors of all the maximal cliques of I is called the *clique matrix* of I. The connection between perfect graphs and perfect matrices has been investigated by Fulkerson [15], and Chvatal [6].

Theorem 2.1. *A (0, 1) matrix is perfect if and only if the undominated rows of A form the clique-node matrix of its intersection graphs I_A.*

Hence in the study of perfection one has just to consider matrices with no duplicated or dominated rows. Note that testing whether a matrix is a clique matrix can be done in polynomial time, using a theorem of Gilmore (see [2, p. 396]).

Let G be the bipartite graph of a matrix A and I_A the corresponding intersection graph. An odd hole H of G is said to be *imperfect* if there exists no node in V^-, strongly adjacent to H and having at least three neighbors in H.

Since the only perfect graphs containing no clique of size greater than two are bipartite graphs, we have the following.

Remark 2.1. Let G be a bipartite graph containing an imperfect odd hole H. If H has length 6, then the matrix represented by G is not a clique matrix. If H has length greater than 6, then the graph I_A has an odd chordless cycle. Hence G is not perfect.

It is easy to see that G is linear if and only if I_A is $(K_4 - e)$-free. Since every node strongly adjacent to an odd hole of length 6 induces a cycle of length 4, we have the following.

Remark 2.2. A linear perfect graph can not contain an odd hole of length 6.

We now interpret the results of Section 2.2 in terms of the intersection graph: Let I be a graph not containing $K_4 - e$ as induced subgraph. Suppose the associated clique matrix A_I is perfect but not balanced. Then I contains a clique-articulation; i.e., the induced subgraph of I obtained by removing the nodes belonging to some clique, is disconnected. It is well known [6] that the reverse operation preserves perfection. Hence, to test the perfection of I, it is enough to test the perfection of the component graphs.

Furthermore, if I does not contain $K_4 - e$ as an induced subgraph, the number of maximal cliques of I is bounded by the number of edges of I, hence testing the existence of clique-articulations is trivial. However, for general graphs, there exists a polynomial algorithm to test the existence of clique articulations [22] or more complicated disconnecting sets [14].

3. Properties of balanced graphs

Recall that a bipartite graph is balanced if it does not contain a chordless cycle of length 2 mod 4. In this section we study properties of graphs that are balanced. Most of the results contained in this section also appear in [11] but we repeat them here to make the paper self-contained.

3.1. Starred cycles

Definition 3.1. A cycle C is *starred* if its set of chords satisfies the following properties:

(a) There exists two nodes x and y of C, called the *star nodes* of C such that every chord of C has either node x or node y, but not both, as its end node. A chord with node x (y) as star node is a x-chord (y-chord).

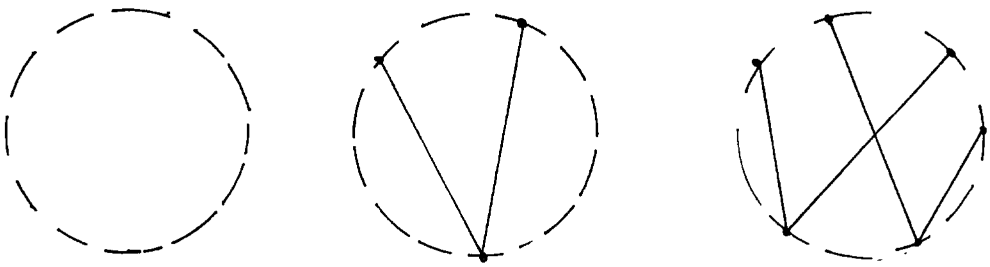

Fig. 2.

(b) No other node of C is the end node of two distinct chords.
(c) No two end nodes of chords are adjacent.

Note that, according to Definition 3.1, all the chords of a starred cycle C can have the same star node, or C can be chordless. Fig. 2 shows three starred cycles.

Remark 3.1. Let x, y be two nodes of a cycle C in a linear graph G, with the property that every chord of C has x or y as an end node. Let P_1, P_2 be the two xy-paths in C. If one of the two paths P_1, P_2 has length less than four and C does not contain an odd hole of length six, then properties (a), (b) and (c) of Definition 3.1 are satisfied.

Theorem 3.1. *Let C be a starred cycle of a balanced graph $G(V^+, V^-; E)$ and T its set of chords. Then C and T satisfy the following relationship:*

$$2\,|T| \equiv |C|.$$

Proof. By induction on the cardinality $|T|$ of the set of chords of C. The theorem is obviously true for $|T| \leq 1$. We consider the following two cases:

Case 1: *C contains a chord (u, x) such that every chord of C other than (u, x) has both its end nodes contained in $C_1 = u, P_1, x, u$ or $C_2 = u, P_2, x, u$ where P_1, P_2 are two disjoint paths connecting u and x in C* (see Fig. 3(a)).

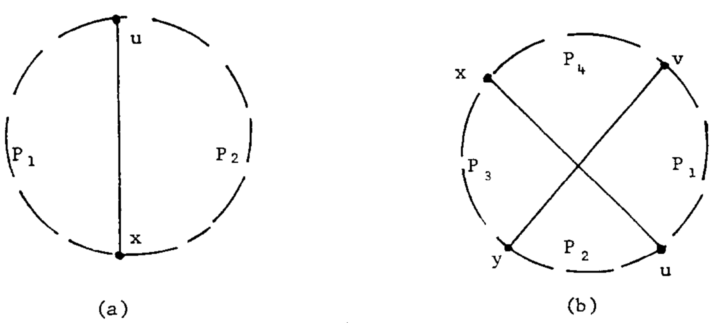

Fig. 3.

Let T_1 and T_2 be the sets of chords of C_1 and C_2 respectively. Then we have that $|T_1| < |T|$ and $|T_2| < |T|$ since $T_1 \cap T_2 = \emptyset$ and $T_1 \cup T_2 = T \setminus \{(u, x)\}$. Therefore by the induction hypothesis, we have

$$2\,|T_1| \equiv |C_1| \quad \text{and} \quad 2\,|T_2| \equiv |C_2|.$$

Furthermore, we have

$$2 + |C| = |C_1| + |C_2| \quad \text{and} \quad 1 + |T| = |T_1| + |T_2|.$$

These relationships imply:

$$2\,|T| \equiv |C|.$$

Case 2: *No chord satisfying the assumption of Case* 1 *exists.*

This implies that there exists two chords (u, x) and (v, y) of C, such that their end nodes appear in the order x, v, u, y when C is traversed in one direction, see Fig. 2(b).

