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Existence of a global attractor for the parabolic equation with nonlinear Laplacian principal part in an unbounded domain

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Abstract

In this paper, we study the long-time behavior of solutions for the parabolic equation with nonlinear Laplacian principal part in R^n. We prove the existence of a global $(L_2(R^n), L_\infty(R^n))$-attractor when $n \leqslant p$ and the existence of a global $(L_2(R^n), L_{np/(n-p)}(R^n))$-attractor when $n > p$.

Keywords: Global attractors; Asymptotic compactness

1. Introduction

The subject of investigation of this paper is the existence of a global attractor for the following initial-value problem:

$$u_t - \sum_{i=1}^{n} \frac{\partial}{\partial x_i}\left(|u_{x_i}|^{p-2}u_{x_i}\right) + \lambda|u|^{p-2}u + f(x,u) = g(x), \quad (t,x) \in R_+ \times R^n, \quad (1)$$

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$$u(0,x)=u_0(x), \quad x\in R^n, \tag{2}$$

where $p>2$, $\lambda>0$, $g(\cdot)\in L_{q_0}(R^n)$,

$$q_0\in\begin{cases}\left[\frac{np}{np-n+p},p'\right], & n>p,\\ (1,p'], & n=p,\\ [1,p'], & n<p,\end{cases}\qquad \left(\frac{1}{p'}+\frac{1}{p}=1\right)$$

and $f(\cdot,\cdot)$ satisfies the following conditions:

$$f(\cdot,\cdot)\in C^1\big(R^{n+1}\big),\qquad f(\cdot,0)=0,\qquad f'_u(x,u)\geqslant k(x),\quad \forall(x,u)\in R^{n+1}, \tag{3}$$

$$\big|f'_u(x,u)\big|\leqslant c\big(\big|k(x)\big|+|u|^{p-2}+|u|^r\big),$$
$$r\geqslant p-2,\ (n-p)r\leqslant n(p-2)+2p, \tag{4}$$

for some $k(\cdot)\in L_\infty(R^n)\cap L_{p/(p-2)}(R^n)$.

The existence of a global attractor for Eq. (1) in a bounded domain, when $\lambda=0$, $g(\cdot)\equiv 0$ and $f=ku$, was studied in [1, p. 158]. Attractors in bounded domains for degenerate parabolic equations with p-Laplacian and for porous medium type equations were investigated in [2–7] and the references therein. In bounded domains, the asymptotic compactness of the solutions—which plays an important role for the existence of a global attractor—follows from the compactness of the Sobolev embeddings. This method cannot be applied to unbounded domains, since in that case the embeddings are no longer compact.

In [8–10] global attractors for abstract evolution equations with monotone principal part were studied. The results of these articles are also well applicable to bounded domains, because either compact embeddings of spaces or the compactness of the semigroup was assumed in the said articles.

The existence of global attractors for nondegenerate parabolic equations in unbounded domains was investigated in [11,12] and has been established in weighted spaces. Later, these results were extended for degenerate parabolic equations in unbounded domains (see, for example, [13,14]). However, when studying in weighted spaces the initial data and forcing term are usually assumed to be in corresponding weighted spaces. The idea of using weights to prove the existence of global attractors in unbounded domains is also used in [15] for nondegenerate parabolic equations. Some authors on the other hand, have preferred studying attractors in spaces of bounded continuous functions (see, for example, [16,17]).

The existence of the global attractor in $L_2(R^n)$ for single nondegenerate reaction diffusion equation with forcing term from $L_2(R^n)$ was—to our knowledge—first proved in [18]. In that article the author used a suitable cut-off function for the proof (see also [19]).

In this paper, using a similar idea, we will show that the solutions of (1)–(2) are uniformly small at infinity for large time. This fact plays a key role in our result.

The paper is organized as follows: In Section 2, we derive some estimates and prove some lemmas which will be used for the proof of the asymptotic compactness. In Section 3, we present the proof of the asymptotic compactness and then establish our main result (Theorem 2).

2. Preliminaries

We denote the norms in $W_p^1(R^n)$ and $L_p(R^n)$ by $\|\cdot\|_{1,p}$ and $\|\cdot\|_p$, respectively, and the inner product in $L_2(R^n)$ by $\langle,\rangle$. We also define the operator

$$A\varphi = -\sum_{i=1}^{n} \frac{\partial}{\partial x_i}\left(|\varphi_{x_i}|^{p-2}\varphi_{x_i}\right) + \lambda|\varphi|^{p-2}\varphi + f(x,\varphi) - k(x)\varphi + g$$

acting from $L_p(0,T;W_p^1(R^n))$ to $L_{p'}(0,T;W_{p'}^{-1}(R^n))$. Then we can reduce problem (1)–(2) to the problem:

$$u_t + Au + ku = 0, \qquad u(0) = u_0. \tag{5}$$

It is easy to verify that A is bounded, hemicontinuous, monotone, and $A + kI$ is a pseudomonotone operator from $L_p(0,T;W_p^1(R^n))$ to $L_{p'}(0,T;W_{p'}^{-1}(R^n))$. Since

$$W_p^1(R^n) \cap W_{p'}^{-1}(R^n) \subset L_2(R^n) \subset W_p^1(R^n) + W_{p'}^{-1}(R^n),$$

for $u_0 \in L_2(R^n)$ there exist $w(\cdot) \in L_p(0,T;W_p^1(R^n) \cap W_{p'}^{-1}(R^n))$, $w_t(\cdot) \in L_{p'}(0,T;W_p^1(R^n) + W_{p'}^{-1}(R^n))$ such that $w(0) = u_0$. Let $w_t(t) = w_1(t) + w_2(t)$, where $w_1(\cdot) \in L_{p'}(0,T;W_p^1(R^n))$ and $w_2(\cdot) \in L_{p'}(0,T;W_{p'}^{-1}(R^n))$. Defining $v(\cdot) = w(t) - \int_0^t w_1(\tau)\,d\tau$, we obtain that $v(\cdot) \in L_p(0,T;W_p^1(R^n)$, $v_t(\cdot) \in L_{p'}(0,T;W_{p'}^{-1}(R^n))$ and $v(0) = u_0$. Thus for every $u_0 \in L_2(R^n)$ the problem (5) or (1)–(2) under condition (3)–(4) has a unique solution $u \in L_\infty(0,T;L_2(R^n)) \cap L_p(0,T;W_p^1(R^n))$, $u_t \in L_{p'}(0,T;W_{p'}^{-1}(R^n))$ (see, for example, [20, Theorem 7.1, p. 232]), which satisfies the following equality:

