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Interior penalty discontinuous Galerkin method for Maxwell's equations: Energy norm error estimates

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Abstract

We develop the symmetric interior penalty discontinuous Galerkin (DG) method for the time-dependent Maxwell equations in second-order form. We derive optimal a priori error estimates in the energy norm for smooth solutions. We also consider the case of low-regularity solutions that have singularities in space.

Keywords: Discontinuous Galerkin methods; Maxwell equations; A priori error analysis

1. Introduction

The development of new more sophisticated algorithms for the numerical solution of Maxwell's equations is dictated by increasingly complex applications in electromagnetics. In 1966, Yee [28] introduced the first and probably most popular method, the finite difference time domain (FDTD) scheme, which is simple and efficient. However, the FDTD scheme can only be applied on structured (Cartesian) grids and suffers from the inaccurate representation of the solution on curved boundaries (staircase approximation) [2,25]. Moreover, higher order FDTD methods are generally difficult to implement near interfaces and boundaries.

In contrast, finite element methods (FEMs) easily handle complex boundaries and unstructured grids, even when higher order discretizations are used. They also provide rigorous a posteriori error estimates which are useful for local adaptivity and error control. Different FE discretizations of Maxwell's equations are available, such as the edge elements of Nédélec [20], the node-based first order formulation of Lee and Madsen [16], the node-based curl–curl formulation of Paulsen and Lynch [21], or the node-based least-squares FEM by Jiang et al. [14]—see also Monk [17].

Edge elements are probably the most satisfactory from a theoretical point of view [18], in particular because they correctly represent singular behavior at reentrant corners. However, they are less attractive for time-dependent computations, because the solution of a linear system is required at every time iteration. Indeed, in the case of triangular or tetrahedral edge elements, the entries of the diagonal matrix resulting from mass-lumping are not necessarily strictly positive [8]; therefore, explicit time stepping cannot be used in general. In contrast, nodal elements naturally lead to a

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fully explicit scheme when mass-lumping is applied both in space and time [8], but cannot correctly represent corner singularities in general.

Discontinuous Galerkin (DG) FEM offer an attractive alternative to edge elements for the numerical solution of Maxwell's equations, in particular for time-dependent problems. Not only do they accommodate elements of various types and shapes, irregular non-matching grids, and even locally varying polynomial order, and hence offer great flexibility in the mesh design, but they also lead to (block-) diagonal mass matrices and therefore yield fully explicit, inherently parallel methods when coupled with explicit time stepping. Indeed, the mass matrix arising from a DG discretization is always block-diagonal, with block size equal to the number of degrees of freedom per element; hence, it can be inverted at very low computational cost. In fact, for constant material coefficients, the mass matrix is truly diagonal for a judicious choice of (locally orthogonal) shape functions. Because continuity across element interfaces is weakly enforced merely by adding suitable bilinear forms (so-called numerical fluxes) to the standard variational formulation, the implementation of DG–FE methods is straightforward within existing FE software libraries.

For first-order hyperbolic systems, various DG FEM are available. In [7], for instance, Cockburn and Shu use a DGFEM in space combined with a Runge–Kutta scheme in time to discretize hyperbolic conservation laws. In [15], Kopriva et al. developed discontinuous Galerkin methods, which combine high-order spectral elements with a fourth order low-storage Runge–Kutta scheme. Warburton [26], and Hesthaven and Warburton [11] used a similar approach for their Runge–Kutta discontinuous Galerkin (RKDG) method, which combines high-order spatial accuracy with a fourth order low-storage Runge–Kutta scheme. While successful, their scheme does not conserve energy due to upwinding. Fezoui et al. [9] used central fluxes instead, yet the convergence rate of their scheme remains sub-optimal. Recently, Chen et al. developed a high-order RKDG method for Maxwell's equations in first-order hyperbolic form, which achieves high-order convergence both in space and time by using a strong stability preserving (low storage) SSP–RK scheme [3]. By using locally divergence-free polynomials Cockburn, Li, and Shu developed a locally divergence-free DG method for the first-order Maxwell system [6].

For the second order (scalar) wave equation Rivière and Wheeler proposed a nonsymmetric formulation, which required additional stabilization for optimal convergence [23,24]. A symmetric interior penalty DG FEM was first proposed by the authors in [10], where optimal convergence rates in the energy norm and in the L^2 norm were shown; the usefulness of the method was also demonstrated via numerical experiments. Recently, Chung and Engquist [4] proposed a hybrid DG/continuous FE approach for the acoustic wave equation.

Here, we propose and analyze the symmetric interior penalty DG method for the spatial discretization of Maxwell's equations in *second order* form. After stating the model problem in Section 2, we describe the interior penalty DG variational formulation in Section 3. In Section 4, we state optimal a-priori error bounds in the energy norm. In the case of solutions with smoothness beyond H^1, the error bound (Theorem 2) holds for arbitrary DG–FE discretizations, whereas in the case of lower regularity, the error bound (Theorem 3) only holds for conforming meshes. All proofs and technical approximation results are provided in Section 5. Finally, we end with some concluding remarks in Section 6.

2. Model problem

The evolution of a time-dependent electromagnetic field $\mathbf{E}(\mathbf{x}, t)$, $\mathbf{H}(\mathbf{x}, t)$ propagating through a linear isotropic medium is determined by Maxwell's equations

$$\varepsilon \mathbf{E}_t = \nabla \times \mathbf{H} - \sigma \mathbf{E} + \mathbf{j},$$
$$\mu \mathbf{H}_t = -\nabla \times \mathbf{E}.$$

Here, the coefficients μ, ε, and σ denote the relative magnetic permeability, the relative electric permittivity, and the conductivity of the medium, respectively. The source term $\mathbf{j}$ corresponds to the applied current density. By eliminating the magnetic field $\mathbf{H}$, Maxwell's equations reduce to a second-order vector wave equation for the electric field $\mathbf{E}$

$$\varepsilon \mathbf{E}_{tt} + \sigma \mathbf{E}_t + \nabla \times (\mu^{-1} \nabla \times \mathbf{E}) = \mathbf{j}_t.$$

If the electric field is eliminated instead, one easily finds that the magnetic field $\mathbf{H}$ satisfies a similar vector wave equation.

Thus, we shall consider the following model problem: find the (electric or magnetic) field $\mathbf{u}(\mathbf{x}, t)$, which satisfies

$$\begin{aligned}
&\varepsilon \mathbf{u}_{tt} + \sigma \mathbf{u}_t + \nabla \times (\mu^{-1} \nabla \times \mathbf{u}) = \mathbf{f} \quad \text{in } \Omega \times J,\\
&\mathbf{n} \times \mathbf{u} = \mathbf{0} \quad \text{in } \Gamma \times J,\\
&\mathbf{u}|_{t=0} = \mathbf{u}_0 \quad \text{on } \Omega,\\
&\mathbf{u}_t|_{t=0} = \mathbf{v}_0 \quad \text{on } \Omega.
\end{aligned} \tag{1}$$

Here, $J = (0, T)$ is a finite time interval and Ω is a bounded Lipschitz polyhedron in $\mathbb{R}^3$ with boundary $\Gamma = \partial\Omega$ and outward unit normal $\mathbf{n}$. For simplicity, we assume Ω to be simply connected and Γ to be connected. The right-hand side $\mathbf{f}$ is a given source term in $L^2(J; L^2(\Omega)^3)$, where $L^p(J; H^s(\Omega))$ denotes the standard Bochner space of (time-dependent) functions whose $\|\cdot\|_{s,\Omega}$ Sobolev-norm is p-integrable in time. The standard inner product in $L^2(\Omega)^3$ is denoted by $(\mathbf{u}, \mathbf{v}) := \int_\Omega \mathbf{u} \cdot \mathbf{v} \, d\mathbf{x}$.

