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Generalizing Cotilting Dualities

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A generalization of cotilting bimodules and cotilting dualities is studied. Moreover, we investigate the structure and the properties of the classes involved in this kind of dualities.

Cotilting bimodules over arbitrary rings give rise to a theory which naturally generalizes Morita duality. Finitely generated cotilting modules were first studied by Colby [Cb1] and Colby and Fuller [CbF] in case of noetherian rings. Later on, Colby [Cb2] investigated the more general class of generalized Morita dualities, which is a class of representable dualities. The first results on cotilting modules and bimodules over arbitrary rings appeared more recently in [CDT, CTT, C, CF]. In particular, Colpi in 1998 [C] proved that a cotilting bimodule induces a pair of dualities between the torsion and the torsion free classes of suitable large subcategories, by means of a "cotilting theorem" [C, Theorem 6]. But neither Colby [Cb2] nor Colpi [C] proved the existence of a natural transformation between the pair of functors acting on the torsion classes.

In this article, we prove a similar theorem to [C, Theorem 6] which holds in a more general setting. In particular, we focus on the construction of a natural morphism γ giving rise to a duality between the torsion classes of the subcategories we are dealing with. Moreover, we investigate the structure and the properties of the classes which are involved in the dualities described by the cotilting theorem.

Throughout this article, we denote by R and S two arbitrary associative rings with unit and by Mod-R and S-Mod we denote the categories of all unitary right R- and left S-modules, respectively. Given a module U, we denote by $\mathrm{Cogen}(U)$ the class of all modules cogenerated by U and

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by $\mathrm{Rej}_U(-)$ the *reject* radical, defined by $\mathrm{Rej}_U(M) = \bigcap\{\mathrm{Ker}(f) \mid f \in \mathrm{Hom}(M, U)\}$.

Given a bimodule ${}_SU_R$, we denote by Δ both the contravariant functors $\mathrm{Hom}_R(-, U)$ and $\mathrm{Hom}_S(-, U)$ and by Γ both the contravariant functors $\mathrm{Ext}_R(-, U)$ and $\mathrm{Ext}_S(-, U)$. By Δ^2, (respectively Γ^2), we mean both the composition $\Delta_R \circ \Delta_S$ and $\Delta_S \circ \Delta_R$ ($\Gamma_R \circ \Gamma_S$ and $\Gamma_S \circ \Gamma_R$, respectively). For any module M, we denote by δ_M the evaluation morphism defined by δ_M: $M \to \Delta^2(M)$, $x \mapsto [\,\xi \mapsto (x)]$. Since the functors Δ_R and Δ_S are right adjoint with the morphisms δ as unities, the identity $\Delta(\delta_L) \circ \delta_{\Delta(L)} = \mathrm{id}_{\Delta}(L)$ holds for any module L. Moreover, given a module L, $\mathrm{Ker}\,\delta_L = \mathrm{Rej}_U(L)$.

A module M belongs to $\mathrm{Cogen}(U)$ if and only if δ_M is a monomorphism. If δ_M is an isomorphism, we say that M is Δ-reflexive. From the previous identity, it follows that $\Delta(M) \in \mathrm{Cogen}(U)$ for any module M and that $\Delta(M)$ is Δ-reflexive if so is M.

Finally, as in [C], we introduce the following classes of right R-modules (respectively, left S-modules).

(i) $\mathscr{Y} = \{M \mid M \text{ is } \Delta\text{-reflexive}\} \subseteq \mathrm{Cogen}(U)$;

(ii) $\overline{\mathscr{Y}} = \{L \mid L \leq M, \text{ for some } M \in \mathscr{Y}\}$;

(iii) $\mathscr{C} = \{M \mid M \cong L/N, \text{ for some } L, N \in \mathscr{Y}\}$;

(iv) $\mathscr{X} = \mathscr{C} \cap \mathrm{Ker}\,\Delta$.

For further notation, we refer to [AF] or [W].