$$\frac{1}{2}\|u(t)\|_2^2 + \int_s^t \left[\sum_{i=1}^{n}\|u_{x_i}(\tau)\|_p^p + \lambda\|u(\tau)\|_p^p + \int_{R^n}\left(f\left(x,u(\tau)\right) - g\right)u(\tau)\,dx\right]d\tau$$
$$= \frac{1}{2}\|u(s)\|_2^2, \quad \forall t \geqslant s \geqslant 0, \tag{6}$$

and for every $u_0 \in W_p^1(R^n)$ the problem (1)–(2) under condition (3)–(4) has a unique solution $u \in L_\infty(0,T;W_p^1(R^n))$, $u_t \in L_2(0,T;L_2(R^n))$, which satisfies the following inequality:

$$\frac{1}{p}\sum_{i=1}^{n}\|u_{x_i}(t)\|_p^p + \frac{\lambda}{p}\|u(t)\|_p^p + \int_{R^n}\left(F\left(x,u(t)\right) - gu(t)\right)dx + \int_s^t \|u_t(\tau)\|_2^2\,d\tau$$
$$\leqslant \frac{1}{p}\sum_{i=1}^{n}\|u_{x_i}(s)\|_p^p + \frac{\lambda}{p}\|u(s)\|_p^p + \int_{R^n}\left(F\left(x,u(s)\right) - gu(s)\right)dx, \quad \forall t \geqslant s \geqslant 0, \tag{7}$$

where $F(x,u) = \int_0^u f(x,v)\,dv$. Taking into account (3)–(4) from (6)–(7), we obtain that

$$\|u(t)\|_{1,p} \leqslant \frac{1}{t-s}c\left(\|u(s)\|_2\right) + \mathcal{R}_0, \quad \forall t > s \geqslant 0, \tag{8}$$

where $c(\cdot)$ is a monotone increasing function and $\mathcal{R}_0$ is a constant, which depends on λ, k and g. Formally differentiating (1) with respect to t, then multiplying by u_t and integrating over $(0,t)\times R^n$, we have

$$\frac{1}{2}\|u_t(t)\|_2^2+(p-1)\sum_{i=1}^n\int_s^t\int_{R^n}|u_{x_i}|^{p-2}u_{tx_i}^2\,dx\,d\tau+\lambda(p-1)\int_s^t\int_{R^n}|u|^{p-2}u_t^2\,dx\,d\tau$$
$$+\int_s^t\int_{R^n}f_u'(x,u)u_t^2\,dx\,d\tau\leqslant\frac{1}{2}\|u_t(s)\|_2^2,\quad \text{a.e. } t\geqslant s>0, \tag{9}$$

which together with (7) yields

$$\|u_t(t)\|_2\leqslant\left(1+\frac{1}{t-s}\right)c\big(\|u(s)\|_{1,p}\big),\quad \text{a.e. } t>s\geqslant 0. \tag{10}$$

So from (6)–(7) we obtain that the solution operator $S(t)u_0=u(t)$, $t\in R_+$, of the problem (1)–(2) generates a semigroup on the spaces $L_2(R^n)$ and $W_p^1(R^n)$, which satisfies the following properties:

(I) $S(t):L_2(R^n)\to L_2(R^n)$ for every $t\geqslant 0$ ($S(t):W_p^1(R^n)\to W_p^1(R^n)$ for every $t\geqslant 0$) and $S(t):L_2(R^n)\to W_p^1(R^n)$ for every $t>0$; $S(0)v=v$ for every $v\in L_2(R^n)$ or $v\in W_p^1(R^n)$.
(II) $S(t+s)=S(t)S(s)$ for $t\geqslant 0$, $s\geqslant 0$.
(III) $S(t)v\to S(t_0)v$ weakly in $L_2(R^n)\cap W_p^1(R^n)$ (in $W_p^1(R^n)$) as $t\to t_0$ for every $v\in L_2(R^n)$ ($v\in W_p^1(R^n)$).

Let $\mathfrak{B}=\{x\colon\ x\in W_p^1(R^n),\ \|x\|_{1,p}\leqslant\mathcal{R}_0+1\}$. Then the following lemma follows from (8).

Lemma 1. *Let us assume the conditions* (3)–(4) *are satisfied and B is a bounded subset of $L_2(R^n)$. Then there exists $t_0=t_0(B)$ such that $S(t)B\subset\mathfrak{B}$ for every $t\geqslant t_0$.*

Definition. A set $\mathcal{A}\subset W_p^1(R^n)$ is called a global $(L_2(R^n), L_q(R^n))$-attractor of the semigroup $S(t)$, if it has the following properties:

(1) $\mathcal{A}$ is compact in $L_q(R^n)$ and is bounded in $W_p^1(R^n)$.
(2) $S(t)\mathcal{A}=\mathcal{A}$ for every $t\geqslant 0$.
(3) $\lim_{t\to+\infty}\sup_{v\in B}\inf_{u\in\mathcal{A}}\|S(t)v-u\|_q=0$ for every bounded subset B of $L_2(R^n)$.

To prove the asymptotic compactness of solutions, we will need the following lemma.