The functions $\mathbf{u}_0$ and $\mathbf{v}_0$ are prescribed initial data with $\mathbf{u}_0 \in H_0(\text{curl}; \Omega)$ and $\mathbf{v}_0 \in L^2(\Omega)^3$, where $H_0(\text{curl}; \Omega)$ denotes the subspace of functions in

$$H(\text{curl}; \Omega) = \{\mathbf{v} \in L^2(\Omega)^3 \colon \nabla \times \mathbf{v} \in L^2(\Omega)^3\},$$

which have zero tangential component on $\partial\Omega$, the boundary of Ω. Furthermore, we assume that μ, ε and σ are scalar positive functions that satisfy

$$0 < \mu_\star \leqslant \mu(\mathbf{x}) \leqslant \mu^\star < \infty, \quad 0 < \varepsilon_\star \leqslant \varepsilon(\mathbf{x}) \leqslant \varepsilon^\star < \infty, \quad \mathbf{x} \in \Omega,$$

and

$$0 \leqslant \sigma(\mathbf{x}) \leqslant \sigma^\star < \infty, \quad \mathbf{x} \in \Omega,$$

respectively. For simplicity, we also assume that μ is piecewise constant.

3. DG discretization

We shall now discretize Maxwell's equations in space using the interior penalty DG method. First, we consider shape-regular meshes $\mathcal{T}_h$ that partitions the domain Ω into disjoint tetrahedral or affine hexahedral elements $\{K\}$, such that $\overline{\Omega} = \bigcup_{K \in \mathcal{T}_h} \overline{K}$. The diameter of element K is denoted by h_K, and the mesh size h is given by $h = \max_{K \in \mathcal{T}_h} h_K$. We assume that the partition is aligned with the discontinuities of the coefficient μ and that the local mesh sizes are of bounded variation, that is, there exists a positive constant κ, which depends only on the shape-regularity of the mesh, such that $\kappa h_K \leqslant h_{K'} \leqslant \kappa^{-1} h_K$, for all neighboring elements K and K'. We denote by $\mathcal{F}_h^{\mathcal{I}}$ the set of all interior faces, by $\mathcal{F}_h^{\mathcal{B}}$ the set of all boundary faces, and set $\mathcal{F}_h = \mathcal{F}_h^{\mathcal{I}} \cup \mathcal{F}_h^{\mathcal{B}}$.

For a given partition $\mathcal{T}_h$ of Ω and an approximation order $\ell \geqslant 1$, we wish to approximate $\mathbf{u}(\cdot, t)$ in the finite element space

$$\mathbf{V}^h := \{\mathbf{v} \in L^2(\Omega)^3 \colon \mathbf{v}|_K \in \mathcal{S}^\ell(K)^3, K \in \mathcal{T}_h\},$$

where $\mathcal{S}^\ell(K)$ is the space $\mathcal{P}^\ell(K)$ of polynomials of total degree at most ℓ on K, if K is a tetrahedron, and the space $\mathcal{Q}^\ell(K)$ of polynomials of degree at most ℓ in each variable on K, if K is a parallelepiped.

We consider the following (semi-discrete) DG formulation of (1): find $u^h \colon \overline{J} \times V^h \to \mathbb{R}$ such that

$$\begin{aligned}
&(\varepsilon \mathbf{u}_{tt}^h, \mathbf{v}) + (\sigma \mathbf{u}_t^h, \mathbf{v}) + a_h(\mathbf{u}^h, \mathbf{v}) = (\mathbf{f}, \mathbf{v}), \quad \mathbf{v} \in \mathbf{V}^h, \quad t \in J,\\
&\mathbf{u}^h|_{t=0} = \mathbf{\Pi}_h \mathbf{u}_0,\\
&\mathbf{u}_t^h|_{t=0} = \mathbf{\Pi}_h \mathbf{v}_0.
\end{aligned} \tag{2}$$

Here, $\mathbf{\Pi}_h$ denotes the L^2-projection onto $\mathbf{V}^h$, while the discrete bilinear form a_h, defined on $\mathbf{V}^h \times \mathbf{V}^h$, is given by

$$a_h(\mathbf{u},\mathbf{v}) := \sum_{K\in\mathscr{T}_h}\int_K \mu^{-1}(\nabla\times\mathbf{u})\cdot(\nabla\times\mathbf{v})\,\mathrm{d}\mathbf{x} - \sum_{f\in\mathscr{F}_h}\int_f [\![\mathbf{u}]\!]_T\cdot\{\{\mu^{-1}\nabla\times\mathbf{v}\}\}\,\mathrm{d}A$$
$$- \sum_{f\in\mathscr{F}_h}\int_f [\![\mathbf{v}]\!]_T\cdot\{\{\mu^{-1}\nabla\times\mathbf{u}\}\}\,\mathrm{d}A + \sum_{f\in\mathscr{F}_h}\int_f \mathtt{a}[\![\mathbf{u}]\!]_T\cdot[\![\mathbf{v}]\!]_T\,\mathrm{d}A.$$

We denote by $[\![\mathbf{v}]\!]_T$ and $\{\{\mathbf{v}\}\}$, respectively, the tangential jumps and averages of a DG function $\mathbf{v}$ across interior faces; cf. [12,13]. On boundary faces we set $[\![\mathbf{v}]\!]_T := \mathbf{n}\times\mathbf{v}$ and $\{\{\mathbf{v}\}\} := \mathbf{v}$.

The function $\mathtt{a}$ penalizes the jumps of $\mathbf{u}$ and $\mathbf{v}$ over the faces of the triangulation. To define it, we first introduce the function $\mathtt{h}$ and $\mathtt{m}$ by

$$\mathtt{h}|_f = \begin{cases} \min\{h_K, h_{K'}\}, & f\in\mathscr{F}_h^{\mathscr{I}},\quad F = \partial K\cap\partial K', \\ h_K, & f\in\mathscr{F}_h^{\mathscr{B}},\quad f=\partial K\cap\partial\Omega, \end{cases}$$

$$\mathtt{m}|_f = \begin{cases} \min\{\mu_K, \mu_{K'}\}, & f\in\mathscr{F}_h^{\mathscr{I}},\quad f = \partial K\cap\partial K', \\ \mu_K, & f\in\mathscr{F}_h^{\mathscr{B}},\quad f=\partial K\cap\partial\Omega. \end{cases}$$

Here, we denote by μ_K the restriction of the piecewise coefficient μ to element K. On each $f\in\mathscr{F}_h$, we then set

$$\mathtt{a}|_f := \alpha\mathtt{m}^{-1}\mathtt{h}^{-1}.$$

In Lemma 5 we shall show that there is a positive constant $\alpha_{\min}$, independent of the local mesh sizes and the coefficient μ, such that for $\alpha\geqslant\alpha_{\min}$ the bilinear form a_h is coercive. Hence the DG approximation of (1) is well defined. We note that larger values of α result in a more restrictive CFL condition in (explicit) time discretizations of (2).