Let P_1, P_2, P_3 and P_4 be respectively the paths connecting v to u, u to y, y to x, and x to v, as shown in Fig. 3(b). Let T_1 and T_2 be the sets of chords of the cycles:

$$C_1 = v, P_1, u, x, P_3, y, v \quad \text{and} \quad C_2 = v, P_4, x, u, P_2, y, v.$$

As a consequence of properties (a), (b) and (c) of Definition 3.1, we have that none of the following edges: (v, u), (u, y), (y, x) or (x, v) is an edge or a chord of C. This implies that $T_1 \cap T_2 = \emptyset$ and $T_1 \cup T_2 = T \setminus \{(u, x), (v, y)\}$. Therefore we have

$$|T_1| + |T_2| = |T| - 2 \quad \text{and} \quad |C_1| + |C_2| = |C| + 4.$$

Since $|T_1| < |T|$ and $|T_2| < |T|$, by induction hypothesis we have

$$2\,|T_1| \equiv |C_1| \quad \text{and} \quad 2\,|T_2| \equiv |C_2|.$$

The above relationships imply

$$2\,|T| \equiv |C|. \qquad \square$$

3.2. Expanded cycles

Consider a graph G consisting of a starred cycle C plus two additional nodes b and d that are adjacent to C. Suppose C contains just one star node s and b is adjacent to d and s but not adjacent to any other node of C. We define the triple (C, b, d) to be an *expanded cycle*, see Fig. 4. Let T be the set of chords of C and T_d be the set of edges joining d to a node of C.

Lemma 3.1. *Let G be a linear balanced graph containing an expanded cycle (C, b, d) with star node s. Then $|T_d|$ is even.*

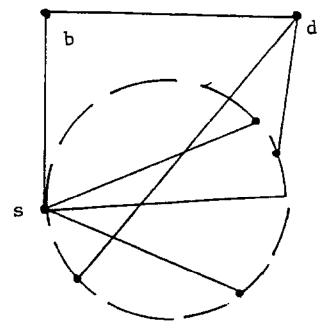

Fig. 4.

Proof. Suppose $|T_d|$ is odd. Then there exists a node $x \in C$ adjacent to d. Let P_1 and P_2 be the two xs-paths forming C. Consider the two cycles $C_1 = x, P_1, s, b, d, x$ and $C_2 = x, P_2, s, b, d, x$. All the possible chords of C_1 and C_2 must have either node s or d as an end node. By Remark 3.1, it follows that C_1 and C_2 are starred cycles with star nodes s and d. Let T_1 and T_2 be the chords of C_1 and C_2 respectively. Then by Theorem 3.1 we have $2|T_1| \equiv |C_1|$, $2|T_2| \equiv |C_2|$ and $2|T| \equiv |C|$. Clearly $T \cap T_d = \emptyset$ and $|C| = |C_1| + |C_2| - 6$. As T_1, T_2 and (x, d) partition $T \cup T_d$, it follows that

$$\begin{aligned} 2|T_d| &= 2|T \cup T_d| - 2|T| = 2|T_1| + 2|T_2| + 2 - 2|T| \\ &\equiv |C_1| + |C_2| - |C| + 2 = 6 + 2 \equiv 0. \end{aligned}$$

Hence $|T_d|$ is even. □

Remark 3.2. Suppose (C, v_1, v_2) with $v_2 \in V^+$ is an expanded cycle containing an odd hole. If every node in V^- that is strongly adjacent to C is adjacent to exactly two nodes of C and is non-adjacent to v_2, then (C, v_1, v_2) contains an imperfect odd hole.

3.3. Starred wheels

A cycle C and a node $\notin C$ form a *wheel* (C, v) if v is adjacent to at least two nodes of C. Node v is the *hub* of the wheel and edges (v, v_i), where $v_i \in C$, are the *spokes* (rays) of the wheel. The $v_{i-1}v_i$-path of C that does not contain any other neighbor of v is called a *sector* S_i of the wheel. The corresponding cycle C_i is formed by v_{i-1}, S_i, v_i, v, v_{i-1}. Two sectors are said to be *adjacent* if they have a neighbor of v in common. Two spokes (v_i, v) and (v_j, v) are adjacent if (C, v) contains a sector with end nodes v_i and v_j.

If C is chordless then (C, v) is a chordless wheel. Clearly a balanced chordless wheel must have an even number of spokes.

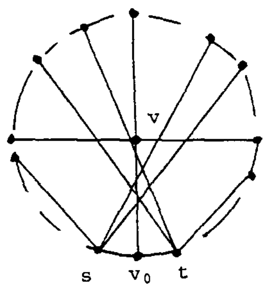

Fig. 5.

Definition 3.2. A wheel (C, v) is a *starred wheel* if C is a starred cycle with star nodes s and t satisfying the following two conditions, (see Fig. 5):

(a) The star nodes s and t are adjacent to v_0 where (v_0, v) is a spoke of the wheel.

(b) The number of s-chords (t-chords) of C having an end node in the sector containing t (s) is even.

Theorem 3.2. *Let G be a linear balanced bipartite graph containing a starred wheel (C, v) with star nodes s and t. Let S_i, $i = 1, 2, \ldots, n$ be the n sectors of the starred wheel with $s \in S_1$, $t \in S_n$. Then the following holds:*

(i) *The number, n, of sectors (spokes) is even.*

(ii) *All the s-chords (t-chords) have end nodes in odd (even) numbered sectors only. Each odd (even) numbered sector S_j, $1 < j < n$, has an even number of end nodes of s-chords (t-chords).*

We first prove the following claim.

Claim 1. *Each sector S_j, $1 < j < n$ has an even number of end nodes of s-chords and an even number of end nodes of t-chords.*

Proof. Suppose sector S_j with end nodes v_{j-1} and v_j contradicts Claim 1. Without any loss in generality, suppose S_j contains an odd number of s-chords. The cycle $C_j = v, v_{j-1}, S_j, v_j, v$ defines an expanded cycle (C_j, v_0, s) with star node v. By Lemma 3.1 the number of s-chords in S_j must be even, and Claim 1 is proven. □

Proof of Theorem 3.2. (i) Let T_s and T_t be the set of s-chords and t-chords respectively of C. Similarly let T_s^i and T_t^i be the sets of s-chords and t-chords respectively of C with end node in S_i. Then,

$$T_s = \bigcup_{i=1}^{n} T_s^i \quad \text{and} \quad T_s^i \cap T_s^j = \emptyset \quad \text{for all } i \neq j.$$

Similarly we have

$$T_t = \bigcup_{i=1}^{n} T_t^i \quad \text{and} \quad T_t^i \cap T_t^j = \emptyset \quad \text{for all } i \neq j.$$

As a consequence of Theorem 3.1 it follows that

$$2\,|T_s \cup T_t| \equiv |C|, \qquad 2\,|T_s^1| \equiv |C_1| \quad \text{and} \quad 2\,|T_t^n| \equiv |C_n|.$$

By Claim 1, we have

$$2\,|T_s^i| \equiv 2\,|T_t^i| \equiv 0 \quad \text{for } 1 < i < n.$$

Furthermore, by condition (b) of Definition 3.7, we have $|T_j^n| \equiv 2$, $|T_t^1| \equiv 0$. Since C_i, $i = 2, \ldots, n-1$ are chordless, we have $|C_i| \equiv 0$, $i = 2, \ldots, n-1$. Then

$$|C| \equiv \sum_{i=1}^{n} |S_i| \equiv \sum_{i=1}^{n} |C_i| - 2n \equiv \sum_{i=1}^{n} 2\,|T_s^i| + |T_t^i| - 2n \equiv |C| - 2n.$$

Hence $2n \equiv 0$ and n is even.