Lemma 2. *Let us assume that conditions* (3)–(4) *are satisfied. If $u_0^m\to u_0$ weakly in $W_p^1(R^n)$ as $m\to\infty$, then*

$$S(t)u_0^m\to S(t)u_0\qquad *\text{-weakly in } L_\infty\big(0,T;W_p^1\big(R^n\big)\big),$$

$$\frac{\partial}{\partial t}S(t)u_0^m \to \frac{\partial}{\partial t}S(t)u_0 \quad \text{weakly in } L_2\big(0,T;L_2\big(R^n\big)\big),$$
$$S(t)u_0^m \to S(t)u_0 \quad \text{weakly in } W_p^1\big(R^n\big) \text{ for every } t \geqslant 0.$$

Proof. Let $u_m(t) = S(t)u_0^m$ and $u(t) = S(t)u_0$. Since $u_0^m \to u_0$ weakly in $W_p^1(R^n)$, the sequence $\{u_0^m\}$ is bounded in $W_p^1(R^n)$. Thus from (3), (4), and (7), the sequences $\{u_m(t)\}$ and $\{\frac{\partial}{\partial t}u_m(t)\}$ are bounded in $L_\infty(0,T;W_p^1(R^n))$ and in $L_2(0,T;L_2(R^n))$, respectively. Consequently, there is a subsequence $\{m_k\}$ such that

$$\begin{cases} u_{m_k}(t) \to v(t) & *\text{-weakly in } L_\infty(0,T;W_p^1(R^n)),\\ \frac{\partial}{\partial t}u_{m_k}(t) \to \frac{\partial}{\partial t}v(t) & \text{weakly in } L_2(0,T;L_2(R^n)),\\ Au_{m_k}(t) \to \chi(t) & \text{weakly in } L_{p'}(0,T;W_{p'}^{-1}(R^n)) \cap L_2(0,T;L_2(R^n)). \end{cases} \tag{11}$$

Let us show that $Av = \chi$. Let $\varphi(\cdot) \in C^\infty(R^n)$ be such that $0 \leqslant \varphi(x) \leqslant 1$ and

$$\varphi(x) = \begin{cases} 0, & |x| \geqslant 2,\\ 1, & |x| \leqslant 1, \end{cases}$$

furthermore define $\varphi_r(x) = \varphi(\frac{x}{r})$. Then from $(11)_1$–$(11)_2$ we have

$$\varphi_r u_{m_k}(t) \to \varphi_r v(t) \quad \text{strongly in } L_2\big(0,T;L_2\big(R^n\big)\big). \tag{12}$$

Thus from (11)–(12) we obtain

$$\begin{aligned} &\lim_{k\to\infty}\int_0^T \big\langle Au_{m_k}(t) - Av(t), \varphi_r u_{m_k}(t) - \varphi_r v(t)\big\rangle\,dt\\ &\quad = \lim_{k\to\infty}\int_0^T \big\langle Au_{m_k}(t), \varphi_r u_{m_k}(t)\big\rangle\,dt - \int_0^T \big\langle \chi(t), \varphi_r v(t)\big\rangle\,dt\\ &\quad = \lim_{k\to\infty}\int_0^T \big\langle Au_{m_k}(t) - \chi(t), \varphi_r u_{m_k}(t) - \varphi_r v(t)\big\rangle\,dt = 0. \end{aligned} \tag{13}$$

On the other hand, since

$$\begin{aligned} &\int_0^T \big\langle Au_{m_k}(t) - Av(t), \varphi_r u_{m_k}(t) - \varphi_r v(t)\big\rangle\,dt\\ &\quad \geqslant c\int_0^T \big\|\varphi_r^{1/p}\big(u_{m_k}(t) - v(t)\big)\big\|_{1,p}^p\,dt\\ &\qquad + \sum_{i=1}^n \int_0^T \left\langle \left|\frac{\partial}{\partial x_i}u_{m_k}\right|^{p-2}\frac{\partial}{\partial x_i}u_{m_k} - \left|\frac{\partial}{\partial x_i}v\right|^{p-2}\frac{\partial}{\partial x_i}v, \big(u_{m_k}(t) - v(t)\big)\frac{\partial}{\partial x_i}\varphi_r\right\rangle dt, \end{aligned}$$

taking into account (11) and (13) in the last inequality, we get

$$u_{m_k}(t) \to v(t) \quad \text{strongly in } L_p\big(0, T; W_p^1\big(B(0, r)\big)\big)$$

and consequently

$$Au_{m_k}(t) \to Av(t) \quad \text{strongly in } L_{p'}\big(0, T; W_{p'}^{-1}\big(B(0, r)\big)\big), \tag{14}$$

where $B(0,r) = \{x\colon\ x \in R^n,\ |x| \leqslant r\}$. Since r is an arbitrary positive number, $(11)_3$ and (14) yield $Av(t) = \chi(t)$. Therefore passing to the limit and taking into account (11), we have

$$v_t + Av + kv = 0, \qquad v(0) = u_0$$

and by the uniqueness of solutions, we find $u(t) = v(t)$. This shows that any subsequence of $\{S(t)u_0^m\}$ has a $*$-weakly convergent subsequence in $L_\infty(0, T; W_p^1(R^n))$ and the limit of any such subsequence is equal to $S(t)u_0$. Therefore, the sequence $\{S(t)u_0^m\}$ $*$-weakly converges to $S(t)u_0$ in $L_\infty(0, T; W_p^1(R^n))$ and the sequence $\{\frac{\partial}{\partial t}S(t)u_0^m\}$ weakly converges to $\{\frac{\partial}{\partial t}S(t)u_0\}$ in $L_2(0, T; L_2(R^n))$, which yield that $S(t)u_0^m \to S(t)u_0$ weakly in $L_2(R^n) + W_p^1(R^n)$ for every $t \in [0, T]$. On the other hand, according to (7), for every $t \in [0, T]$ the sequence $\{S(t)u_0^m\}$ is bounded in $W_p^1(R^n)$. Thus we obtain $S(t)u_0^m \to S(t)u_0$ weakly in $W_p^1(R^n)$ for every $t \in [0, T]$. □

3. Asymptotic compactness and a global attractor

In this section, we shall show the asymptotic compactness of solutions and then establish existence of a global attractor. To this end, we first prove the following lemmas.