1. Definitions. We say that a right R-module U_R is *cotilting* if it satisfies the condition $\mathrm{Cogen}\, U_R = \mathrm{Ker}\, \Gamma_R$ (see [CDT, Definition 1.6, Proposition 1.7]) and is *quasi-cotilting* if $\mathrm{Cogen}\, U_R \subseteq \mathrm{Ker}\, \Gamma_R$.

A *locally quasi-cotilting bimodule* is a bimodule ${}_SU_R$ such that $\overline{\mathscr{Y}} \subseteq \mathrm{Ker}\,\Gamma$ on both sides.

A *quasi-cotilting bimodule* is a bimodule ${}_SU_R$ which is quasi-cotilting on both sides.

A *cotilting bimodule* is a faithfully balanced bimodule ${}_SU_R$ which is cotilting on both sides.

2. Examples. The following example is due to G. D'Este: let k be an algebraically closed field and let S be the k-algebra given by the quiver $1 \underset{\beta}{\overset{\alpha}{\rightleftarrows}} 2 \overset{\gamma}{\leftarrow} 3$ with relation $\alpha\beta = 0$ and $\beta\gamma = 0$. Let ${}_SU = {}^{1}_{2} \oplus {}^{2}_{1} \oplus {}^{3}_{2}$ and $R \cong k \oplus k$ be the k-algebra given by the quiver 4. 5.. Then ${}_SU_R$ is a bimodule such that: (1) the indecomposable module ${}^{1}_{2} \in \mathrm{Cogen}({}_SU) \setminus \mathrm{Ker}\,\mathrm{Ext}^1_S(-, U)$; (2) $2 \in \mathrm{Ker}\,\mathrm{Ext}^1_S(-, U)$; (3) the projective S-module ${}^{3}_{2}$ is Δ-reflexive and it is the only indecomposable Δ-reflexive module. Therefore, ${}_SU_R$ is a bimodule such that ${}_SU$ satisfies the condition ${}_S\mathscr{Y} \subseteq \mathrm{Ker}\,\Gamma_S$ but it is not quasi-cotilting.

Any partial cotilting module [CTT, Definition 2.5] is a quasi-cotilting module, so [CTT, Proposition 2.15] gives examples of quasi-cotilting modules which are not cotilting.

Moreover, any GMD bimodule in the sense of Colby [Cb2, Definition 1, Proposition 3] is a locally quasi-cotilting bimodule and any weakly cotilting bimodule in the sense of Tonolo [T, Section 2] is a quasi-cotilting bimodule.

3. *Remark.* As proved in [C, Lemma 2], if U_R is a cotilting module, then $\operatorname{Im}\Delta_S \subseteq \operatorname{Ker}\Gamma_R$ and $(\operatorname{Ker}\Delta_R, \operatorname{Ker}\Gamma_R)$ is a torsion theory in Mod-R. For a given module $M_R \in \operatorname{Cogen} U_R$, it follows that M is Δ-reflexive if and only if $\Delta(M)$ is Δ-reflexive (see [C, Proposition 5]).

If U_R is a quasi-cotilting module, since $\operatorname{Cogen}(U_R) \subseteq \operatorname{Ker}\Gamma_R$, the radical $\operatorname{Rej}_U(-)$ is still idempotent and so $(\operatorname{Ker}\Delta_R, \operatorname{Cogen} U_R)$ is a torsion theory in Mod-R. It still holds that $\operatorname{Im}\Delta_S \subseteq \operatorname{Ker}\Gamma_R$; nevertheless, if U_R is not cotilting, for a module M such that $M \in \operatorname{Cogen} U_R$ and $\Delta(M) \in \mathscr{Y}$, we cannot use Colpi's argument to prove that $M \in \mathscr{Y}$.

For analogous reasons, if U_R is locally quasi-cotilting, we do not know whether $\operatorname{Cogen}(U_R)$ is a torsion free class in Mod-R or whether $\operatorname{Im}\Delta_S \subseteq \operatorname{Ker}\Gamma_R$.