(ii) In view of Claim 1, because of symmetry, it is enough to show that all the s-chords have end nodes in odd numbered sectors only. Suppose the contrary. Then there exists a chord (s, u) where u is in an even numbered sector. Without loss of generality, we assume that u is in the first such node encountered when traversing C clockwise starting from node v_0 and let w be the last neighbor of s encountered before u. Let P be the wu-subpath of C not containing v_0. Suppose v_i and v_j (possibly $v_j = v_i$) are respectively the first and last neighbors of v encountered when traversing P starting from node w. Then both the wv_i and v_ju-subpaths, P_1 and P_2 respectively, of P are of length $0 \bmod 4$ since $s, w, P_1, v_i, v, v_0, s$ and $s, v_0, v, v_j, P_2, u, s$ are chordless cycles. Now the v_iv_j-subpath of P has length $0 \bmod 4$ since it contains an odd number of neighbors of v. This implies that P is of length $0 \bmod 4$ and the cycle s, w, P, u, s is an odd hole which contradicts the assumption that G is balanced. $\square$

Theorem 3.3. *Let (C, v) be a starred wheel with $v \in V^+$. Suppose that every node in V^- that is strongly adjacent to C has exactly two neighbors in C and is not adjacent to v. Then either (C, v) has an even number of sectors and the chords of C satisfy Theorem 3.2 or (C, v) contains an imperfect odd hole.*

Proof. By using Lemma 3.1 and Remark 3.2, it is easily verified that the odd holes identified in the proof of Theorem 3.2 are imperfect since every node in V^- that is strongly adjacent to C has exactly two neighbors in C and is not adjacent to v. Hence Theorem 3.2 must hold for otherwise (C, v) generates an imperfect odd hole. Thus the theorem follows. $\square$

4. The strong perfect graph conjecture for $(k_4 - e)$-free graphs

An odd cycle of a bipartite graph is said to be *minimal* if no proper subset of its nodes induces an odd cycle. In this section we characterize a property of minimal odd cycles of a linear balanced graph.

Lemma 4.1. *Every minimal odd cycle of a linear graph has at most one chord.*

Proof. Conforti and Rao [9] have shown that, for every pair of chords (u_1, v_1) and (u_2, v_2) of a minimal odd cycle C, the nodes must appear in the order v_1, u_2, u_1, v_2 when C is traversed clockwise or counter-clockwise. Moreover, nodes u_1 and v_2 as well as nodes u_2 and v_1 must be adjacent in C. Hence nodes v_1, u_1, v_2, u_2 induce a cycle of length 4. □

Given a cycle C, let $C^+ = \{v_i \mid v_i \in V^+ \cap C\}$ and $C^- = \{v_i \mid v_i \in V^- \cap C\}$. A cycle C is said to be *node-minimal* if there exists no other cycle H such that $H^+ \subset C^+$ or $H^- \subset C^-$. Recall that a node v not in C is strongly adjacent to C if v has two or more neighbors in C. We now state the following property of node-minimal odd cycles.

Lemma 4.2. *Let G be a linear graph containing an odd cycle C with a unique chord. Then C is a node-minimal cycle only if no node is strongly adjacent to C.*

Proof. Conforti and Rao [10] have shown that a node strongly adjacent to C must have two neighbors in C which in turn have a common neighbor in C. Hence the graph contains a cycle of length four. □

Lemma 4.3. *Every node-minimal odd cycle C of a linear perfect graph has exactly one chord.*

Proof. By Lemma 4.1, the odd cycle C can not have more than one chord. Assume C has no chord, then by Remark 2.1, C has a strongly adjacent node in V^-, a contradiction to Lemma 4.2. □

We now prove that given a linear balanced graph G containing a node-minimal odd cycle with a unique chord (u, v), $N(u) \cup N(v)$ is an articulation set of G. This result will be used in Theorem 4.2 to prove the validity of Conjecture 1.2 for linear graphs.

Let C be a node-minimal odd cycle in $G(V^+, V^-; E)$ with chord (u, v). Let a, b and c, d be the nodes of C that are adjacent in C to u and v respectively as in Fig. 6. We use the following notation:

$$C = u, a, P_1, c, v, d, P_2, b,$$

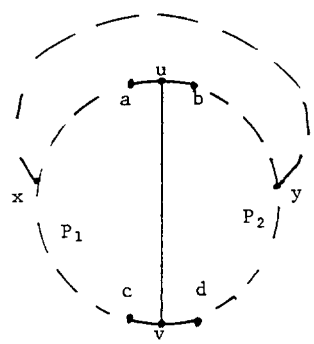

Fig. 6.

where P_1 (P_2) is a path of length at least five having nodes a and c (b and d) as end nodes.

Let P be a chordless path having as end nodes, a node $x \in P_1$ but $x \neq a, c$ and a node $y \in P_2$ but $y \neq b, d$. Furthermore, no node of C is an intermediate node of P. We define the subpaths P_{xa}, P_{xc} of P_1 having nodes x, a and x, c as end nodes and the subpaths P_{by}, P_{dy} and P_2 having nodes b, y and d, y as nodes, see Fig. 6.

Finally, we define the following four cycles:

$$C_{ab} = x,\ P_{xa},\ a,\ u,\ b,\ P_{by},\ y,\ P,\ x$$

$$C_{cd} = x,\ P_{xc},\ c,\ v,\ d,\ P_{dy},\ y,\ P,\ x$$

$$C_{ad} = x,\ P_{xa},\ a,\ u,\ v,\ d,\ P_{dy},\ y,\ P,\ x$$

$$C_{cb} = x,\ P_{xc},\ c,\ v,\ u,\ b,\ P_{by},\ y,\ P,\ x$$

We can now state our result.

Theorem 4.1. *Let (u, v) be the chord of a node-minimal odd cycle of a linear balanced graph G. Then $N(u) \cup N(v)$ is an articulation set of G.*

Proof. Let $C = u, a, P_1, c, v, d, P_2, b, u$ be a node-minimal cycle with (u, v) as chord. With nodes of $N(u) \cup N(v)$ removed, let P be a shortest path connecting P_1 to P_2 and let x, y be the end nodes of P. Let nodes x_1, y_1 be the neighbors of x and y in P. As a consequence of Lemma 4.2 nodes x_1, y_1 are not strongly adjacent to C and every other intermediate node of P can be adjacent to at most one node in the set $\{a, b, c, d\}$. From the path P and cycle C define the four cycles C_{ab}, C_{cd}, C_{ad}, C_{cb}. As a consequence of Remark 3.1, the above four cycles are starred cycles with nodes a, b; c, d; a, d; c, b as star nodes respectively.