Lemma 3. *Assume the conditions* (3)–(4) *are satisfied, and B is a bounded subset of $W_p^1(R^n)$. If $\{\theta_m\}$ is a sequence in B weakly converging to θ in $W_p^1(R^n)$, then for any $\varepsilon > 0$ and $T > 0$ there exists a $T_0 = T_0(\varepsilon, T, B)$ such that whenever $t \geqslant T_0$,*

$$\limsup_{m\to\infty} \big\| S(t+T)\theta_m - S(t)\theta_m - S(t+T)\theta - S(t)\theta \big\|_2 \leqslant \varepsilon. \tag{15}$$

Proof. Let $v \in W_p^1(R^n)$. From (7) we have

$$\int_0^t \big\| S(\tau+T)v - S(\tau)v \big\|_2^2 \, d\tau \leqslant T^2 \int_0^t \int_0^1 \left\| \frac{\partial}{\partial t} S(\tau + \xi T)v \right\|_2^2 d\xi\, d\tau \leqslant T^2 c\big(\|v\|_{1,p}\big). \tag{16}$$

From (5) we obtain

$$\frac{1}{2}\big\| S(t+T)v - S(t)v \big\|_2^2 + \int_s^t \big\langle AS(\tau+T)v - AS(\tau)v, S(\tau+T)v - S(\tau)v \big\rangle \, d\tau$$

$$+\int_s^t \big\langle k\big(S(\tau+T)v - S(\tau)v\big), S(\tau+T)v - S(\tau)v\big\rangle\, d\tau$$
$$= \frac{1}{2}\big\|S(s+T)v - S(s)v\big\|_2^2 \tag{17}$$

for every $t \geqslant s \geqslant 0$ and $v \in W_p^1(R^n)$. Using (16) and (17), we find

$$\frac{t}{2}\big\|S(t+T)\theta_m - S(t)\theta_m\big\|_2^2$$
$$+ \int_0^t \int_s^t \big\langle AS(\tau+T)\theta_m - AS(\tau)\theta_m, S(\tau+T)\theta_m - S(\tau)\theta_m\big\rangle\, d\tau\, ds$$
$$+ \int_0^t \int_s^t \big\langle k\big(S(\tau+T)\theta_m - S(\tau)\theta_m\big), S(\tau+T)\theta_m - S(\tau)\theta_m\big\rangle\, d\tau\, ds$$
$$\leqslant \frac{T^2}{2} c\big(\|B\|_{1,p}\big), \tag{18}$$

where $t \geqslant s \geqslant 0$ and $\|B\|_{1,p} = \sup_{v\in B} \|v\|_{1,p}$.

By Lemma 2 and compact embedding theorems, we have

$$\lim_{m\to\infty} \int_s^t \big\langle AS(\tau)\theta_m - AS(\tau)\theta, \varphi_r\big(S(\tau)\theta_m - S(\tau)\theta\big)\big\rangle\, d\tau = 0$$

and consequently

$$\lim_{m\to\infty} \int_s^t \big\|S(\tau)\theta_m - S(\tau)\theta\big\|^p_{W_p^1(B(0,r))}\, d\tau = 0, \tag{19}$$

which yields

$$\liminf_{m\to\infty} \int_0^t \int_s^t \big(AS(\tau+T)\theta_m - AS(\tau)\theta_m, S(\tau+T)\theta_m - S(\tau)\theta_m\big)\, d\tau\, ds$$
$$\geqslant \int_0^t \int_s^t \big\langle AS(\tau+T)\theta - AS(\tau)\theta, S(\tau+T)\theta - S(\tau)\theta\big\rangle\, d\tau\, ds. \tag{20}$$

As since as $k(\cdot) \in L_\infty(R^n)$, from (16) and (17) we have

$$\int_s^t \big\|S(\tau+T)v - S(\tau)v\big\|^p_{1,p}\, d\tau \leqslant T^2 c\big(\|B\|_{1,p}\big), \quad \forall v \in B,$$

and taking into account the last inequality and (19), we obtain

$$\liminf_{m\to\infty}\int_0^t\int_s^t\big\langle k\big(S(\tau+T)\theta_m-S(\tau)\theta_m\big),S(\tau+T)\theta_m-S(\tau)\theta_m\big\rangle\,d\tau\,ds$$
$$\geqslant\int_0^t\int_s^t\big\langle k\big(S(\tau+T)\theta-S(\tau)\theta\big),S(\tau+T)\theta-S(\tau)\theta\big\rangle\,d\tau\,ds. \tag{21}$$

Also, taking into account (17), (20) and (21) in (18), we have

$$\frac{t}{2}\big\|S(t+T)\theta_m-S(t)\theta_m\big\|_2^2-\frac{t}{2}\big\|S(t+T)\theta-S(t)\theta\big\|_2^2\leqslant T^2c\big(\|B\|_{1,p}\big),$$

which together with Lemma 1 yields (15). □

Lemma 4. *Assume conditions* (3) *and* (4) *hold. Let B be a bounded subset of $W_p^1(R^n)$, $t_m\to\infty$, $\{\tau_m\}_{m=1}^\infty\subset[0,T]$ and $\{\theta_m\}_{m=1}^\infty\subset B$. Then for any $\varepsilon>0$ there exists $r_0=r_0(\varepsilon,T)$ such that whenever $r\geqslant r_0$,*

$$\big\|S(t_m+\tau_m)\theta_m-S(t_m)\theta_m\big\|_{W_p^1(R^n\setminus B(0,r))}\leqslant\varepsilon. \tag{22}$$

Proof. Since B is bounded, by (7) we have $\sup_{t\geqslant0}\sup_{\theta\in B}\|S(t)\theta\|_{1,p}<\infty$. Therefore, there exists a bounded subset B_0 of $W_p^1(R^n)$ such that $S(t)\theta\in B_0$, for every $t\geqslant0$ and $\theta\in B$. Thus $\{S(t_m+\tau)\theta_m-S(t_m)\theta_m\}_{m=1}^\infty$ has a subsequence $b_k:=S(t_{m_k}+\tau)\theta_{m_k}-S(t_{m_k})\theta_{m_k}$ weakly converging in $W_p^1(R^n)$ to an $a\in W_p^1(R^n)$. From Lemma 3 we have that, if $\{\varphi_\nu\}_{\nu=1}^\infty\subset B_0$ and $\varphi_\nu\to\varphi$ weakly in $\mathcal{H}$, then for any $\varepsilon>0$ there exist a $T_0=T_0(\varepsilon,\tau,B_0)$ such that

$$\limsup_{\nu\to\infty}\big\|S(T_0+\tau)\varphi_\nu-S(T_0)\varphi_\nu-S(T_0+\tau)\varphi+S(T_0)\varphi\big\|_2\leqslant\varepsilon. \tag{23}$$