The following technical lemmas point out some properties of the classes $\mathscr{Y}$ and $\mathscr{C}$, when U is a locally quasi-cotilting bimodule.

4. LEMMA. *Let ${}_SU_R$ be a locally quasi-cotilting bimodule and let* $0 \to L \to M \xrightarrow{f} N \to 0$ *be an exact sequence*;

(i) *if* $L, N \in \mathscr{Y}$, *then* $M \in \mathscr{Y}$;

(ii) *if* $M, N \in \mathscr{Y}$, *then* $L \in \mathscr{Y}$;

(iii) *if* ${}_SU_R$ *is quasi-cotilting*, $M \in \mathscr{Y}$, *and* $N \in \operatorname{Cogen} U$, *then* L *and* $N \in \mathscr{Y}$.

Proof. (i) As $N, \Delta(L) \in \mathscr{Y} \subseteq \operatorname{Ker}\Gamma$, we get the commutative diagram with exact rows:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & L & \longrightarrow & M & \longrightarrow & N & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \delta_L} & & \downarrow{\scriptstyle \delta_M} & & \downarrow{\scriptstyle \delta_N} & & \\ 0 & \longrightarrow & \Delta^2(L) & \longrightarrow & \Delta^2(M) & \longrightarrow & \Delta^2(N) & \longrightarrow & 0 \end{array} \qquad (*)$$

Since δ_L and δ_N are isomorphisms, it follows that δ_M is an isomorphism.

(ii) Since δ_N is an isomorphism, for the naturality of the morphism δ, it follows that $\Delta^2(f)$ is epic. Hence, as $N \in \operatorname{Ker}\Gamma$, we get the commutative diagram with exact rows,

$$\begin{tikzcd} 0 \arrow[r] & L \arrow[r] \arrow[d, "\delta_L"] & M \arrow[r, "f"] \arrow[d, "\delta_M", "\cong"'] & N \arrow[r] \arrow[d, "\delta_N", "\cong"'] & 0 \\ 0 \arrow[r] & \Delta^2(L) \arrow[r] & \Delta^2(M) \arrow[r, "\Delta^2(f)"] & \Delta^2(N) \arrow[r] & 0 \end{tikzcd}$$

It follows that δ_L is an isomorphism.

(iii) If ${}_SU_R$ is quasi-cotilting, given an exact sequence $0 \to L \to M \to N \to 0$ in Cogen U, we get the commutative diagram $(*)$. It follows that δ_M is an isomorphism if and only if δ_N and δ_L are both isomorphisms. ∎

5. Lemma. *Let ${}_SU_R$ be a locally quasi-cotilting bimodule and let $M \in \mathscr{C}$. Then*, $\Delta(M) \in \mathscr{Y}$, $\Gamma(M) \in \mathscr{X}$, *and* δ_M *is an epimorphism.*

Proof. From the exact sequence $0 \to Y_1 \overset{k}{\to} Y_2 \overset{\varphi}{\to} M \to 0$ where $Y_1, Y_2 \in \mathscr{Y}$, we get the two exact sequences $0 \to \Delta(M) \to \Delta(Y_2) \overset{\alpha}{\to} C \to 0$ and $0 \to C \overset{\beta}{\to} \Delta(Y_1) \to \Gamma(M) \to 0$ where $\Delta(k) = \beta \circ \alpha$. As ${}_SU_R$ is locally quasi-cotilting, the first sequence is in Ker Γ. Hence, we get the following commutative diagram with exact rows,

$$\begin{tikzcd} 0 \arrow[r] & \Delta(M) \arrow[r] \arrow[d, "\delta_{\Delta(M)}"] & \Delta(Y_2) \arrow[r] \arrow[d, "\delta_{\Delta(Y_2)}"] & C \arrow[r] \arrow[d, "\delta_C"] & 0 \\ 0 \arrow[r] & \Delta^3(M) \arrow[r] & \Delta^3(Y_2) \arrow[r, "\Delta^2(\alpha)"] & \Delta^2(C) \arrow[r] & \Gamma\Delta^2(M) \end{tikzcd}$$