Remark 1. When the interior penalty DG method is used for time-dependent computations, the FE solution consists of a superposition of discrete eigenmodes. Because of symmetry, the energy of the semi-discrete formulation (2) is conserved, so that all the modes neither grow nor decay. For eigenvalue computations, Buffa and Perugia [1] recently proved that the interior penalty DG discretization of the Maxwell operator is free of spurious modes: the discrete spectrum will eventually converge to the continuous spectrum, as $h\to 0$. Nonetheless, on any fixed mesh some of the discrete eigenmodes will not correspond to physical modes. Hesthaven and Warburton [11], and Warburton and Embree [27] showed that larger values of the penalty parameter in central flux or local discontinuous Galerkin (LDG) discretizations increase the separation between spurious and physical eigenmodes. Alternatively, if upwinding is used some of the spurious modes will be damped as well.

Clearly, as the mesh is refined, the energy present in the spurious modes will decrease and eventually vanish, as the numerical solution obtained with the interior penalty DG method converges to the exact solution; see Section 4.

4. A priori error bounds

In this section we state optimal a priori error bounds with respect to the DG energy norm. To that end, we set $\mathbf{V}(h) := H_0(\mathrm{curl};\Omega) + \mathbf{V}^h$ and introduce the semi-norm

$$|\mathbf{v}|_h^2 := \sum_{K\in\mathscr{T}_h}\|\mu^{-1/2}(\nabla\times\mathbf{v})\|_{0,K}^2 + \sum_{f\in\mathscr{F}_h}\|\mathtt{a}^{1/2}[\![\mathbf{v}]\!]_T\|_{0,f}^2.$$

The DG energy norm is then defined by

$$\|\mathbf{v}\|_h^2 := \|\varepsilon^{1/2}\mathbf{v}\|_{0,\Omega}^2 + |\mathbf{v}|_h^2.$$

For functions $\mathbf{v}\in H(\mathrm{curl};\Omega)$ it coincides with the standard energy norm. We further define the norms

$$\|\mathbf{v}\|_{L^p(J;\mathbf{V}(h))} = \begin{cases} \left(\int_J \|\mathbf{v}\|_h^p\,\mathrm{d}t\right)^{1/p}, & 1\leqslant p<\infty, \\ \mathrm{ess\ sup}_{t\in J}\|\mathbf{v}\|_h, & p=\infty, \end{cases}$$

and set

$$|\mathbf{v}|_{L^p(J;\mathbf{V}(h))} = \begin{cases} \left(\int_J |\mathbf{v}|_h^p \,\mathrm{d}t\right)^{1/p}, & 1 \leqslant p < \infty, \\ \operatorname{ess\,sup}_{t\in J} |\mathbf{v}|_h, & p = \infty. \end{cases}$$

Then, we have the following error estimate.

Theorem 2. *Let the analytical solution* $\mathbf{u}$ *of* (1) *satisfy*

$$\mathbf{u} \in L^\infty(J; H^{1+s}(\Omega)^3), \quad \mathbf{u}_t \in L^\infty(J; H^{1+s}(\Omega)^3), \quad \mathbf{u}_{tt} \in L^1(J; H^s(\Omega)^3),$$

for $s > \frac{1}{2}$, *and* $\mathbf{u}^h$ *be the semi-discrete DG approximation with* $\alpha \geqslant \alpha_{\min}$. *Then, the error* $\mathbf{e} = \mathbf{u} - \mathbf{u}^h$ *satisfies*

$$\begin{aligned} \|\varepsilon^{1/2}\mathbf{e}_t\|_{L^\infty(J;L^2(\Omega)^3)} + \|\mathbf{e}\|_{L^\infty(J;\mathbf{V}(h))} \leqslant{} & C\left(\|\varepsilon^{1/2}\mathbf{e}_t(0)\|_{0,\Omega} + |\mathbf{e}(0)|_h\right) \\ & + Ch^{\min\{s,\ell\}}\Big(\|\mathbf{u}\|_{L^\infty(J;H^{1+s}(\Omega)^3)} \\ & + \|\mathbf{u}_t\|_{L^\infty(J;H^{1+s}(\Omega)^3)} + \|\mathbf{u}_{tt}\|_{L^1(J;H^s(\Omega)^3)}\Big), \end{aligned}$$

with a constant $C > 0$ *that is independent of the mesh size.*

In Theorem 2 we implicitly assume that $\mathbf{u}_0 \in H^{1+s}(\Omega)^3$ and $\mathbf{v}_0 \in H^s(\Omega)^3$. Hence, the approximation properties of the L^2-projection in Lemmas 7 and 8 imply that

$$\|\varepsilon^{1/2}\mathbf{e}_t(0)\|_{0,\Omega} \leqslant Ch^{\min\{s,\ell+1\}}\|\mathbf{v}_0\|_{s,\Omega}, \quad |\mathbf{e}(0)|_h \leqslant Ch^{\min\{s,\ell\}}\|\mathbf{u}_0\|_{1+s,\Omega}.$$

As a consequence, Theorem 2 yields optimal convergence of order $\mathcal{O}(h^{\min\{s,l\}})$ in the DG energy norm.

In many instances, solutions to the Maxwell equations have singularities that do not satisfy the regularity assumptions in Theorem 2. Indeed, it is well known that the strongest singularities have smoothness below $H^1(\Omega)^3$. We shall now show that the DG method still converges under weaker yet realistic regularity assumptions provided that the meshes are conforming.

Theorem 3. *Let the analytical solution* $\mathbf{u}$ *of* (1) *satisfy*

$$\mathbf{u},\ \mathbf{u}_t,\ \nabla\times\mathbf{u},\ \nabla\times\mathbf{u}_t \in L^\infty(J; H^s(\Omega)^3) \quad \textit{and} \quad \mathbf{u}_{tt},\ \nabla\times\mathbf{u}_{tt} \in L^1(J; H^s(\Omega)^3),$$

for $s > 1/2$. *Next, let* $\mathcal{T}_h$ *be a conforming triangulation of* Ω *into tetrahedra or hexahedra with edges parallel to the coordinate axes, and* $\mathbf{u}^h$ *be the semi-discrete DG approximation obtained with* $\alpha \geqslant \alpha_{\min}$. *Then the error* $\mathbf{e} = \mathbf{u} - \mathbf{u}^h$ *satisfies*

$$\begin{aligned} \|\varepsilon^{1/2}\mathbf{e}_t\|_{L^\infty(J;L^2(\Omega)^3)} + \|\mathbf{e}\|_{L^\infty(J;\mathbf{V}(h))} \leqslant{} & C\|\varepsilon^{1/2}\mathbf{e}_t(0)\|_{0,\Omega} + |\mathbf{e}(0)|_h \\ & + Ch^{\min\{s,\ell\}}\Big(\|\mathbf{u}\|_{L^\infty(J;H^s(\Omega)^3)} + \|\nabla\times\mathbf{u}\|_{L^\infty(J;H^s(\Omega)^3)} \\ & + \|\mathbf{u}_t\|_{L^\infty(J;H^s(\Omega)^3)} + \|\nabla\times\mathbf{u}_t\|_{L^\infty(J;H^s(\Omega)^3)} \\ & + \|\mathbf{u}_{tt}\|_{L^1(J;H^s(\Omega)^3)} + \|\nabla\times\mathbf{u}_{tt}\|_{L^1(J;H^s(\Omega)^3)}\Big), \end{aligned}$$

with a constant $C > 0$ *that is independent of the mesh size.*

If we additionally assume that $\mathbf{u}_0 \in H^{1+s}(\Omega)^3$ for $t > 0$, the bound in Theorem 3 yields again optimal convergence of the order $\mathcal{O}(h^{\min\{s,\ell\}})$ for the error in the energy norm.

The bounds in Theorems 2 and 3 are proven in the next section.