Let T_a, T_b, T_c, T_d be the sets of edges having one end node in P and nodes a, b, c and d respectively as the other end node. Note that $T_a \cup T_b$, $T_c \cup T_d$, $T_a \cup T_d$ and $T_c \cup T_b$ are the sets of chords of the starred cycles C_{ab}, C_{cd}, C_{ad}, C_{cb} respectively. There are two cases to consider.

Case 1: *Both end nodes x, y of P belong either to V^+ or to V^-.*
Since C is an odd cycle, we have

$$|C_{ab}| \equiv 2 + |C_{cd}|.$$

Furthermore, since $|C_{ab}| + |C_{cd}| + 2 \equiv |C_{ad}| + |C_{cb}|$,

$$|C_{ad}| \equiv |C_{cb}|.$$

The above relationships, together with Theorem 3.1, imply the following inconsistent set of equations:

$$2(|T_a| + |T_b|) \equiv 2 + 2(|T_c| + |T_d|),$$
$$2(|T_a| + |T_d|) \equiv 2(|T_c| + |T_b|).$$

Case 2: *An end node of P belongs to V^+ and the other end node belongs to V^-.*
We can derive the following relationships:

$$|C_{ab}| \equiv |C_{cd}|, \qquad |C_{cd}| \equiv 2 + |C_{cb}|.$$

These relations, together with Theorem 3.1, imply the following inconsistent set of equations:

$$2(|T_a| + |T_b|) \equiv 2(|T_c| + |T_d|),$$
$$2(|T_a| + |T_d|) \equiv 2 + 2(|T_c| + |T_d|).$$

The proof is now complete. □

Remark 4.1. Let C be a node-minimal odd cycle with one chord (u, v). Suppose there exists a chordless path P connecting $P_1\backslash\{a, c\}$ to $P_2\backslash\{b, d\}$, and not containing a node in $N(u) \cup N(v)$. Then the subgraph induced by the nodes in $C \cup P$ contains an odd hole.

Let $N^2(v)$ be the set of nodes connected go v by a path of length at most two. That is,

$$N^2(v) = \bigcup_{u_i \subset N(v)} N(u_i).$$

We now use Remark 4.1 to prove the following.

Theorem 4.2. *Let (u, v), $v \in V^+$ be the unique chord of a node-minimal odd cycle in a linear graph G not containing an imperfect odd hole. Then $N^2(v)$ is an articulation set of G.*

Proof. We give a proof for the following stronger statement: *Every path connecting a node in $P_1\backslash\{a, c\}$ to a node in $P_2\backslash\{b, d\}$ must contain an intermediate node belonging to $N^2(v)$.*

Assume not and let $P = x, x_1, P^*, y_1, y$ be a shortest path contradicting the above statement. Nodes u, v, c, d belong to $N(v)$, hence can not be adjacent to any node in P^*. The only neighbors in C of nodes x_1, y_1 are x and y respectively and every other node of P^* can be adjacent to node a or b but not both. By Remark 4.1, the graph induced by the nodes in $C \cup P$ contains an odd hole H. Since G is perfect, there must exist a node $x \in V^-$ adjacent to at least three nodes of H. Since C has no strongly adjacent nodes, node x has at most one neighbor in C. If $x \in N(v)$, x can not have any neighbor in H. If $x \notin N(v)$, x can have at most one neighbor in H, since P is a shortest path. Hence in either case, x has at most two neighbors in H, contradicting the fact that G is perfect. □

A matrix A is said to be *critically imperfect* if A is not perfect but every submatrix obtained from A by deleting a column is perfect.

We now translate in terms of bipartite graphs a result of Chvatal [7].

Theorem 4.3 [7]. *The bipartite graph G associated with a critically imperfect matrix A, can not contain an articulation set $N^2(v)$, where $v \in V^+$.*

We can now state our main result.

Theorem 4.4. *A $(K_4 - e)$-free graph is perfect if an only if it does not contain an odd hole.*

Proof. Let I be a $(K_4 - e)$ graph that is not perfect, but every induced subgraph of I is perfect. Construct the bipartite graph G of the (critically imperfect) clique-node matrix A_I of I. G must contain odd cycles, else the matrix A_I is totally unimodular and hence perfect. G can not contain an imperfect odd hole, otherwise by Remark 2.1, I contains an odd hole. However, by Theorem 4.2, G contains a node v of V^+ such that $N^2(v)$ is an articulation set of G, a contradiction to Theorem 4.3. □

5. Minimal odd hole

An odd hole H has the *minimality* property, or is *minimal* if no subset of its nodes together with at most three nodes not in H induces an odd hole of smaller cardinality. Given a linear perfect bipartite graph G, in this section we study the properties of nodes strongly adjacent to a minimal odd hole H (in G) having a node $v \in V^-$ as the center of the chordless wheel (H, v). Some of these results are contained in [12] but we repeat them here for the sake of completeness.

With respect to H, we now define the following sets.

$$A_+ = \{u \in V^+ \mid u \text{ is strongly adjacent to } H\},$$

$$A_- = \{u \in V^- \mid u \text{ is strongly adjacent to } H\},$$

$$A_+(v) = \{u \in V^+ \mid u \text{ is strongly adjacent to } H \text{ but not adjacent to } v \in A_-\}.$$

Let H be a minimal odd hole and (H, v) be an odd chordless wheel with $v \in A_-$.

Lemma 5.1. *Suppose $u \in A_-$ and $u \neq v$. Then u has an odd number of neighbors in H, and in any sector of (H, v), node u cannot have only one neighbor which is not adjacent to v. Moreover exactly one node of H is a neighbor of both u and v.*

Proof. From the minimality property of H and linearity it follows immediately that node u cannot have an even number of neighbors in H and cannot have an odd number, greater than 1, of neighbors in any one sector of H. Furthermore, only one node of H can be adjacent to both u and v, else nodes u and v belong to a cycle of length 4. Next we show that, in any sector, node u cannot have a unique neighbor which is not a neighbor of v. This in turn implies that exactly one node of H is a neighbor of both u and v and completes the proof of the lemma.

Suppose in sector S_k, node u has a unique neighbor u_k which is not a neighbor of v. Let v_{k-1} and v_k be the end nodes of S_k, P_1 and P_2 be the $v_{k-1}u_k$ and $v_k u_k$-subpaths of S_k respectively, see Fig. 7.