Since $\delta_{\Delta(M)}$ and δ_C are monic and $\delta_{\Delta(Y_2)}$ is an isomorphism, from the Snake lemma it follows that $\delta_{\Delta(M)}$ is an isomorphism. Therefore, $\Gamma\Delta^2(M) = 0$, since $\Delta^2(M)$ is reflexive. It follows that also δ_C is an isomorphism. Thus, $\Delta(M)$, $C \in \mathscr{Y}$, and then $\Gamma(M) \in \mathscr{C}$. Moreover, since δ_{Y_1}, δ_{Y_2} are isomorphisms and k is monic, also $\Delta^2(k) = \Delta(\alpha) \circ \Delta(\beta)$, and hence $\Delta(\beta)$, is monic. Thus, from the exact sequence $0 \to \Delta\Gamma(M) \to \Delta^2(Y_1) \xrightarrow{\Delta(\beta)} \Delta(C) \to \Gamma^2(M) \to 0$ we get $\Gamma(M) \in \operatorname{Ker} \Delta$. Finally, as $C \in \operatorname{Ker} \Gamma$, we obtain the commutative diagram,

$$\begin{tikzcd} Y_2 \arrow[r, "\varphi"] \arrow[d, "\delta_{Y_2}", "\cong"'] & M \arrow[r] \arrow[d, "\delta_M"] & 0 \\ \Delta^2(Y_2) \arrow[r] & \Delta^2(M) \arrow[r] & 0 \end{tikzcd}$$

from which we get that δ_M is an epimorphism. ∎

6. Lemma. *Let ${}_SU_R$ be a locally quasi-cotilting bimodule and let* $0 \to L \to M \to N \to 0$ *be an exact sequence*;

(a) *if* $M \in \mathscr{Y}$, *then* $N \in \mathscr{C}$ *if and only if* $L \in \mathscr{Y}$;

(b) *if* $L \in \mathscr{Y}$, *then* $N \in \mathscr{C}$ *if and only if* $M \in \mathscr{C}$;

(c) *if* $M \in \mathscr{C}$, *then* $L \in \mathscr{C}$ *if and only if* $N \in \mathscr{C}$;

(d) *if* ${}_SU_R$ *is quasi-cotilting and* $M \in \mathscr{C}$, *then* $N \in \mathscr{Y}$ *if and only if* $N \in \mathrm{Cogen}(U)$.

Proof. If $N \in \mathscr{C}$, let $0 \to Y_1 \to Y_2 \to N \to 0$ be an exact sequence where $Y_1, Y_2 \in \mathscr{Y}$. We obtain the commutative diagram with exact rows and columns,

$$\begin{tikzcd}
 & & 0 & 0 & \\
0 \arrow[r] & L \arrow[r] & M \arrow[u] \arrow[r, "f_2"] & N \arrow[u] \arrow[r] & 0 \\
0 \arrow[r] & L \arrow[u, equal] \arrow[r] & P \arrow[u] \arrow[r] & Y_2 \arrow[u, "f_1"'] \arrow[r] & 0 \\
 & & Y_1 \arrow[u] \arrow[r, equal] & Y_1 \arrow[u] & \\
 & & 0 \arrow[u] & 0 \arrow[u] &
\end{tikzcd}$$

where P is the pullback of f_1, f_2.

Looking at the first column, if $M \in \mathscr{Y}$, from Lemma 4(i) it follows that $P \in \mathscr{Y}$. Hence, if we apply Lemma 4(ii) to the second row, we get $L \in \mathscr{Y}$ and so (a) is proved.

If $L \in \mathscr{Y}$, looking at the second row, it follows that $P \in \mathscr{Y}$, thus looking at the first column we conclude that $M \in \mathscr{C}$.