For $t_{m_k}\geqslant T_0$, since $S(t_{m_k}-T_0)\theta_{m_k}\in B_0$, there is a subsequence $\{k_\nu\}$ such that $\{S(t_{m_{k_\nu}}-T_0)\theta_{m_{k_\nu}}\}$ weakly converges to some φ in $W_p^1(R^n)$. Then by Lemma 2, the sequence $b_{k_\nu}:=\{S(T_0+\tau)S(t_{m_{k_\nu}}-T_0)\theta_{m_{k_\nu}}-S(T_0)S(t_{m_{k_\nu}}-T_0)\theta_{m_{k_\nu}}\}$ weakly converges to $S(T_0+\tau)\varphi-S(T_0)\varphi$ in $W_p^1(R^n)$. Hence from the uniqueness of the limit we find that $a=S(T_0+\tau)\varphi-S(T_0)\varphi$. Taking $\varphi_\nu=S(t_{m_{k_\nu}}-T_0)\theta_{m_{k_\nu}}$ in (23), we obtain $\limsup_{\nu\to\infty}\|b_{k_\nu}-a\|_2\leqslant\varepsilon$ and consequently $\liminf_{k\to\infty}\|b_k-a\|_2=0$. In other words, the sequence $\{S(t_m+\tau)\theta_m-S(t_m)\theta_m\}_{m=1}^\infty$ has a subsequence strongly convergent in $L_2(R^n)$ and consequently the sequence $\{S(t_m+\tau_m)\theta_m-S(t_m)\theta_m\}_{m=1}^\infty$ also has a subsequence strongly convergent in $L_2(R^n)$. It can be seen in a similar way that every subsequence of $\{S(t_m+\tau_m)\theta_m-S(t_m)\theta_m\}_{m=1}^\infty$ has a subsequence strongly convergent in $L_2(R^n)$. Thus the set $\{S(t_m+\tau_m)\theta_m-S(t_m)\theta_m\}_{m=1}^\infty$ is relatively compact in $L_2(R^n)$ and consequently for any $\varepsilon>0$ there exists $r_0=r_0(\varepsilon,T)$ such that whenever $r\geqslant r_0$,

$$\big\|S(t_m+\tau_m)\theta_m-S(t_m)\theta_m\big\|_{L_2(R^n\setminus B(0,r))}\leqslant\varepsilon \tag{24}$$

for every m. On the other hand, from (1), (9) and (10) we have that for every $v\in W_p^1(R^n)$,

$$\frac{\partial}{\partial t}S(t)v\in L_\infty\big(\delta,\infty;L_2\big(R^n\big)\big),\qquad\frac{\partial^2}{\partial t^2}S(t)v\in L_{p'}\big(0,T;W_{p'}^{-1}\big(R^n\big)\big),$$

from which follows that $(\frac{\partial}{\partial t}S(t)v, \psi)$ is continuous on $[\delta, \infty)$, for every $w \in L_2(R^n)$ and $\delta > 0$ (see, for example, [21, Lemma 8.1, p. 275]), i.e., $\frac{\partial}{\partial t}S(t)v$ has a trace in $L_2(R^n)$ for every $t > 0$. Then multiplying the equality

$$\frac{\partial}{\partial t}\big(S(t_m+\tau_m)\theta_m - S(t_m)\theta_m\big) + AS(t_m+\tau_m)\theta_m - AS(t_m)\theta_m$$
$$= kS(t_m+\tau_m)\theta_m - kS(t_m)\theta_m$$

by $(1-\varphi_r)(S(t_m+\tau_m)\theta_m - S(t_m)\theta_m)$ and taking into account (24), we get inequality (22). □

Let $S_0(t)$ denote the semigroup generated by problem

$$\tilde{u}_t - \sum_{i=1}^{n}\frac{\partial}{\partial x_i}\big(|\tilde{u}_{x_i}|^{p-2}\tilde{u}_{x_i}\big) + \lambda|\tilde{u}|^{p-2}\tilde{u} + f(x,\tilde{u}) - k(x)\tilde{u} = 0, \quad (t,x) \in R_+ \times R^n,$$
$$\tilde{u}(0,x) = u_0(x), \quad x \in R^n.$$

We now establish the estimate for $S_0(t)$.

Lemma 5. *Assume conditions* (3) *and* (4) *hold. Let B be a bounded subset of $W_p^1(R^n)$. Then*

$$\int_0^t \Big\| \big|S_0(\tau)\theta\big|^{\frac{2p-2}{p}} \Big\|_{1,p}^p \, d\tau \leqslant c\big(\|B\|_{1,p}\big), \quad \forall t \geqslant 0, \tag{25}$$

for every $\theta \in B$.

Proof. By (7), we obtain that

$$\int_0^t \left\| \frac{\partial}{\partial t}\big(S_0(\tau)\theta\big)\right\|_2^2 d\tau \leqslant c\big(\|B\|_{1,p}\big), \quad \forall t \geqslant 0, \tag{26}$$

for every $\theta \in B$. Now let us consider the following elliptic problem:

$$-\sum_{i=1}^{n}\frac{\partial}{\partial x_i}\big(|w_{x_i}|^{p-2}w_{x_i}\big) + \lambda|w|^{p-2}w + f(x,w) - k(x)w = h(x), \quad x \in R^n, \tag{27}$$

where $h \in L_2(R^n) \cap W_{p'}^{-1}(R^n)$. Let $h_m \in C_0^\infty(R^n)$, such that $h_m \to h$ in $L_2(R^n) \cap W_{p'}^{-1}(R^n)$ and let w_n is the solution of

$$-\sum_{i=1}^{n}\frac{\partial}{\partial x_i}\big(\big|(w_m)_{x_i}\big|^{p-2}(w_m)_{x_i}\big) + \lambda|w_m|^{p-2}w_m + f(x,w_m) - k(x)w_m = h_m(x). \tag{28}$$