Since S_k is of length $2 \bmod 4$, one of these paths is of length $0 \bmod 4$ and the other is of length $2 \bmod 4$. Now, since node u is strongly adjacent to H, it has a neighbor in another sector, say S_l, having one end node v_l distinct from v_{k-1} and v_k. Let u_l be the neighbor of u closest to v_l (u_l may be coincident with v_l) in sector S_l. Clearly, u_l is non-adjacent to and not coincident with v_{k-1} and v_k. Let P_l be the $u_l u_l$-subpath of S_l. Now using the fact that H contains at least three

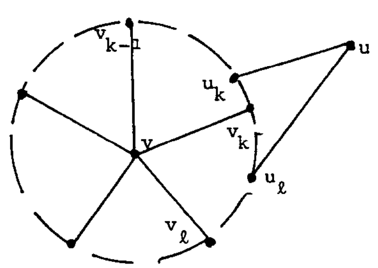

Fig. 7.

sectors, an easy counting argument shows that one of the two cycles

$$C_1 = u,\ u_l,\ P_l,\ v_l,\ v,\ v_{k-1},\ P_1,\ u_k,\ u; \qquad C_2 = u,\ u_l,\ P_l,\ v_l,\ v,\ v_k,\ P_2,\ u_k,\ u$$

is an odd hole of smaller cardinality than H.

This contradicts the minimality of H and the lemma follows.

Lemma 5.2. *All nodes in A_- have a common neighbor in H.*

Proof. Choose a node $v \in A_-$ as the hub of a wheel (H, v). Then by Lemma 5.1, for any other node $u \in A_-$ there exists a node of H, say v_1, which is a neighbor of both u and v. Suppose now, for another node, say $t \in A_-$, node $v_2 \neq v_1$, is a neighbor of both t and v. By considering node u as the hub of the wheel (H, u) and reapplying Lemma 5.1, we have that u and t have a common neighbor, say $x \neq v_1, v_2$, in H. Clearly, since G is linear, x cannot be a neighbor of v. Moreover, since $u \in A_-$, we must have $|H| \geq 18$. Now the cycle v, v_1, u, x, t, v_2, v is an odd hole of length 6. contradicting the minimality of H. Hence the lemma follows. □

Let v_1 be the node of H which is adjacent to all the nodes in A_-. Suppose u and v are two nodes in A_-. Assume that when H is traversed counter-clockwise starting from node v_1, the first node of H which is adjacent to either u or v is a node which is ajdacent to u. Consider the wheel (H, v) and let $S_1, S_2, \ldots, S_{2n+1}$ be the sectors of (H, v) with sector S_i, $1 \leq i \leq 2n+1$, having nodes v_i, v_{i+1} as end nodes where $v_{2n+2} = v_1$; see Fig. 8. The neighbors of u in H satisfy the following lemma.

Lemma 5.3. *Node u has a positive even number of neighbors in sectors S_1 and S_{2n+1}. In the other odd numbered sectors of (H, v), node u has an even number of neighbors. The even numbered sectors of (H, v) contain no neighbors of u.*

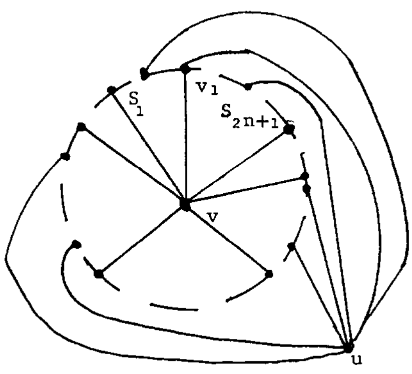

Fig. 8.

Proof. By definition, node u has at least two neighbors in S_1. Then by minimality of H, node u must have a positive even number of neighbors in S_1. Now, by Lemma 5.1 and the minimality of H, node u must have an even number of neighbors in S_i, $2 \leq i \leq 2n$. Again by Lemma 5.1, node u must have an odd number of neighbors in H. This implies that the sector S_{2n+1} has at least two neighbors of u. Then, by the minimality of H, node u must have a positive even number of neighbors in S_{2n+1}. Thus in order to complete the proof of the lemma, we have to show that an even numbered sector of (H, v) has no neighbor of u.

Suppose now S_i is the smallest indexed even numbered sector in which u has neighbors. Let u_k, $k = 1, 2, \ldots, 2l$ be the neighbors of u in S_i such that u_1 is closest to the end node v_i and u_{2l} is closest to the end node v_{i+1} of S_i. Suppose S_j with $j < i$ is the largest indexed odd numbered sector in which u has neighbors. Let u^* be the neighbor of u in S_j such that u^* is closest to the end node v_{j+1} of S_j. Now both the u^*v_{j+1} and v_iu_1-subpaths, P_1 and P_2 of S_j and S_i respectively must be of length 0 mod 4 since the cycles

$$C_1 = u, u^*, P_1, v_{j+1}, v, v_1, u \quad \text{and} \quad C_2 = u, v_1, v, v_i, P_2, u_1, u$$

are chordless and have smaller cardinality than H. Now let Q be $v_{j+1}v_i$-subpath of H not containing any neighbor of u. Then the odd hole $\hat{C} = u, u^*, P_1, v_{j+1}, Q, v_i, P_2, u_1, u$ has smaller cardinality than H. Hence u can not have any neighbors in an even numbered sector of (H, v). This completes the proof of the lemma. □

Remark 5.1. In sector S_i (i odd) of (H, v) let u_k, $k = 1, 2, \ldots, 2m$ be the neighbors of a node $u \in A_-$. Let v_i and v_{i+1} be the end nodes of S_i with u_1 closest to v_i and u_{2m} closest to v_{i+1}. The length of the v_iu_1-subpath and the $u_{2m}v_{i+1}$-subpath of S_1 is 0 mod 4. For $1 \leq k \leq 2m - 1$ the length of the u_ku_{k+1}-subpath in S_i is 2 mod 4.

Let $v \in A_-$ be the node satisfying the following property.

Property 5.1. When traversing H counter-clockwise starting from node v_1, a neighbor (other than v_1) of every node in $A_- \setminus \{v\}$ is encountered before encountering a neighbor (other than v_1) of v.

Consider now the wheel (H, v) where v satisfies Property 5.1. We then have the following corollary to Lemma 5.3.

Corollary 5.1. *Every node satisfies Lemma* 5.3 *and Remark* 5.1 *with respect to* (H, v).

Definition 5.1. Given a minimal odd hole H, a total order relation between all the nodes in A_- is defined by saying that for $u_i, u_j \in A_-$ we have $u_i \gg u_j$ if a neighbor of u_j is encountered before encountering a neighbor of u_i when H is traversed counter-clockwise starting from the node v_i.

Suppose $v \in A_-$ satisfies Property 5.1. Then considering the wheel (H, v) we have the following.

Lemma 5.4. *Let $t, u \in A_-$ with $t \gg u$. Then in any one sector S_j, if u has a neighbor, then t also has a neighbor in the same sector. In sector S_i let t_k, $k = 1, 2, \ldots, 2l$; u_k, $k = 1, 2, \ldots, 2m$ be the neighbors of t and u respectively. In the sector S_i, among the neighbors of t (u), suppose that t_1 (u_1) is closest to v_i and t_{2l} (u_{2m}) is closest to v_{i+1}. Then the lengths of t_1v_i and $t_{2l}v_{i+1}$-subpaths of S_i are shorter than the lengths of u_1v_i and $u_{2m}v_{i+1}$-subpaths of S_i respectively.*

Proof. Suppose in sector S_k, u has a neighbor but t does not. By Lemma 5.3, nodes t and u must have an even number of neighbors in the odd numbered sectors of (H, v) and no neighbors in the even numbered sectors. Consequently, S_k must be an odd numbered sector. Now consider the wheel (H, t). It follows that node u has a neighbor in an even numbered sector of (H, t), thereby contradicting Lemma 5.3. By the same argument, the latter part of the lemma also follows. □

We next examine the structure of nodes in A_+. By symmetry we have the following.