To prove (c) and the second implication in (b), let us suppose that $M \in \mathscr{C}$. Then, we get the following commutative diagram with exact rows and columns,

$$\begin{tikzcd}
 & & 0 & 0 & & \\
0 \arrow[r] & L \arrow[r] & M \arrow[u] \arrow[r] & N \arrow[u] \arrow[r] & 0 & \\
 & & Y_2 \arrow[u] \arrow[r, equal] & Y_2 \arrow[u] & & \\
 & 0 \arrow[r] & Y_1 \arrow[u] \arrow[r] & K \arrow[u] \arrow[r] & L \arrow[r] & 0 \\
 & & 0 \arrow[u] & 0 \arrow[u] & &
\end{tikzcd} \qquad (*)$$

where Y_1 and Y_2 are in $\mathscr{Y}$. Looking at the last row, assuming $L \in \mathscr{C}$, from (b) it follows that $K \in \mathscr{Y}$ and so $N \in \mathscr{C}$. Similarly, if $N \in \mathscr{C}$, from the second column we get $K \in \mathscr{Y}$ and so $L \in \mathscr{C}$.

Now, let us suppose ${}_SU_R$ to be a quasi-cotilting bimodule and $M \in \mathscr{C}$. If $N \in \operatorname{Cogen}(U)$, since the second column of $(*)$ is in $\operatorname{Cogen}(U)$, by Lemma 4(iii), it follows that $N \in \mathscr{Y}$. ∎

With the following construction (see also [C, Theorem 6]), for any module in $\mathscr{C}$ we define a new morphism $\gamma_M\colon \Gamma^2(M) \to M$. The restriction of γ to $\mathscr{X}$ will be a natural isomorphism giving rise to a duality between the classes $\mathscr{X}$.

7. *Construction of* γ. Let ${}_SU_R$ be a locally quasi-cotilting bimodule, $M \in \mathscr{C}$ and let $0 \to Y_1 \overset{\alpha}{\to} Y_2 \overset{\beta}{\to} M \to 0$ be an exact sequence with $Y_1, Y_2 \in \mathscr{Y}$. If $L = \operatorname{Coker} \Delta(\beta)$, we get the two exact sequences,

$$0 \to \Delta(M) \xrightarrow{\Delta(\beta)} \Delta(Y_2) \overset{\pi}{\to} L \to 0 \qquad 0 \to L \overset{j}{\to} \Delta(Y_1) \to \Gamma(M) \to 0,$$

where $\Delta(\alpha) = j \circ \pi$, $L \in \operatorname{Ker} \Gamma$ and, by Lemma 5, $\Gamma(M) \in \mathscr{X}$. Hence, we obtain the two short exact sequences,

$$0 \to \Delta(L) \xrightarrow{\Delta(\pi)} \Delta^2(Y_2) \xrightarrow{\Delta^2(\beta)} \Delta^2(M) \to 0,$$
$$0 \to \Delta^2(Y_1) \xrightarrow{\Delta(j)} \Delta(L) \overset{\eta}{\to} \Gamma^2(M) \to 0,$$

where η is the connecting homomorphism. Since $\Delta^2(\alpha) = \Delta(\pi) \circ \Delta(j)$, we get the following commutative diagram where the upper and the bottom row are exact. So

$$\begin{tikzcd}
0 \arrow[r] & Y_1 \arrow[r, "\alpha"'] \arrow[d, "\delta_{Y_1}", "\cong"'] & Y_2 \arrow[r, "\beta"] \arrow[d, "\delta_{Y_2}", "\cong"'] & M \arrow[r] \arrow[d, "\delta_M"] & 0 \\
 & \Delta^2(Y_1) \arrow[r, "\Delta^2(\alpha)"] \arrow[d, equal] & \Delta^2(Y_2) \arrow[r, "\Delta^2(\beta)"] & \Delta^2(M) \arrow[r] & 0 \\
0 \arrow[r] & \Delta^2(Y_1) \arrow[r, "\Delta(j)"] & \Delta(L) \arrow[r, "\eta"] \arrow[u, "\Delta(\pi)"'] & \Gamma^2(M) \arrow[r] & 0 \\
 & & 0 \arrow[u] & &
\end{tikzcd}$$