Then $w_n \to w$ in $W_p^1(R^n)$. On the other hand, using techniques of [10,22] it is easy to show that $w_m \in L_\infty(R^n)$. So $|w_m|^{p-2}w_m \in L_2(R^n)$, and multiplying both sides of (28) by $|w_m|^{p-2}w_m$, we obtain

$$\sum_{i=1}^{n}\left\|\frac{\partial}{\partial x_i}|w_m|^{\frac{2p-2}{p}}\right\|_p^p + \lambda\left\||w_m|^{\frac{2p-2}{p}}\right\|_p^p \leqslant c\|h_m\|_2^2, \tag{29}$$

from which follows that there is a subsequence $\{w_{m_\nu}\}$ such that

$$|w_{m_\nu}|^{\frac{2p-2}{p}} \to v \quad \text{weakly in } L_p(R^n),$$
$$\frac{\partial}{\partial x_i}|w_m|^{\frac{2p-2}{p}} \to \frac{\partial}{\partial x_i}v \quad \text{weakly in } L_p(R^n),$$

which together with

$$w_n \to w \quad \text{strongly in } W_p^1(R^n)$$

yields $v = |w|^{(2p-2)/p}$, and consequently from (29) we have the following estimate for the solution of (27):

$$\sum_{i=1}^{n}\left\|\frac{\partial}{\partial x_i}|w|^{\frac{2p-2}{p}}\right\|_p^p + \lambda\left\||w|^{\frac{2p-2}{p}}\right\|_p^p \leqslant c\|h\|_2^2.$$

The last inequality and (26) give us (25). □

Lemma 6. *Assume the conditions* (3)–(4) *are satisfied, and B is a bounded subset of $W_p^1(R^n)$. Then for any $\varepsilon > 0$ there exists $r_0 = r_0(\varepsilon, B)$ such that whenever $r \geqslant r_0$ and $t > 0$,*

$$\frac{1}{t}\int_0^t \left\|S(\tau)\theta - S_0(\tau)\theta\right\|_{W_p^1(R^n\setminus B(0,r))}^p d\tau \leqslant \varepsilon \tag{30}$$

for every $\theta \in B$.

Proof. Let $\theta \in B$, $u(t) = S(t)\theta$, $\tilde{u}(t) = S_0(t)\theta$ and $v(t) = u(t) - \tilde{u}(t)$. Then multiplying both sides of equality

$$v_t + \sum_{i=1}^{n}\frac{\partial}{\partial x_i}\left(|u_{x_i}|^{p-2}u_{x_i} - |\tilde{u}_{x_i}|^{p-2}\tilde{u}_{x_i}\right) + \lambda\left(|u|^{p-2}u - |\tilde{u}|^{p-2}\tilde{u}\right)$$
$$+ f(x,u) - f(x,\tilde{u}) + k(x)\tilde{u} = g(x)$$

by $(1-\varphi_r)v$ and integrating over $(0,t)\times R^n$, we have

$$\left\|(1-\varphi_r)^{1/2}v(t)\right\|_2^2 + \int_0^t \left\|v(\tau)\right\|_{W_p^1(R^n\setminus B(0,2r))}^p d\tau$$
$$\leqslant c\left(\|B\|_{1,p}\right)\left[\|k\|_{L_{\frac{p}{p-2}}(R^n\setminus B(0,r))}^{\frac{p}{p-2}} + \|g\|_{L_q(R^n\setminus B(0,r))}^{p'} + \frac{1}{r}\right]t, \quad \forall t \geqslant 0,$$

which yields (30). □

Lemma 7. *Assume the conditions* (3)–(4) *are satisfied, and B is a bounded subset of $W_p^1(R^n)$. Then the set $\bigcup_{t\geqslant 0} S(t)B$ is relatively compact in $W_p^1(B(0,r))$.*

Proof. Let $\{S(t_m)\theta_m\}_{m=1}^{\infty} \subset \bigcup_{t\geqslant 0} S(t)B$. Since by (7) the set $\{S(t_m)\theta_m\}_{m=1}^{\infty}$ is bounded in $W_p^1(R^n)$, from Sobolev compact embedding theorems we have that the set $\{\varphi_r S(t_m)\theta_m\}_{m=1}^{\infty}$ is relatively compact in $L_p(R^n)$ and $L_2(R^n)$. Thus there exists a subsequence $\{\varphi_r S(t_{m_\nu})\theta_{m_\nu}\}_{\nu=1}^{\infty}$, which converges strongly in $L_p(R^n)$ and $L_2(R^n)$. From (5) we have

$$\begin{aligned}&\big\langle AS(t_{m_\nu})\theta_{m_\nu} - AS(t_{m_\mu})\theta_{m_\mu}, \varphi_r S(t_{m_\nu})\theta_{m_\nu} - \varphi_r S(t_{m_\mu})\theta_{m_\mu}\big\rangle \\ &\quad\leqslant \big\|kS(t_{m_\nu})\theta_{m_\nu} - kS(t_{m_\mu})\theta_{m_\mu}\big\|_2 \big\|\varphi_r S(t_{m_\nu})\theta_{m_\nu} - \varphi_r S(t_{m_\mu})\theta_{m_\mu}\big\|_2 \\ &\qquad+ \left\|\frac{\partial}{\partial t} S(t_{m_\nu})\theta_{m_\nu} - \frac{\partial}{\partial t} S(t_{m_\mu})\theta_{m_\mu}\right\|_2 \big\|\varphi_r S(t_{m_\nu})\theta_{m_\nu} - \varphi_r S(t_{m_\mu})\theta_{m_\mu}\big\|_2\end{aligned}$$

and consequently

$$\begin{aligned}&\big\|S(t_{m_\nu})\theta_{m_\nu} - S(t_{m_\mu})\theta_{m_\mu}\big\|_{W_p^1(B(0,r))} \\ &\quad\leqslant c_3\big(\|B\|_{1,p}\big)\big\|\varphi_r S(t_{m_\nu})\theta_{m_\nu} - \varphi_r S(t_{m_\mu})\theta_{m_\mu}\big\|_2 \\ &\qquad+ \frac{1}{r} c_4\big(\|B\|_{1,p}\big)\big\|\varphi_r S(t_{m_\nu})\theta_{m_\nu} - \varphi_r S(t_{m_\mu})\theta_{m_\mu}\big\|_p,\end{aligned}$$

which gives relative compactness of the set $\bigcup_{t\geqslant 0} S(t)B$ in $W_p^1(B(0,r))$. □

Now, based on the results established above, we can prove the asymptotic compactness of $S(t)$, which is included in the following theorem.