5. Proofs of Theorems 2 and 3

5.1. Extension of the DG form and stability properties

The bilinear DG form a_h, while well-defined on $\mathbf{V}^h$, is not well defined on the larger space $\mathbf{V}(h)$. To extend the DG form to $\mathbf{V}(h)$, we follow an approach similar to [22] and introduce the auxiliary form

$$\begin{aligned}\widetilde{a}_h(\mathbf{u},\mathbf{v}) = {} & \sum_{K\in\mathscr{T}_h}\int_K \mu^{-1}(\nabla\times\mathbf{u})\cdot(\nabla\times\mathbf{v})\,\mathrm{d}\mathbf{x} - \sum_{f\in\mathscr{F}_h}\int_f [\![\mathbf{u}]\!]_T\cdot\{\{\mu^{-1}\mathbf{\Pi}_h(\nabla\times\mathbf{v})\}\}\,\mathrm{d}A \\ & - \sum_{f\in\mathscr{F}_h}\int_f [\![\mathbf{v}]\!]_T\cdot\{\{\mu^{-1}\mathbf{\Pi}_h(\nabla\times\mathbf{u})\}\}\,\mathrm{d}A + \sum_{f\in\mathscr{F}_h}\int_f \mathtt{a}[\![\mathbf{u}]\!]_T\cdot[\![\mathbf{v}]\!]_T\,\mathrm{d}A,\end{aligned}$$

where we recall that $\mathbf{\Pi}_h$ is the L^2-projection onto $\mathbf{V}^h$. Note that $\widetilde{a}_h$ coincides with a_h on $\mathbf{V}^h\times\mathbf{V}^h$ and is well defined on $H_0(\mathrm{curl};\Omega)\times H_0(\mathrm{curl};\Omega)$. This follows from the following result.

Lemma 4. *For $\mathbf{v}\in\mathbf{V}(h)$ and $\mathbf{z}\in L^2(\Omega)^3$ there holds*

$$\sum_{f\in\mathscr{F}_h}\int_f [\![\mathbf{v}]\!]_T\{\{\mu^{-1}\mathbf{\Pi}_h\mathbf{z}\}\}\,\mathrm{d}A \leqslant C_{\mathrm{inv}}\alpha^{-1/2}\left(\sum_{f\in\mathscr{F}_h}\|\mathtt{a}^{1/2}[\![\mathbf{v}]\!]_T\|_{0,f}^2\right)^{1/2}\|\mu^{-1/2}\mathbf{z}\|_{0,\Omega},$$

with a constant C_{inv} that only depends on the shape-regularity of the mesh and the approximation order ℓ.

Proof. By the Cauchy–Schwarz inequality and the definition of the stabilization function $\mathtt{a}$ we have

$$\begin{aligned}\sum_{f\in\mathscr{F}_h}\int_f [\![\mathbf{v}]\!]_T\{\{\mu^{-1}\mathbf{\Pi}_h\mathbf{z}\}\}\,\mathrm{d}A \leqslant {} & \alpha^{-1/2}\left(\sum_{f\in\mathscr{F}_h}\|\mathtt{a}^{1/2}[\![\mathbf{v}]\!]_T\|_{0,f}^2\right)^{1/2} \\ & \times\left(\sum_{f\in\mathscr{F}_h}\|\mathtt{m}^{1/2}\mathtt{h}^{1/2}\{\{(\mu^{-1}\mathbf{\Pi}_h\mathbf{z})\}\}\|_{0,f}^2\right)^{1/2}.\end{aligned}$$

Using the definition of $\mathtt{m}$ and $\mathtt{h}$ and the assumption that μ is piecewise constant, we can bound the last term above by

$$\begin{aligned}\sum_{f\in\mathscr{T}_h}\|\mathtt{m}^{1/2}\mathtt{h}^{1/2}\{\{\mu^{-1}\mathbf{\Pi}_h\mathbf{z}\}\}\|_{0,f}^2 \leqslant {} & \sum_{K\in\mathscr{T}_h} h_K\mu_K\|\mu_K^{-1}\mathbf{\Pi}_h\mathbf{z}\|_{0,\partial K}^2 \\ = {} & \sum_{K\in\mathscr{T}_h} h_K\|\mathbf{\Pi}_h(\mu_K^{-1/2}\mathbf{z})\|_{0,\partial K}^2.\end{aligned}$$

Recalling the inverse inequality

$$\|\mathbf{w}\|_{0,\partial K}^2 \leqslant C_{\mathrm{inv}}^2 h_K^{-1}\|\mathbf{w}\|_{0,K}^2,\quad \mathbf{w}\in(\mathscr{S}^\ell(K))^3,$$

with a constant C_{inv} that only depends on the shape-regularity of the mesh and the approximation order ℓ, and using the stability of the L^2-projection, we obtain $\sum_{K\in\mathscr{T}_h} h_K\|\mathbf{\Pi}_h(\mu_K^{-1/2}\mathbf{z})\|_{0,\partial K}^2 \leqslant C_{\mathrm{inv}}^2\|\mu^{-1/2}\mathbf{z}\|_{0,\Omega}^2$. This completes the proof. □

We are now ready to show the continuity and coercivity of $\widetilde{a}_h$ on $\mathbf{V}(h)$.

Lemma 5. *Set $\alpha_{\min} = 4C_{\mathrm{inv}}^2$, with C_{inv} denoting the constant from Lemma 4. For $\alpha\geqslant\alpha_{\min}$ we have*

$$|\widetilde{a}_h(\mathbf{u},\mathbf{v})|\leqslant C_{\mathrm{cont}}|\mathbf{u}|_h|\mathbf{v}|_h,\quad a_h(\mathbf{v},\mathbf{v})\geqslant C_{\mathrm{coer}}|\mathbf{v}|_h^2,\quad \mathbf{u},\mathbf{v}\in\mathbf{V}(h),$$

with $C_{\mathrm{cont}}=\sqrt{2}$ and $C_{\mathrm{coer}}=\frac{1}{2}$.

Proof. The continuity of $\widetilde{a}_h$ is a straightforward application of the result in Lemma 4 and the Cauchy–Schwarz inequality. The coercivity property of $\widetilde{a}_h$ follows similarly by employing Lemma 4 and the geometric–arithmetic mean inequality:

$$\widetilde{a}_h(\mathbf{u},\mathbf{u}) \geqslant (1-\alpha^{-1/2}C_{\mathrm{inv}}) \left(\sum_{K\in\mathcal{T}_h} \|\mu^{-1/2}(\nabla\times\mathbf{u})\|^2_{0,K} + \sum_{f\in\mathcal{F}_h} \|\mathsf{a}^{1/2}[\![\mathbf{u}]\!]_T\|^2_{0,f} \right),$$

which proves the coercivity of $\widetilde{a}_h$ with $C_{\mathrm{coer}} = \frac{1}{2}$ provided that $\alpha \geqslant \alpha_{\min}$.

5.2. Error equation

We shall use the form $\widetilde{a}_h$ as the basis of our error analysis, similarly to the approach in [12,13]. To do so, we define for $\mathbf{v}\in\mathbf{V}(h)$

$$r_h(\mathbf{u};\mathbf{v}) = \sum_{f\in\mathcal{F}_h}\int_f [\![\mathbf{v}]\!]_T \cdot \{\{\mu^{-1}(\nabla\times\mathbf{u}) - \mu^{-1}\mathbf{\Pi}_h(\nabla\times\mathbf{u})\}\}\,\mathrm{d}A. \tag{3}$$

In order for $r_h(\mathbf{u};\mathbf{v})$ to be well defined, we also need to assume that $\nabla\times\mathbf{u}\in H^s(\Omega)^3$ for $s>\frac{1}{2}$.