Since $\Delta(\pi)$ is monic, $\operatorname{Ker} \beta \circ \delta_{Y_2}^{-1} \circ \Delta(\pi) = \operatorname{Im} \Delta(j)$. It follows that $\beta \circ \delta_{Y_2}^{-1} \circ \Delta(\pi)$ induces a monomorphism $\gamma_M\colon \Gamma^2(M) \to M$ with

$$\begin{aligned}\operatorname{Im}(\gamma_M) &= \beta\big(\delta_{Y_2}^{-1}(\operatorname{Im} \Delta(\pi))\big) = \beta\big(\delta_{Y_2}^{-1}(\operatorname{Ker} \Delta^2(\beta))\big) = \operatorname{Ker} \delta_M \\ &= \operatorname{Rej}(M).\end{aligned}$$

Now, we are going to prove a lemma which is crucial to show the naturality of the morphism γ defined above. The proof is inspired by [Cb1, Proposition 2.2].

8. LEMMA. *Let $M, M' \in \mathscr{C}$ and let $f \in \operatorname{Hom}(M, M')$. Let $\gamma_M, \gamma_{M'}$ be two monomorphisms obtained, as in 7, from the exact sequences $0 \to Y_1 \to Y_2 \to M \to 0$ and $0 \to Y_1' \to Y_2' \to M' \to 0$ with $Y_1, Y_2, Y_1', Y_2' \in \mathscr{Y}$. If there exists $g \in \operatorname{Hom}(Y_2, Y_2')$ which extends f, then $f \circ \gamma_M = \gamma_{M'} \circ \Gamma^2(f)$.*

Proof. By assumption, we get the exact commutative diagram defining h,

$$\begin{tikzcd} 0 \arrow[r] & Y_1 \arrow[r, "\alpha"] \arrow[d, "h"] & Y_2 \arrow[r, "\beta"] \arrow[d, "g"] & M \arrow[r] \arrow[d, "f"] & 0 \\ 0 \arrow[r] & Y_1' \arrow[r, "\alpha'"] & Y_2' \arrow[r, "\beta'"] & M' \arrow[r] & 0 \end{tikzcd} \qquad (*)$$

We use the notation of 7. Since η is epic, to get the claim, we prove that

$$f \circ \gamma_M \circ \eta = \gamma_{M'} \circ \Gamma^2(f) \circ \eta.$$

From $(*)$ we obtain the exact commutative diagram defining ν,

$$\begin{tikzcd} 0 \arrow[r] & \Delta(M) \arrow[r, "\Delta(\beta)"] & \Delta(Y_2) \arrow[r, "\pi"] & L \arrow[r] & 0 \\ 0 \arrow[r] & \Delta(M') \arrow[r, "\Delta(\beta')"] \arrow[u, "\Delta(f)"] & \Delta(Y_2') \arrow[r, "\pi'"] \arrow[u, "\Delta(g)"] & L' \arrow[r] \arrow[u, "\nu"] & 0 \end{tikzcd}$$

where $L = \operatorname{Coker} \Delta(\beta)$, $L' = \operatorname{Coker} \Delta(\beta')$, $\Delta(\alpha) = j \circ \pi$, $\Delta(\alpha') = j' \circ \pi'$. Moreover, we get the following diagram with exact rows,

$$\begin{tikzcd} 0 \arrow[r] & L \arrow[r, "j"] \arrow[d, "\nu"] & \Delta(Y_1) \arrow[r] \arrow[d, "\Delta(h)"] & \Gamma(M) \arrow[r] \arrow[d, "\Gamma(f)"] & 0 \\ 0 \arrow[r] & L' \arrow[r, "j'"] & \Delta(Y_1') \arrow[r] & \Gamma(M') \arrow[r] & 0 \end{tikzcd} \qquad (**)$$

which is commutative since

$$\begin{aligned} j \circ \nu \circ \pi' &= j \circ \pi \circ \Delta(g) = \Delta(\alpha) \circ \Delta(g) \\ &= \Delta(g \circ \alpha) = \Delta(\alpha' \circ h) = \Delta(h) \circ j' \circ \pi'. \end{aligned}$$