Theorem 1. *Assume the conditions* (3)–(4) *are satisfied, and B is a bounded subset of $W_p^1(R^n)$. Then*

(i) *the set $\{S(t_m)\theta_m\}_{m=1}^{\infty}$ is relatively compact in $L_{np/(n-p)}(R^n)$, if $n > p$;*
(ii) *the set $\{S(t_m)\theta_m\}_{m=1}^{\infty}$ is relatively compact in $L_\infty(R^n)$, if $n \leqslant p$,*

where $t_m \to \infty$ and $\{\theta_m\}_{m=1}^{\infty} \subset B$.

Proof. (i) Let $n > p$ and $B_0 = \bigcup_{t\geqslant 0} S(t)B$. By (7), we have B_0 is a bounded subset of $W_p^1(R^n)$. From (25) it follows that

$$\int_0^t \big\|S_0(\tau)\theta\big\|_{\frac{2(p-1)n}{n-p}}^{2(p-1)} d\tau \leqslant c\big(\|B_0\|_{1,p}\big), \quad \forall t \geqslant 0, \tag{31}$$

for every $\theta \in B_0$. On the other hand, from interpolation theorems we have

$$\big\|S_0(\tau)\theta\big\|_{\frac{pn}{n-p}} \leqslant c\big\|S_0(\tau)\theta\big\|_{\frac{2(p-1)n}{n-p}}^{1-s} \big\|S_0(\tau)\theta\big\|_p^s, \tag{32}$$

where $s = \frac{(n-p)(p-2)}{n(p-2)+p^2}$. From (31)–(32) and (7) we obtain

$$\int_0^t \left\| S_0(\tau)\theta \right\|^p_{\frac{pn}{n-p}} \, d\tau \leqslant c\big(\|B_0\|_{1,p}\big) t^{1-\frac{p(1-s)}{2(p-1)}}, \quad \forall t \geqslant 0, \tag{33}$$

for every $\theta \in B_0$. The last inequality together with (30) yields that for every $\varepsilon > 0$ there exist $r_0 = r_0(\varepsilon, B_0)$ and $T_0 = T_0(\varepsilon, B_0)$ such that whenever $r \geqslant r_0$ and $t \geqslant T_0$

$$\frac{1}{t} \int_0^t \left\| S(\tau)\theta \right\|^p_{L_{\frac{pn}{n-p}}(R^n \setminus B(0,r))} \, d\tau \leqslant \varepsilon,$$

for every $\theta \in B_0$, and consequently there is a sequence $\{\tau_m\}_{m=1}^{\infty} \subset [0, T_0]$ such that

$$\left\| S(\tau_m + t_m)\theta_m \right\|^p_{L_{\frac{pn}{n-p}}(R^n \setminus B(0,r))} \leqslant \varepsilon. \tag{34}$$

From (22), (34) and Lemma 7 we obtain relative compactness of the set $\{S(t_m)\theta_m\}_{m=1}^{\infty}$ in $L_{np/(n-p)}(R^n)$.

(ii) Let $n \leqslant p$. Since in the case $n < p$ the theorem follows from embedding $W^1_p(R^n) \subset L_\infty(R^n)$, we present the proof for $n = p$. In this case by repeating above procedure, we obtain that the set $\{S(t_m)\theta_m\}_{m=1}^{\infty}$ is relatively compact in $L_q(R^n)$ for every $q > p$. Let $q_0' = \frac{q_0}{q_0-1}$ and choose q such that $q > q_0'$, $\frac{q-2}{2p} > 1$ and $\frac{q-q_0'}{pq_0'} > 1$. Also let $u_m(x) = S(t_m)\theta_m$ and $\varphi_m(x) = \max\{u_m(x) - r, 0\}$. From (1) we have

$$\begin{aligned} &-\sum_{i=1}^n \frac{\partial}{\partial x_i}\big(\left|(u_m)_{x_i}\right|^{p-2}(u_m)_{x_i}\big) + \lambda |u_m|^{p-2} u_m + f(x, u_m) \\ &\quad = g(x) - \frac{\partial}{\partial t} S(t_m)\theta_m. \end{aligned} \tag{35}$$

Multiplying (35) by $\varphi_m(x)$ and integrating over R^n, we have

$$\|\varphi_m\|^p_{1,p} \leqslant c_1\big(\|B_0\|_{1,p}\big)\|\varphi_m\|_2 + c_2\|\varphi_m\|_{q_0'}, \quad m = 1, 2, \ldots.$$

Letting $A^m_r = \{x\colon\ x \in R^n,\ u_m(x) \geqslant r\}$ and using the last inequality, we obtain

$$\begin{aligned} \|\varphi_m\|^p_1 &\leqslant \big(\operatorname{mes} A^m_r\big)^{\frac{p(q-1)}{q}} \|\varphi_m\|^p_q \leqslant \big(\operatorname{mes} A^m_r\big)^{\frac{p(q-1)}{q}} c\|\varphi_m\|^p_{1,p} \\ &\leqslant c_1\big(\|B_0\|_{1,p}\big)\big(\operatorname{mes} A^m_r\big)^{\frac{p(q-1)}{q}} \|\varphi_m\|_q \big(\operatorname{mes} A^m_r\big)^{\frac{(q-2)}{2q}} \\ &\quad + c_2\|\varphi_m\|_q \big(\operatorname{mes} A^m_r\big)^{\frac{p(q-1)}{q}} \big(\operatorname{mes} A^m_r\big)^{\frac{q-q_0'}{qq_0'}} \\ &\leqslant c_3\big(\|B_0\|_{1,p}\big)\|u_m\|_{1,p}\Big[\big(\operatorname{mes} A^m_r\big)^{p(1+\delta_1)} + \big(\operatorname{mes} A^m_r\big)^{p(1+\delta_2)}\Big], \end{aligned} \tag{36}$$

where $\delta_1 = \frac{q-2}{2pq} - \frac{1}{q}$ and $\delta_2 = \frac{q-q_0'}{pqq_0'} - \frac{1}{q}$. Since

$$\operatorname{mes} A^m_r \leqslant \frac{1}{r^p}\|u_m\|^p_p$$

letting $\delta = \min\{\delta_1, \delta_2\}$, from (36), we obtain

$$\|\varphi_m\|_1 \leqslant c\big(\|B_0\|_{1,p}, r_0\big)\big(\operatorname{mes} A_r^m\big)^{(1+\delta)} \tag{37}$$

for every $r \geqslant r_0 > 0$ and $m = 1, 2, \ldots$. Applying Lemma 5.1 from [22] to (37), we have

$$\begin{aligned} u_m(x) &\leqslant r_0 + \frac{1+\delta}{\delta}\big[c\big(\|B_0\|_{1,p}, r_0\big)\big]^{\frac{1}{1+\delta}}\Bigg[\int\limits_{A_{r_0}^m} (u_m - r_0)\,dx\Bigg]^{\frac{\delta}{1+\delta}} \\ &\leqslant c\big(\|B_0\|_{1,p}, r_0, \delta\big) \quad \text{for a.e. } x \in R^n \text{ and } m = 1, 2, \ldots. \end{aligned} \tag{38}$$