Lemma 6. *Let the analytical solution* $\mathbf{u}$ *of* (1) *satisfy*

$$\nabla\times\mathbf{u}\in L^\infty(J;H^s(\Omega)^3), \quad \mathbf{u}_t,\ \mathbf{u}_{tt}\in L^1(J;L^2(\Omega)^3),$$

for $s>\frac{1}{2}$. *Let* $\mathbf{u}^h$ *be the semi-discrete DG approximation obtained with* $\alpha\geqslant\alpha_{\min}$. *Then the error* $\mathbf{e}=\mathbf{u}-\mathbf{u}^h$ *satisfies*

$$(\varepsilon\mathbf{e}_{tt},\mathbf{v}) + (\sigma\mathbf{e}_t,\mathbf{v}) + \widetilde{a}_h(\mathbf{e},\mathbf{v}) = r_h(\mathbf{u};\mathbf{v}), \quad \mathbf{v}\in\mathbf{V}^h,\ a.e.\ in\ J.$$

Proof. Since $[\![\mathbf{u}]\!]_T = \mathbf{0}$ across all faces, we have

$$\widetilde{a}_h(\mathbf{u},\mathbf{v}) = \sum_{K\in\mathcal{T}_h}\int_K \mu^{-1}(\nabla\times\mathbf{u})\cdot(\nabla\times\mathbf{v})\,\mathrm{d}\mathbf{x} - \sum_{f\in\mathcal{F}_h}\int_f [\![\mathbf{v}]\!]_T\cdot\{\{\mu^{-1}\mathbf{\Pi}_h(\nabla\times\mathbf{u})\}\}\,\mathrm{d}A.$$

Integration by parts then leads to

$$\begin{aligned}\widetilde{a}_h(\mathbf{u},\mathbf{v}) = &\sum_{K\in\mathcal{T}_h}\int_K (\nabla\times(\mu^{-1}(\nabla\times\mathbf{u})))\cdot\mathbf{v}\,\mathrm{d}\mathbf{x} + \sum_{f\in\mathcal{F}_h}\int_f [\![\mathbf{v}]\!]_T\cdot\{\{\mu^{-1}\,(\nabla\times\mathbf{u})\}\}\,\mathrm{d}A \\ &- \sum_{f\in\mathcal{F}_h}\int_f [\![\mathbf{v}]\!]_T\cdot\{\{\mu^{-1}\mathbf{\Pi}_h(\nabla\times\mathbf{u})\}\}\,\mathrm{d}A.\end{aligned}$$

Therefore, we conclude that

$$\begin{aligned}(\varepsilon\mathbf{u}_{tt},\mathbf{v}) + (\sigma\mathbf{u}_t,\mathbf{v}) + \widetilde{a}_h(\mathbf{u},\mathbf{v}) &= (\varepsilon\mathbf{u}_{tt} + \sigma\mathbf{u}_t + \nabla\times(\mu^{-1}\nabla\times\mathbf{u}),\mathbf{v}) + r_h(\mathbf{u};\mathbf{v}) \\ &= (\mathbf{f},\mathbf{v}) + r_h(\mathbf{u};\mathbf{v}),\end{aligned}$$

where in the last step we have used the fact that $\mathbf{u}$ solves (1). This immediately yields the desired error equation.

5.3. Approximation results

In this section, we provide the approximation results that we need to prove Theorems 2 and 3.

To begin we recall the approximation properties of the L^2-projection; see [5]. Here, we denote by $|\cdot|_{1,D}$ the standard semi-norm on the Sobolev space $H^1(D)^3$.

Lemma 7. *Let $K \in \mathcal{T}_h$. Then*

(i) *For* $\mathbf{v} \in H^s(K)^3$, $s \geqslant 0$, *we have*

$$\|\mathbf{v} - \mathbf{\Pi}_h \mathbf{v}\|_{0,K} \leqslant C h_K^{\min\{s,\ell+1\}} \|\mathbf{v}\|_{s,K}.$$

(ii) *For* $\mathbf{v} \in H^{1+s}(K)^3$, $s > 0$, *we have*

$$|\mathbf{v} - \mathbf{\Pi}_h \mathbf{v}|_{1,K} \leqslant C h_K^{\min\{s,\ell\}} \|\mathbf{v}\|_{1+s,K}.$$

(iii) *For* $\mathbf{v} \in H^s(K)^3$, $s > \frac{1}{2}$, *we have*

$$\|\mathbf{v} - \mathbf{\Pi}_h \mathbf{v}\|_{0,\partial K} \leqslant C h_K^{\min\{s-1/2,\ell+1/2\}} \|\mathbf{v}\|_{s,K}.$$

The constants C are independent of the local mesh sizes and only depend on the shape-regularity of the mesh, the approximation order ℓ, and the regularity exponent s.

The approximation properties in Lemma 7 imply the following result.

Lemma 8. *Let* $\mathbf{u} \in H^{1+s}(\Omega)^3$, *for* $s > \frac{1}{2}$. *Then we have*

$$\|\mathbf{u} - \mathbf{\Pi}_h \mathbf{u}\|_h \leqslant C_A h^{\min\{s,\ell\}} \|\mathbf{u}\|_{1+s,\Omega},$$

with a constant C_A that is independent of the mesh size and only depends on α, the bounds for the coefficients μ and ε, the shape-regularity of the mesh, the constant κ of the mesh variation, and the approximation order ℓ.

Similarly, the approximation properties for the L^2-projection and the Cauchy–Schwarz inequality imply that $r_h(\mathbf{u};\mathbf{v})$ in (3) can be bounded as follows; cf. [13, Proposition 6.2] or [12, Lemma 4.9].

Lemma 9. *Let $\mathbf{u}$ be such that* $\nabla \times \mathbf{u} \in H^s(\Omega)^3$, *for* $s > \frac{1}{2}$. *Then,* $r_h(\mathbf{u};\mathbf{v})$, *defined in* (3), *satisfies*

$$|r_h(\mathbf{u};\mathbf{v})| \leqslant C_R h^{\min\{s,\ell+1\}} |\mathbf{v}|_h \|\nabla \times \mathbf{u}\|_{s,\Omega}, \quad \mathbf{v} \in \mathbf{V}(h),$$

with a constant C_R that is independent of the mesh size and only depends on α, the bounds for the coefficient μ, the shape-regularity of the mesh, the constant κ of the mesh variation, and the approximation order ℓ.

Consequently, we also obtain the following result.

Lemma 10. *Let $\mathbf{u}$ satisfy*

$$\nabla \times \mathbf{u} \in L^\infty(J; H^s(\Omega)^3), \quad \nabla \times \mathbf{u}_t \in L^\infty(J; H^s(\Omega)^3),$$

for $s > \frac{1}{2}$. *Let* $\mathbf{v} \in C^0(\overline{J}; \mathbf{V}^h)$ *and* $\mathbf{v}_t \in L^\infty(\overline{J}; \mathbf{V}^h)$. *Then there holds*

$$\int_J |r_h(\mathbf{u};\mathbf{v}_t)|\,\mathrm{d}t \leqslant C_R h^{\min\{s,\ell+1\}} |\mathbf{v}|_{L^\infty(J;\mathbf{V}(h))} \cdot (2\|\nabla \times \mathbf{u}\|_{L^\infty(J;H^s(\Omega)^3)} + T\|\nabla \times \mathbf{u}_t\|_{L^\infty(J;H^s(\Omega)^3)}),$$

with C_R denoting the constant from Lemma 9.