Remark 5.2. All the nodes in A_+ have a common neighbor in H.

We now study the structure of nodes in $A_+(v)$.

Lemma 5.5. *A node $u \in A_+(v)$ has only one neighbor u^* in a sector S_i of (H, v) having v_i and v_{i+1} as end nodes; node u^* is adjacent to one end node say v_i, of S_i. Node u has an even number of neighbors $u_1, u_2, \ldots, u_{2m}$, in sector S_{i-1} having v_{i-1} and v_i as end nodes. No other sector of (H, v) contains a neighbor of u.*

Proof. As a consequence of the minimality property of H, node u has an odd number of neighbors in H. If u has an odd number of neighbors in any sector of (H, v), then this number is 1. Furthermore, since $u \in V^+$, no neighbor of u can belong to two sectors. We now divide the proof of the lemma into the following three claims.

Claim 1. *No two sectors of (H, v) can contain a positive even number of neighbors of u.*

Proof. Suppose u has neighbors u_i^j, $j = 1, 2, \ldots, 2l$ in sector S_i and neighbors u_k^j, $j = 1, 2, \ldots, 2m$ in sector S_k. Let v_{i-1} and v_i be the end nodes of S_i with u_i^1 closest to v_{i-1} and u_i^{2l} closest to v_i. Similarly, let v_{k-1} and v_k be the end nodes of S_k with u_k^1 closest to v_{k-1} and u_k^{2m} closest to v_k; see Fig. 9. We can assume without loss of generality that $v_{i-1} \neq v_{k-1}, v_k$ and $v_i \neq v_k$. Let P_i (P_k) be the $v_{i-1}u_i^1$-subpath $(v_k u_k^{2m}$-subpath) of S_i (S_k). Now the two uv-subpaths in the even hole $C^* = v, v_{i-1}, P_i, u_i^1, u, u_k^{2m}, P_k, v_k, v$ are such that one of them is of length 1 mod 4 and the other is of length 3 mod 4. Let Q_i be the $u_i^{2l}v_i$-subpath of S_i. Then the path $P = u_i^{2l}, Q_i, v_i, v$ closes an odd hole with one of the two uv-subpaths in C^*. It is easily verified that this odd hole is of smaller cardinality than H. Hence the claim follows. □

Claim 2. *No two sectors of* (H, v) *contain exactly one neighbor of* u.

Proof. Suppose node u has exactly one neighbor u_i in S_i and u_k in S_k. Since u has an odd number of neighbors in H, it must have exactly one neighbor u_l in another sector S_l, $l \neq i, k$. Let v_{i-1} and v_i be the end nodes of S_i, v_{k-1} and v_k be the end nodes of S_k, v_{l-1} and v_l be the end nodes of S_l; see Fig. 10. Since $u \in V^+$, the pair of $u_i v_{i-1}$ and $u_i v_i$-subpaths of S_i are both of length 1 mod 4 or 3 mod 4. Similarly, the pair of $u_k v_{k-1}$ $(u_l v_{l-1})$ and $u_k v_k$ $(u_l v_l)$-subpaths of S_k (S_l) are both of length 1 mod 4 or 3 mod 4. Now considering the three pairs of paths in the three sectors S_i, S_k and S_l, at least two of the pairs of paths must be such that the length of each of the four paths (constituting the two pairs of paths) is 1 mod 4 or 3 mod 4. Without loss of generality, we can assume that the $u_i v_{i-1}$ and $u_i v_i$-subpaths of S_i and the $u_k v_{k-1}$ and $u_k v_k$-subpaths of S_k are each of length 1 mod 4 (3 mod 4). Again without loss of generality, we assume that $v_{i-1} \neq v_{k-1}, v_k$ and $v_i \neq v_k$. Now, let P_i (P_k) be the $u_i v_{i-1}$-subpath $(u_k v_k$-subpath) of S_i (S_k). Consider now the odd hole $C^* = u, u_i, P_i, v_{i-1}, v, v_k, P_k, u_k, u$. A simple counting argument shows that C^* is of smaller cardinality than H. Hence the claim follows. □

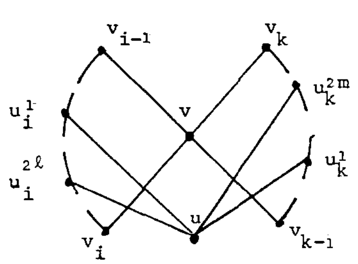

Fig. 9.

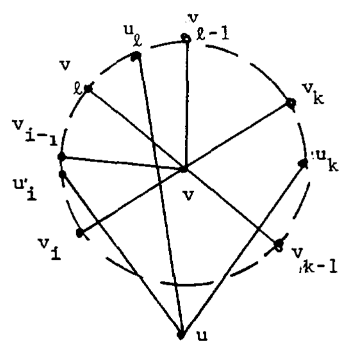

Fig. 10.

As a consequence of Claims 1 and 2, node u must have a positive even number of neighbors in exactly one sector and only one neighbor in exactly one other sector. Let S_l be the sector of (H, v) in which node u has a positive even number of neighbors and S_i be the sector containing the only neighbor u^* of node u.

Claim 3. *S_i and S_l are adjacent sectors and u^* is adjacent to the common node between these two sectors.*

Proof. Let $u_1, u_2, \ldots, u_{2m}$ be the neighbors of u in S_l, as depicted in Fig. 11. Without loss of generality we assume that $v_l \neq v_i$, v_{i+1} and $v_{k+1} \neq v_{i+1}$. Then if S_1 and S_l are adjacent, nodes v_{l+1} and v_i coincide. Suppose the contrary. Let P_1 and P_2 be the u_1v_l and $u_{2m}v_{l+1}$-subpaths of S_l respectively. Let Q_1 and Q_2 be the u^*v_{i+1} and u^*v_i-subpaths of S_i respectively.