Since $v_m(x) = -u_m(x)$ is the solution of the equation similar to (35) repeating the above procedure, we obtain

$$v_m(x) \leqslant c\big(\|B_0\|_{1,p}, r_0, \delta\big) \quad \text{for a.e. } x \in R^n \text{ and } m = 1, 2, \ldots,$$

which together with (38) yields $u_m \in L_\infty(R^n)$ and $\sup_m \|u_m\|_\infty < \infty$.

Similarly letting $\varphi_{mk}(x) = \max\{u_m(x) - u_k(x) - r, 0\}$, we get

$$\begin{aligned} \|\varphi_{mk}\|_1 &\leqslant c\big(\|B_0\|_{1,p}\big)\big(\operatorname{mes} A_r^{mk}\big)^{(1+\nu)}\|\varphi_{mk}\|_q^{1/p} \\ &\leqslant c\big(\|B_0\|_{1,p}\big)\big(\operatorname{mes} A_r^{mk}\big)^{(1+\nu)}\|u_m - u_k\|_q^{1/p}, \end{aligned} \tag{39}$$

where $A_r^{mk} = \{x\colon\ x \in R^n,\ u_m(x) - u_k(x) \geqslant r\}$ and $\nu = \frac{q-2}{2pq} - \frac{1}{q}$. Applying Lemma 5.1 from [22] to (39), we have

$$\begin{aligned} &u_m(x) - u_k(x) \\ &\quad\leqslant r_0 + \frac{1+\delta}{\delta}\big[c\big(\|B_0\|_{1,p}\big)\big]^{\frac{1}{1+\delta}}\|u_m - u_k\|_q^{\frac{1}{p(1+\delta)}}\Bigg[\int\limits_{A_{r_0}^{mk}} (u_m - u_k - r_0)\,dx\Bigg]^{\frac{\delta}{1+\delta}} \\ &\quad\leqslant r_0 + c\big(\|B_0\|_{1,p}, r_0, \delta\big)\|u_m - u_k\|_q^{\frac{1}{p(1+\delta)}} \quad \text{for a.e. } x \in R^n, \end{aligned}$$

and consequently

$$\|u_m - u_k\|_\infty \leqslant r_0 + c\big(\|B_0\|_{1,p}, r_0, \delta\big)\|u_m - u_k\|_q^{\frac{1}{p(1+\delta)}} \tag{40}$$

for every $r_0 > 0$. Since as mentioned above the set $\{u_m\}_{m=1}^\infty$ is relatively compact in $L_q(R^n)$, from (40) follows that this set is relatively compact in $L_\infty(R^n)$. □

Let

$$\mathcal{A} = \bigcap_{\tau \geqslant 0}\Bigg[\bigcup_{t \geqslant \tau} S(t)\mathfrak{B}\Bigg],$$

where $[\cdot]$ is the weak closure in $W_p^1(R^n)$ and $\mathfrak{B}$ is from Lemma 1.

Now we can prove our main result.

Theorem 2. *Assume the conditions* (3)–(4) *are satisfied. Then*

(i) $\mathcal{A}$ *is a global* $(L_2(R^n), L_{np/(n-p)}(R^n))$*-attractor of* $S(t)$*, if* $n > p$*;*
(ii) $\mathcal{A}$ *is a global* $(L_2(R^n), L_\infty(R^n))$*-attractor of* $S(t)$*, if* $n \leqslant p$*.*

Proof. To prove the invariantness of $\mathcal{A}$ let us first show that

$$\varphi \in \mathcal{A} \quad \Rightarrow \quad \exists t_m \to \infty \quad \text{and} \quad \exists \{\theta_m\}_{m=1}^{\infty} \subset \mathfrak{B} \text{ such that}$$
$$S(t_m)\theta_m \to \varphi \text{ weakly in } W_p^1(R^n). \tag{41}$$

Let $n > p$ (the proof is similar in the case $n \leqslant p$). From Theorem 1 we have that for any $\varepsilon > 0$ there exist $t_\varepsilon > 0$ and $r_\varepsilon > 0$ such that

$$\left\| \bigcup_{t \geqslant t_\varepsilon} S(t)\mathfrak{B} \right\|_{L_{\frac{pn}{n-p}}(R^n \setminus B(0, r_\varepsilon))} \leqslant \frac{\varepsilon}{3}.$$

On the other hand, by Lemma 7 the set $\bigcup_{t \geqslant t_\varepsilon} S(t)\mathfrak{B}$ is relatively compact in $W_p^1(B(0, r_\varepsilon))$. Then, since $\varphi \in [\bigcup_{t \geqslant t_\varepsilon} S(t)\mathfrak{B}]$, there exists $\psi_\varepsilon \in \bigcup_{t \geqslant t_\varepsilon} S(t)\mathfrak{B}$ such that we have $\|\varphi - \psi_\varepsilon\|_{pn/(n-p)} \leqslant \varepsilon$. Consequently there exist $t_m \to \infty$ and $\{\theta_m\}_{m=1}^{\infty} \subset \mathfrak{B}$ such that $S(t_m)\theta_m \to \varphi$ strongly in $L_{pn/(n-p)}(R^n)$. Taking into account boundedness of the sequence $\{S(t_m)\theta_m\}_{m=1}^{\infty}$ in $W_p^1(R^n)$, we obtain (41).

Since by Lemma 2 the operator $S(t)$ is weakly continuous in $W_p^1(R^n)$, from (41) we find that $S(t)\mathcal{A} = \mathcal{A}$ for every $t \geqslant 0$. The attracting property of $\mathcal{A}$ follows from Lemma 1 and Theorem 1. □

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