Proof. Using integration by parts, we have

$$\begin{aligned}\int_J r_h(\mathbf{u};\mathbf{v}_t)\,\mathrm{d}t &= \int_J \sum_{f\in\mathscr{F}_h}\int_f [\![\mathbf{v}_t]\!]_T\cdot\{\{\mu^{-1}(\nabla\times\mathbf{u})-\mu^{-1}\mathbf{\Pi}_h(\nabla\times\mathbf{u})\}\}\,\mathrm{d}A\,\mathrm{d}t\\ &= -\int_J \sum_{f\in\mathscr{F}_h}\int_f [\![\mathbf{v}]\!]_T\cdot\{\{\mu^{-1}(\nabla\times\mathbf{u}_t)-\mu^{-1}\mathbf{\Pi}_h(\nabla\times\mathbf{u}_t)\}\}\,\mathrm{d}A\,\mathrm{d}t\\ &\quad+\left[\sum_{f\in\mathscr{F}_h}\int_f [\![\mathbf{v}]\!]_T\cdot\{\{\mu^{-1}(\nabla\times\mathbf{u})-\mu^{-1}\mathbf{\Pi}_h(\nabla\times\mathbf{u})\}\}\,\mathrm{d}A\right]_{t=0}^{t=T}\\ &= -\int_J r_h(\mathbf{u}_t;\mathbf{v})\,\mathrm{d}t+[r_h(\mathbf{u};\mathbf{v})]_{t=0}^{t=T}.\end{aligned}$$

Lemma 9 then implies

$$|[r_h(\mathbf{u};\mathbf{v})]_{t=0}^{t=T}|\leqslant 2C_R h^{\min\{s,\ell\}}|\mathbf{v}|_{L^\infty(J;\mathbf{V}(h))}\|\nabla\times\mathbf{u}\|_{L^\infty(J;H^s(\Omega)^3)}.$$

Similarly, using Hölder's inequality,

$$\left|\int_J r_h(\mathbf{u}_t;\mathbf{v})\,\mathrm{d}t\right|\leqslant C_R h^{\min\{s,\ell\}}T|\mathbf{v}|_{L^\infty(J;\mathbf{V}(h))}\|\nabla\times\mathbf{u}_t\|_{L^\infty(J;H^s(\Omega)^3)}.$$

This concludes the proof. □

Finally, we recall an approximation result for the Nédélec interpolant $\mathbf{\Pi}_N$ of the first kind that; see [19,18]. This result is restricted to conforming meshes $\mathscr{T}_h$; cf. Theorem 3.

Lemma 11. *Let $\mathscr{T}_h$ be a conforming triangulation of the domain Ω into tetrahedra or hexahedra, with edges parallel to the coordinate axes, and assume that $\mathbf{u}\in H^s(\Omega)^3$, $\nabla\times\mathbf{u}\in H^s(\Omega)^3$, for $s>\frac{1}{2}$. Then, we have*

$$\|\mathbf{u}-\mathbf{\Pi}_N\mathbf{u}\|_{0,\Omega}+\|\nabla\times(\mathbf{u}-\mathbf{\Pi}_N\mathbf{u})\|_{0,\Omega}\leqslant Ch^{\min\{s,\ell\}}(\|\mathbf{u}\|_{s,\Omega}+\|\nabla\times\mathbf{u}\|_{s,\Omega}),$$

with a constant $C>0$ that is independent of the mesh size and only depends on the shape-regularity of the mesh and the approximation order ℓ.

Since for $\mathbf{u}\in H_0(\mathrm{curl};\Omega)$ the jumps $[\![\mathbf{u}-\mathbf{\Pi}_N\mathbf{u}]\!]_T$ vanish, Lemma 11 implies the following approximation result.

Lemma 12. *Let $\mathscr{T}_h$ be a conforming triangulation of the domain Ω into tetrahedra or hexahedra, with edges parallel to the coordinate axes, and assume that $\mathbf{u}\in H^s(\Omega)^3$, $\nabla\times\mathbf{u}\in H^s(\Omega)^3$, for $s>\frac{1}{2}$. Then, we have*

$$\|\mathbf{u}-\mathbf{\Pi}_N\mathbf{u}\|_h\leqslant C_N h^{\min\{s,\ell\}}(\|\mathbf{u}\|_{s,\Omega}+\|\nabla\times\mathbf{u}\|_{s,\Omega}),$$

with a constant $C_N>0$ that is independent of the mesh size and only depends the bounds for the coefficients μ and ε, the shape-regularity of the mesh and the approximation order ℓ.

5.4. Proof of Theorem 2

Set $\mathbf{e}=\mathbf{u}-\mathbf{u}^h=\eta+\theta$ with $\eta=\mathbf{u}-\mathbf{\Pi}_h\mathbf{u}$ and $\theta=\mathbf{\Pi}_h\mathbf{u}-\mathbf{u}^h$. Using the symmetry of the form $\widetilde{a}_h$ and the error equation in Lemma 6, we obtain for any $t\in J$

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\|\varepsilon^{1/2}\mathbf{e}_t\|_{0,\Omega}^2+\widetilde{a}_h(\mathbf{e},\mathbf{e}))+\|\sigma^{1/2}\mathbf{e}_t\|_{0,\Omega}^2 &= (\varepsilon\mathbf{e}_{tt},\mathbf{e}_t)+\widetilde{a}_h(\mathbf{e},\mathbf{e}_t)+(\sigma\mathbf{e}_t,\mathbf{e}_t)\\ &= (\varepsilon\mathbf{e}_{tt},\eta_t)+\widetilde{a}_h(\mathbf{e},\eta_t)+(\sigma\mathbf{e}_t,\eta_t)+r_h(\mathbf{u};\theta_t).\end{aligned}$$

Integrating this identity over $(0, s)$, $s \in J$, and using the fact that $0 \leqslant \|\sigma^{1/2}\mathbf{e}_t\|_{0,\Omega}^2$ yields

$$\begin{aligned}\frac{1}{2}\|\varepsilon^{1/2}\mathbf{e}_t(s)\|_{0,\Omega}^2 + \frac{1}{2}\widetilde{a}_h(\mathbf{e}(s), \mathbf{e}(s)) \leqslant \frac{1}{2}\|\varepsilon^{1/2}\mathbf{e}_t(0)\|_{0,\Omega}^2 + \frac{1}{2}\widetilde{a}_h(\mathbf{e}(0), \mathbf{e}(0))\\ + \int_0^s (\varepsilon\mathbf{e}_{tt}, \eta_t)\,\mathrm{d}t + \int_0^s \widetilde{a}_h(\mathbf{e}, \eta_t)\,\mathrm{d}t\\ + \int_0^s (\sigma\mathbf{e}_t, \eta_t)\,\mathrm{d}t + \int_0^s r_h(\mathbf{u}; \theta_t)\,\mathrm{d}t.\end{aligned}$$

By integration by parts, we rewrite the third term on the right-hand side above as follows:

$$\int_0^s (\varepsilon\mathbf{e}_{tt}, \eta_t)\,\mathrm{d}t = -\int_0^s (\varepsilon\mathbf{e}_t, \eta_{tt})\,\mathrm{d}t + [(\varepsilon\mathbf{e}_t, \eta_t)]_{t=0}^{t=s}.$$