Finally, from $(**)$, we obtain the commutative diagram,

$$\begin{tikzcd}
0 \arrow[r] & \Delta^2(Y_1) \arrow[r] \arrow[d, "\Delta^2(h)"] & \Delta(L) \arrow[r, "\eta"] \arrow[d, "\Delta(\nu)"] & \Gamma^2(M) \arrow[r] \arrow[d, "\Gamma^2(f)"] & 0 \\
0 \arrow[r] & \Delta^2(Y_1') \arrow[r] & \Delta(L') \arrow[r, "\eta'"] & \Gamma^2(M) \arrow[r] & 0
\end{tikzcd}$$

From the commutativity of all these diagrams and from the naturality of δ, it follows that

$$\begin{aligned}
f \circ \gamma_M \circ \eta &= f \circ \beta \circ \delta_{Y_2}^{-1} \circ \Delta(\pi) = \beta' \circ g \circ \delta_{Y_2}^{-1} \circ \Delta(\pi) \\
&= \beta' \circ \delta_{Y_2'}^{-1} \circ \Delta^2(g) \circ \Delta(\pi) = \beta' \circ \delta_{Y_2'}^{-1} \circ \Delta((\pi \circ \Delta(g))) \\
&= \beta' \circ \delta_{Y_2'}^{-1} \circ \Delta(\nu \circ \pi') = \beta' \circ \delta_{Y_2'}^{-1} \circ \Delta(\pi') \circ \Delta(\nu) \\
&= \gamma_{M'} \circ \eta' \circ \Delta(\nu) = \gamma_{M'} \circ \Gamma^2(f) \circ \eta.
\end{aligned}$$

▮

The following result shows that γ_M is well defined, for any $M \in \mathscr{C}$.

9. Proposition. *Let ${}_S U_R$ be a locally quasi-cotilting bimodule. Let $M \in \mathscr{C}$ and let γ_M be the morphism constructed as in 7. Then, γ_M does not depend on the exact sequence representing M as a factor of modules in $\mathscr{Y}$.*

Proof. Let $0 \to Y_1 \to Y_2 \xrightarrow{f_1} M \to 0$ and let $0 \to Y_1' \to Y_2' \xrightarrow{f_2} M \to 0$ be two exact sequences with $Y_1, Y_2, Y_1', Y_2' \in \mathscr{Y}$. To prove that they define the same morphism γ_M, let us consider the following diagram with exact rows and columns,

$$\begin{tikzcd}
 & & 0 & 0 & 0 \\
0 \arrow[r] & Y_1 \arrow[r] \arrow[d, equal] & Y_2 \arrow[r, "f_1"] \arrow[u] & M \arrow[r] \arrow[u] \arrow[ur] & 0 \\
0 \arrow[r] & Y_1 \arrow[r] & P \arrow[r, "g_2"] \arrow[u, "g_1"] \arrow[ur] & Y_2' \arrow[r] \arrow[u, "f_2"'] & 0 \\
 & L \arrow[ur] & Y_1' \arrow[r, equal] \arrow[u] & Y_1' \arrow[u] & \\
0 \arrow[ur] & & 0 \arrow[u] & 0 \arrow[u] &
\end{tikzcd}$$

where P is the pullback of f_1, f_2. From the second row we get $P \in \mathscr{Y}$ and then, applying Lemma 6 to the diagonal exact sequence $0 \to L \to P \xrightarrow{h} M \to 0$, where $h = f_1 \circ g_1 = f_2 \circ g_2$, it follows $L \in \mathscr{Y}$. Let γ_M' be the morphism associated with the first exact sequence, let γ_M'' be associated with the second one, and let γ_M''' be associated with the diagonal exact sequence. We obtain the two commutative diagrams with exact rows:

$$\begin{tikzcd} 0 \arrow[r] & L \arrow[r] \arrow[d] & P \arrow[r, "h"] \arrow[d, "g_1"] & M \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & Y_1 \arrow[r] & Y_2 \arrow[r, "f_1"] & M \arrow[r] & 0 \end{tikzcd}$$

$$\begin{tikzcd} 0 \arrow[r] & L \arrow[r] \arrow[d] & P \arrow[r, "h"] \arrow[d, "g_2"] & M \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & Y_1' \arrow[r] & Y_2' \arrow[r, "f_2"] & M \arrow[r] & 0 \end{tikzcd}$$