Consider now the chordless cycle $C^* = u, u_1, P_1, v_l, v, v_{i+1}, Q_1, u^*, u$. This cycle must be even for otherwise it would be an odd hole of smaller cardinality than H. Consequently, the two uv-subpaths in C^*,

$$R_1 = u, u_1, P_1, v_l, v \quad \text{and} \quad R_2 = u, u^*, Q_1, v_{i+1}, v$$

must be such that one of them is of length 1 mod 4 and the other is of length 3 mod 4. Note that both R_1 and R_2 have length $\geqslant 3$. If $v_{l+1} \neq v_i$ or u^* is not a neighbor of v_i, then the uv-path $P_3 = u, u_{2m}, P_2, v_{l+1}, v$ closes an odd hole with either R_1 or R_2. It is easily verified that this odd hole is of smaller cardinality than H. This completes the proof of the lemma. □

Remark 5.3. Let $u \in A_+(v)$. Then by Lemma 5.5, node u has only one neighbor, say u^*, in sector S_i with end nodes v_i and v_{i+1}. Node u has an even number of neighbors, say u_i, $i = 1, 2, \ldots, 2m$, in sector S_{i-1} with end nodes v_{i-1} and v_i. Let u_i (u_{2m}) be the neighbor of u closest to v_i (v_{i-1}) in S_{i-1}. Then u^* is a neighbor of v_i. The length of the v_iu_i-subpath of S_{i-1} is 1 mod 4 and the length of the $v_{i-1}u_{2m}$-subpath of S_{i-1} is 3 mod 4. For $j = 1, 2, \ldots, 2m-1$, the length of the u_ju_{j+1}-subpath of S_{i-1} is 2 mod 4.

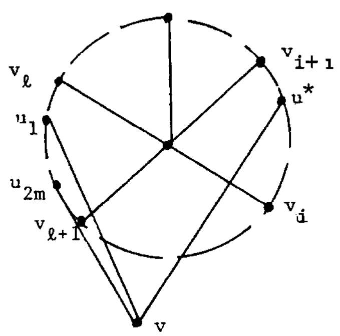

Fig. 11.

Lemma 5.6. *All the nodes in $A_+(v)$ have a common neighbor $u^* \in H$ adjacent to node v_i which is an end node of sector S_i of (H, v). Moreover, nodes in $A_+(v)$ have a positive even number of neighbors in a sector, say S_{i-1}, adjacent to S_i with node v_i as an end node and no neighbors in the other sectors.*

Proof. If $|A_+(v)| = 1$ clearly the lemma follows by Lemma 5.5. Suppose now $|A_+(v)| \geq 2$ and the lemma is false. Let $u, t \in A_+(v)$ be two nodes that do not satisfy Lemma 5.6. By Lemma 5.5, u has one neighbor u^* in S_i having v_i and v_{i+1} as end nodes and an even number of neighbors $u_1, u_2, \ldots, u_{2m}$ in S_{i-1} having v_{i-1} and v_i as end nodes. Similarly, let t have one neighbor t^* in a sector, say S_k, having v_k and v_{k+1} as end nodes and an even number of neighbors $t_1, t_2, \ldots, t_{2l}$ in say S_j where $j = k - 1$ or $k + 1$. Note that if $j = k - 1$, nodes v_{k-1} and v_k are the end nodes of S_j and *if* $j = k + 1$, nodes v_{k+1} and v_{k+2} are the end nodes of S_j. In order to prove the lemma, we have to show that $k = i$ and $j = k - 1$. Suppose not. Since u and t have a common neighbor in H, at least one sector of (H, v) must contain a neighbor of both u and t. There are two cases to consider.

Case 1: *Two sectors contain all the neighbors of u and t.*

In this case if $k = i$ then $j = k - 1$ and the lemma is proven. Suppose now $k = i - 1$. Then $j = k + 1 = i$, see Fig. 12. But this contradicts Remark 5.2 which states that u and t have a common neighbor in H.

Case 2: *Exactly one sector contains one or more neighbors of both u and t.*

Suppose now u^* and t^* are in the same sector. Now since u and t must have a common neighbor in H, it follows that u^* and t^* must coincide, see Fig. 13. But this is impossible since S_{i+1} must have length greater than or equal to 6. Hence u^* and t^* must be in different sectors. Suppose now t^* is in the same sector that contains the nodes u_q, $q = 1, 2, \ldots, 2m$. Now u_{2m} and t^* must coincide, see Fig. 14. But this contradicts Remark 5.3. Hence we must have that the nodes u_q, $q = 1, 2, \ldots, 2m$ and the nodes t_q, $q = 1, 2, \ldots, 2l$ must be in the same sector. In this case $j = k + 1 = i - 1$; see Fig. 15. Note that if (H, v) has only three sectors, then v_{i-2} coincides with v_{i+1}. Let P be the u^*t^*-subpath of H not

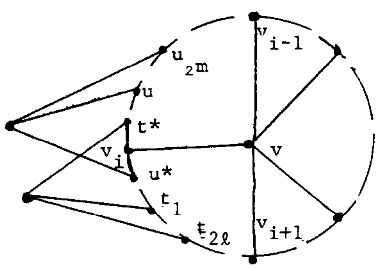

Fig. 12.

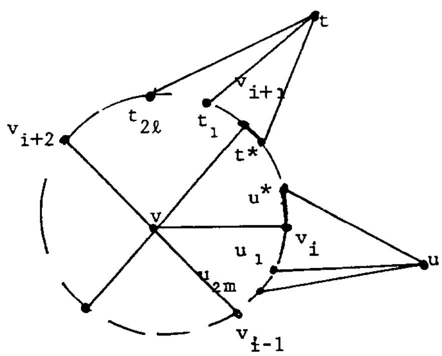

Fig. 13.

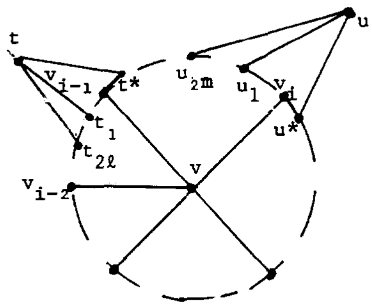

Fig. 14.

containing any neighbor of u or t. Now since u and t must have a common neighbor in H, it follows that one and only one of the nodes u_q, $q = 1, 2, \ldots, 2m$ must coincide with one of the nodes t_q, $q = 1, 2, \ldots, 2l$. Suppose u_r coincides with t_w. Now the cycle u, u^*, P, t^*, t, u_r, u is an odd hole of smaller cardinality than H. This completes the proof of the lemma. □

Next we consider a minimal odd hole H with $|A_-| \geqslant 2$ and $|A_+| \geqslant 1$. Then we have the following.

Lemma 5.7. *Let H be a minimal odd hole with $|A_-| \geqslant 2$. Let v_1 be the node of H adjacent to all the nodes in A_-. Then a node in A_+ is not a neighbor of any of the nodes in A_-. Hence all the nodes in A_+ are adjacent to a node u^* in H. Moreover, nodes v_1 and u^* are adjacent in H.*

Proof. Let v and t be any two nodes in A_-. Node v_1 is a neighbor of both v and t. Consider a node $u \in A_+$. We want to show that u is not a neighbor of v or t.