Taking into account the continuity and coercivity properties of $\widetilde{a}_h$ in Lemma 5, and using standard Hölder inequalities, we conclude that

$$\begin{aligned}\frac{1}{2}\|\varepsilon^{1/2}\mathbf{e}_t(s)\|_{0,\Omega}^2 + \frac{1}{2}C_{\mathrm{coer}}|\mathbf{e}(s)|_h^2 \leqslant \frac{1}{2}\|\varepsilon^{1/2}\mathbf{e}_t(0)\|_{0,\Omega}^2 + \frac{1}{2}C_{\mathrm{cont}}|\mathbf{e}(0)|_h^2\\ + \|\varepsilon^{1/2}\mathbf{e}_t\|_{L^\infty(J;L^2(\Omega)^3)}(\|\varepsilon^{1/2}\eta_{tt}\|_{L^1(J;L^2(\Omega)^3)} + 2\|\varepsilon^{1/2}\eta_t\|_{L^\infty(J;L^2(\Omega)^3)})\\ + C_{\mathrm{cont}}T|\mathbf{e}|_{L^\infty(J;\mathbf{V}(h))}|\eta_t|_{L^\infty(J;\mathbf{V}(h))}\\ + \left|\int_J (\sigma\mathbf{e}_t, \eta_t)\,\mathrm{d}t\right| + \left|\int_J r_h(\mathbf{u}; \theta_t)\,\mathrm{d}t\right|.\end{aligned}$$

Since this inequality holds for any $s \in J$, we obtain

$$\|\varepsilon^{1/2}\mathbf{e}_t\|_{L^\infty(J;L^2(\Omega)^3)}^2 + C_{\mathrm{coer}}|\mathbf{e}|_{L^\infty(J;\mathbf{V}(h))}^2 \leqslant \|\varepsilon^{1/2}\mathbf{e}_t(0)\|_{0,\Omega}^2 + C_{\mathrm{cont}}|\mathbf{e}(0)|_h^2 + T_1 + T_2 + T_3 + T_4,$$

with

$$\begin{aligned}T_1 &= 2\|\varepsilon^{1/2}\mathbf{e}_t\|_{L^\infty(J;L^2(\Omega)^3)}\left(\|\varepsilon^{1/2}\eta_{tt}\|_{L^1(J;L^2(\Omega)^3)} + 2\|\varepsilon^{1/2}\eta_t\|_{L^\infty(J;L^2(\Omega)^3)}\right),\\ T_2 &= 2C_{\mathrm{cont}}T|\mathbf{e}|_{L^\infty(J;\mathbf{V}(h))}|\eta_t|_{L^\infty(J;\mathbf{V}(h))},\\ T_3 &= 2\int_J |(\sigma\mathbf{e}_t, \eta_t)|\,\mathrm{d}t,\\ T_4 &= 2\int_J |r_h(\mathbf{u}; \theta_t)|\,\mathrm{d}t.\end{aligned}$$

Using the geometric-arithmetic mean inequality, the bounds for ε and the approximation results for the L^2-projection in Lemma 7 gives

$$T_1 \leqslant \tfrac{1}{4}\|\varepsilon^{1/2}\mathbf{e}_t\|_{L^\infty(J;L^2(\Omega)^3)}^2 + Ch^{2\min\{s,\ell\}}\left(\|\mathbf{u}_{tt}\|_{L^1(J;H^s(\Omega)^3)}^2 + h^2\|\mathbf{u}_t\|_{L^\infty(J;H^{1+s}(\Omega)^3)}^2\right).$$

Similarly, using the approximation result in Lemma 8,

$$T_2 \leqslant \tfrac{1}{4}C_{\mathrm{coer}}|\mathbf{e}|_{L^\infty(J;\mathbf{V}(h))}^2 + CT^2h^{2\min\{s,\ell\}}\|\mathbf{u}_t\|_{L^\infty(J;H^{1+s}(\Omega)^3)}^2.$$

Due to the bounds for σ and ε we obtain

$$\begin{aligned}T_3 &\leqslant 2T\|\sigma^{1/2}\mathbf{e}_t\|_{L^\infty(J;L^2(\Omega)^3)}\|\sigma^{1/2}\eta_t\|_{L^\infty(J;L^2(\Omega)^3)}\\ &\leqslant \tfrac{1}{4}\|\varepsilon^{1/2}\mathbf{e}_t\|_{L^\infty(J;L^2(\Omega)^3)}^2 + CT^2h^{2\min\{s,\ell\}+2}\|\mathbf{u}_t\|_{L^\infty(J;H^{1+s}(\Omega)^3)}^2.\end{aligned}$$

It remains to bound the term T_4. To do so, we use Lemma 10 and obtain

$$T_4 \leqslant 2C_R h^{\min\{s,\ell\}} |\theta|_{L^\infty(J;\mathbf{V}(h))} \mathscr{R},$$

with

$$\mathscr{R} := (2\|\nabla \times \mathbf{u}\|_{L^\infty(J;H^s(\Omega)^3)} + T\|\nabla \times \mathbf{u}_t\|_{L^\infty(J;H^s(\Omega)^3)}).$$

The triangle inequality, the geometric–arithmetic mean inequality, and the approximation properties in Lemma 8 then yield

$$T_4 \leqslant \tfrac{1}{4} C_{\mathrm{coer}} |\mathbf{e}|^2_{L^\infty(J;\mathbf{V}(h))} + Ch^{2\min\{s,\ell\}} \left(\|\mathbf{u}\|^2_{L^\infty(J;H^{1+s}(\Omega)^3)} + T^2 \|\mathbf{u}_t\|^2_{L^\infty(J;H^{1+s}(\Omega)^3)} \right).$$

Combining the above estimates for T_1, T_2, T_3 and T_4 shows that

$$\begin{aligned} \tfrac{1}{2}\|\varepsilon^{1/2}\mathbf{e}_t\|^2_{L^\infty(J;L^2(\Omega)^3)} + \tfrac{1}{2} C_{\mathrm{coer}} |\mathbf{e}|^2_{L^\infty(J;\mathbf{V}(h))} \leqslant{} & \|\varepsilon^{1/2}\mathbf{e}_t(0)\|^2_{0,\Omega} + C_{\mathrm{cont}} |\mathbf{e}(0)|^2_h \\ & + Ch^{2\min\{s,\ell\}} \left(\|\mathbf{u}_{tt}\|^2_{L^1(J;H^s(\Omega)^3)} \right. \\ & \left. + \|\mathbf{u}_t\|^2_{L^\infty(J;H^{1+s}(\Omega)^3)} + \|\mathbf{u}\|^2_{L^\infty(J;H^{1+s}(\Omega)^3)} \right), \end{aligned}$$

with a constant C that is independent of the mesh size. This proves the desired estimate with respect to the semi-norm $|\cdot|_{L^\infty(J;\mathbf{V}(h))}$. The result for the full $L^\infty(J;\mathbf{V}(h))$-norm is readily obtained by noting that

$$\begin{aligned} \|\varepsilon^{1/2}\mathbf{e}(s)\|_{0,\Omega} &\leqslant \left\| \int_0^s \varepsilon^{1/2}\mathbf{e}_t(t)\,\mathrm{d}t \right\|_{0,\Omega} + \|\varepsilon^{1/2}\mathbf{e}(0)\|_{0,\Omega} \\ &\leqslant T\|\varepsilon^{1/2}\mathbf{e}_t\|_{L^\infty(J;L^2(\Omega)^3)} + \|\varepsilon^{1/2}\mathbf{e}(0)\|_{0,\Omega}. \end{aligned}$$

This concludes the proof of Theorem 2. □

5.5. *Proof of Theorem 3*

The proof of the energy estimate in Theorem 3 follows the lines of the proof of Theorem 2. However, due to the lower spatial regularity of the analytical solution $\mathbf{u}$, we replace the L^2-projection $\mathbf{\Pi}_h$ by the Nédélec interpolant of the first kind $\mathbf{\Pi}_N$ from Lemma 11. Analogously, we use Lemma 11 and Lemma 12 to estimate $\mathbf{u} - \mathbf{\Pi}_N\mathbf{u}$, or time derivatives thereof. With these modifications, the proof of Theorem 3 proceeds exactly as in Theorem 2.

6. Concluding remarks

We have presented and analyzed the symmetric interior penalty DG method for the time-dependent Maxwell equations in second-order form. For smooth solutions, we derive optimal a priori error estimates in the energy norm on general finite element meshes (Theorem 2). On conforming meshes, we derive optimal a priori error estimates in the energy norm for low-regularity solutions that have singularities in space (Theorem 3).

Based on discontinuous finite element spaces, the proposed DG method easily handles elements of various types and shapes, irregular non-matching grids, and even locally varying polynomial order. As continuity is only weakly enforced across mesh interfaces, domain decomposition techniques immediately apply. Since the resulting mass matrix is essentially diagonal, the method is inherently parallel and leads to fully explicit methods when coupled with explicit time integration. Moreover, as the stiffness matrix is symmetric positive definite, the interior penalty DG method shares the following important property with the standard continuous Galerkin approach: the semi-discrete formulation conserves (a discrete version of) the energy for all time; therefore, it is non-dissipative.

Optimal a priori error bounds in the L^2-norm are the subject of ongoing work: here the analysis is more involved and will be reported elsewhere in the near future.

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References

[1] A. Buffa, I. Perugia, Discontinuous Galerkin approximation of the Maxwell eigenproblem, Technical Report PV-24, IMATI-CNR Pavia, 2005, Technical Report.

[2] A.C. Cangellaris, D.B. Wright, Analysis of the numerical error caused by the stair-stepped approximation of a conducting boundary in FDTD simulations of electromagnetic phenomena, IEEE Trans. Antennas Propag. 39 (1991) 1518–1525.

[3] M.-H. Chen, B. Cockburn, F. Reitich, High-order RKDG methods for computational electromagnetics, J. Sci. Comput. 22 (2005) 205–226.

[4] E.T. Chung, B. Engquist, Optimal discontinuous Galerkin methods for wave propagation, preprint.

[5] P.G. Ciarlet, The Finite Element Method for Elliptic Problems, North-Holland, Amsterdam, 1978.

[6] B. Cockburn, F. Li, C.-W. Shu, Locally divergence-free discontinuous Galerkin methods for the Maxwell equations, J. Comput. Phys. 194 (2004) 588–610.

[7] B. Cockburn, C.-W. Shu, TVB Runge–Kutta local projection discontinuous Galerkin method for conservation laws II: General framework, Math. Comput. 52 (186) (1989) 411–435.

[8] A. Elmkies, P. Joly, Finite elements and mass lumping for Maxwell's equations: the 2d case, Numerical Analysis, C.R. Acad. Sci. Paris 324 (1997) 1287–1293.

[9] L. Fezoui, S. Lanteri, S. Lohrengel, S. Piperno, Convergence and stability of a discontinuous Galerkin time-domain methods for the 3D heterogeneous Maxwell equations on unstructured meshes, Modél. Math. Anal. Numér. 39 (6) (2005) 1149–1176.

[10] M.J. Grote, A. Schneebeli, D. Schötzau, Discontinuous Galerkin finite element method for the wave equation, Technical Report 3, Mathematisches Institut, Universität Basel, 2005, SIAM J. Numer. Anal., in press.

[11] J.S. Hesthaven, T. Warburton, Nodal high-order methods on unstructured grids. I. Time-domain solution of Maxwell's equations, J. Comput. Phys. 181 (2002) 186–221.

[12] P. Houston, I. Perugia, A. Schneebeli, D. Schötzau, Interior penalty method for the indefinite time-harmonic Maxwell equations, Numer. Math. 100 (2005) 485–518.

[13] P. Houston, I. Perugia, D. Schötzau, Mixed discontinuous Galerkin approximation of the Maxwell operator, SIAM J. Numer. Anal. 42 (2004) 434–459.

[14] B. Jiang, J. Wu, L.A. Povinelli, The origin of spurious solutions in computational electromagnetics, J. Comput. Phys. 125 (1996) 104.

[15] D. Kopriva, S.L. Woodruff, M.Y. Hussaini, Discontinuous spectral element approximation of Maxwell's equations, in: B. Cockburn, G.E. Karniadakis, C.-W. Shu (Eds.), Discontinuous Galerkin Methods: Theory, Computation and Applications, Lecture Notes in Computer Science Engineering, vol. 11, Springer, Berlin, 2000, pp. 355–362.

[16] R.L. Lee, N.K. Madsen, A mixed finite element formulation for Maxwell's equations in the time domain, J. Comput. Phys. 88 (1990) 284–304.

[17] P. Monk, A comparison of three mixed methods for the time-dependent Maxwell's equations, SIAM J. Sci. Statist. Comput. 13 (1992) 1097–1122.

[18] P. Monk, Finite Element Methods for Maxwell's Equations, Oxford University Press, New York, 2003.

[19] J.C. Nédélec, Mixed finite elements in $\mathbb{R}^3$, Numer. Math. 35 (1980) 315–341.

[20] J.C. Nédélec, A new family of mixed finite elements in $\mathbb{R}^3$, Numer. Math. 50 (1986) 57–81.

[21] K.D. Paulsen, D.R. Lynch, Elimination of vector parasities in finite element Maxwell solutions, IEEE Trans. Microwave Theory Tech. 39 (1991) 395–404.

[22] I. Perugia, D. Schötzau, An *hp*-analysis of the local discontinuous Galerkin method for diffusion problems, J. Sci. Comput. 17 (2002) 561–571.

[23] B. Rivière, M.F. Wheeler, Discontinuous finite element methods for acoustic and elastic wave problems, I: Semidiscrete error estimates, Technical Report 01-02, TICAM, UT, Austin, 2001.

[24] B. Rivière, M.F. Wheeler, Discontinuous finite element methods for acoustic and elastic wave problems, ICM2002-Beijing Satellite Conference on Scientific Computing, Contemporary Mathematics, vol. 329, American Mathematical Society, Providence, RI, 2003, pp. 271–282.

[25] A. Taflove, Computational Electrodynamics: The Finite-Difference Time-Domain Method, Artech House, Boston, 1995.

[26] T. Warburton, Application of the discontinuous Galerkin method to Maxwell's equations using unstructured polymorphic *hp*-finite elements, in: B. Cockburn, G.E. Karniadakis, C.-W. Shu, (Eds.), Discontinuous Galerkin Methods: Theory, Computation and Applications, Lecture Notes in Computer Science Engineering, vol. 11, Springer, Berlin, 2000, pp. 451–458.

[27] T. Warburton, M. Embree, The role of the penalty in the local discontinuous Galerkin method for Maxwell's eigenvalue problem, Comput. Methods Appl. Mech. Eng. 195 (2006) 3205–3223.

[28] K.S. Yee, Numerical solution of initial boundary value problems involving Maxwell's equations in isotropic media, IEEE Trans. Antennas Propag. 14 (1966) 302–307.