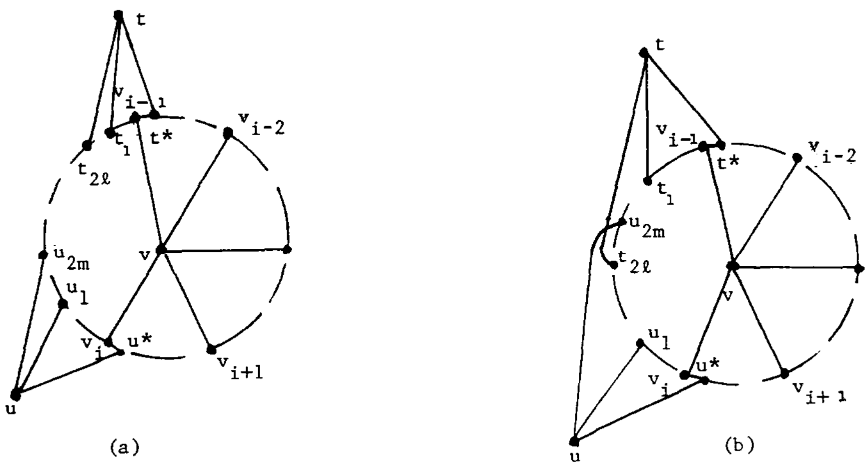

Fig. 15.

First note that u cannot be a neighbor of both v and t for otherwise v, v_1, t, u, v would be a cycle of length 4. Without loss of generality assume that u is a neighbor of t only. Then by Lemma 5.5, u has a neighbor u^* in H, adjacent to node v_i which is an end node of a sector S_i of (H, v). Now if v_i and v_1 coincide, we have v_1, t, u, u^*, v_1 as a cycle of length 4. If v_i and v_1 do not coincide we have v, v_1, t, u, u^*, v_i, v as an odd hole of length 6, contradicting the minimality of H. Consequently, a node in A_+ is not a neighbor of any of the nodes in A_-. Now by Lemma 5.6 all the nodes in A_+ are adjacent to a node u^* in H.

Suppose now nodes v_1 and u^* are not adjacent in H. Let v be a node in A_- that satisfies Property 5.1. Suppose u is a node in A_+ with only one neighbor u^* in S_i and the other neighbors $u_1, u_2, \ldots, u_{2m}$ in S_{i-1}. Suppose v_{i-1} and v_i (v_i and v_{i+1}) are the end nodes of S_{i-1} (S_i). Note that u^* is adjacent to v_i. Let t be another node in A_-. By Corollary 5.1, node t has neighbors in both S_i and S_{i-1} if and only if u^* and v_1 are adjacent, i.e., $v_i = v_1$. Suppose now t has neighbors in at most one sector among S_i and S_{i-1}. We thus have two cases to consider.

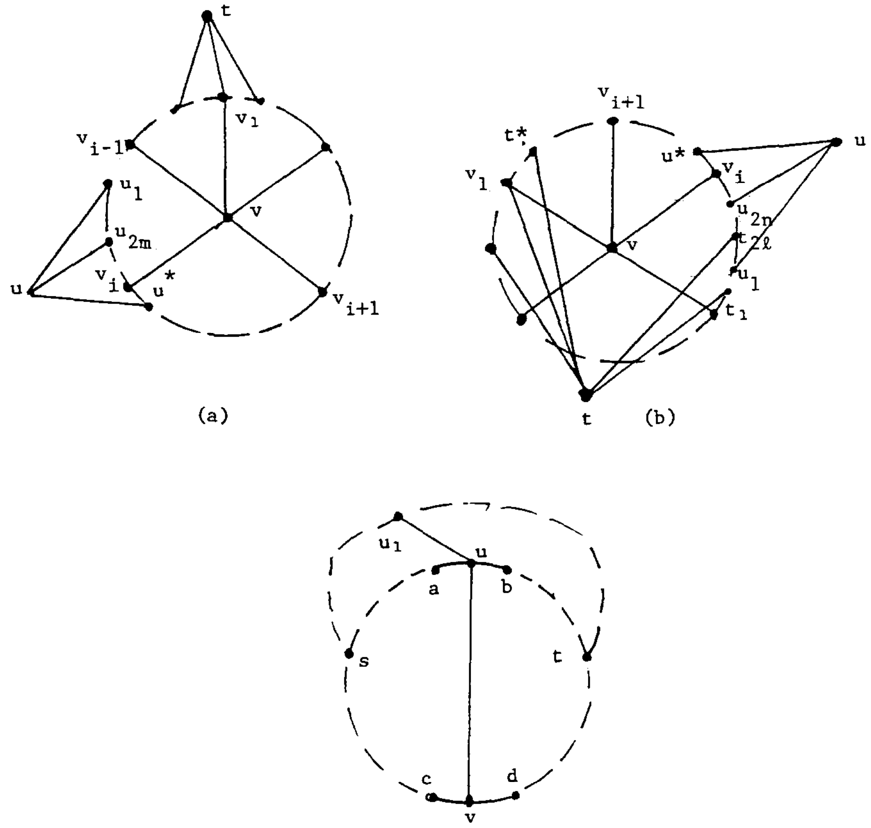

Fig. 16.

Case 1: *Node t has no neighbors in* S_i, S_{i-1}; see Fig. 16(a).

Let P be the u^*u_1-subpath of H not containing an intermediate node which is a neighbor of u. Consider the chordless cycle $C = u, u^*, P, u_1, u$. By Remark 5.3, C must be an even hole. But then t is a node strongly adjacent to C and having an odd number greater than or equal to three neighbors in C. Consequently, the wheel (C, t) must contain an odd hole of smaller cardinality than H.

Case 2: *Node t has neighbors in exactly one of the sectors* S_i *or* S_{i-1}.

If node t has neighbors in S_i, the proof is the same as in Case 1. So we assume that t has neighbors in S_{i-1}; see Fig. 16(b). Now consider the wheel (H, t). Since $u \in A_+(t)$, by Lemma 5.5 it follows that a neighbor of u and a neighbor of t must be adjacent in S_{i-1} (defined with respect to the wheel (H, v)). Let u_j and t_k be the neighbors of u and t respectively such that u_j and t_k are adjacent in S_{i-1}. Let t^* be the first neighbor of t encountered when traversing H from u^* away from v_i. Note that by Remark 5.1 the u^*t^*-subpath P in H not containing v_i has length 1 mod 4. Now the cycle $C = u, u^*, P, t^*, t, t_k, u_j, u$ is an odd hole of smaller cardinality than H. This completes the proof of the lemma. □

Lemma 5.8. *Let H be a minimal odd hole, v be the node in* A_- *that satisfies Property* 5.1. *Suppose* v_1 *is the node of H adjacent to all the nodes in* A_-. *Consider a node u in* A_+ *and a node* $t \in A_- \setminus \{v\}$. *In sector* S_1 (S_{2n+1}) *of* (H, v), *all the neighbors of u are closer to* v_1 *than any neighbor of t.*

Proof. Suppose the lemma is false. By Lemma 5.7, node u is not adjacent to v or t. Then consider the wheel (H, t). Node $u \in A_+(t)$ contradicts Lemma 5.7. □

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