If we apply Lemma 8 with $f = \mathrm{id}$, from the first diagram we conclude $\gamma_M' = \gamma_M'''$ and from the second one we conclude $\gamma_M'' = \gamma_M'''$. Therefore, $\gamma_M' = \gamma_M''$. ∎

10. THEOREM. *Let ${}_SU_R$ be a locally quasi-cotilting bimodule. Then,* γ: $\Gamma^2 \to 1_{\mathscr{C}}$ *is a natural transformation in* $\mathscr{C}$.

Proof. To prove the naturality of γ, we show that the following diagram is commutative for any $M, M' \in \mathscr{C}$ and for any $f \in \mathrm{Hom}(M, M')$:

$$\begin{tikzcd} M \arrow[r, "f"] & M' \\ \Gamma^2(M) \arrow[u, "\gamma_M"] \arrow[r, "\Gamma^2(f)"] & \Gamma^2(M') \arrow[u, "\gamma_M'"'] \end{tikzcd} \qquad (*)$$

Thanks to Proposition 9, we can construct γ_M starting from any exact sequence representing M as a factor of modules in $\mathscr{Y}$. Let $0 \to Y_1' \to Y_2' \xrightarrow{f_1} M' \to 0$ be an exact sequence with $Y_1', Y_2' \in \mathscr{Y}$. If we find an exact sequence $0 \to Y_1 \to Y_2 \to M \to 0$ such that $Y_1, Y_2 \in \mathscr{Y}$ and f can be extended to φ: $Y_2 \to Y_2'$, then we can apply Lemma 8 to prove that $(*)$ commutes. Thus, let P be the pullback of f, f_1. Then, we get the commutative diagram with exact rows:

$$\begin{tikzcd} 0 \arrow[r] & Y_1' \arrow[r] \arrow[d, equal] & P \arrow[r, "g_1"] \arrow[d, "g"'] & M \arrow[r] \arrow[d, "f"] & 0 \\ 0 \arrow[r] & Y_1' \arrow[r] & Y_2' \arrow[r, "f_1"] & M' \arrow[r] & 0 \end{tikzcd}$$

Applying Lemma 6(b) to the first row, it follows that $P \in \mathscr{C}$. Therefore, we get an exact sequence $0 \to \overline{Y}_1 \to \overline{Y}_2 \xrightarrow{h} P \to 0$ with $\overline{Y}_1, \overline{Y}_2 \in \mathscr{Y}$. Finally, we get the commutative diagram with exact rows,

$$\begin{tikzcd} 0 \arrow[r] & \mathrm{Ker}(g_1 \circ h) \arrow[r] \arrow[d] & \overline{Y}_2 \arrow[r, "g_1 \circ h"] \arrow[d, "g \circ h"] & M \arrow[r] \arrow[d, "f"] & 0 \\ 0 \arrow[r] & Y_1' \arrow[r] & Y_2' \arrow[r, "f_1"] & M' \arrow[r] & 0 \end{tikzcd}$$

where, by Lemma 6, $\mathrm{Ker}(g_1 \circ h) \in \mathscr{Y}$. ∎

We are now able to prove the cotilting theorem:

11. THEOREM (Cotilting Theorem). *Let ${}_SU_R$ be a locally quasi-cotilting bimodule. Then*:

(a) $\Delta(\mathscr{C}) \subseteq \mathscr{Y}$ *and* $\Gamma(\mathscr{C}) \subseteq \mathscr{X}$;

(b) *For any $M \in \mathscr{C}$ there is a natural morphism $\gamma_M: \Gamma^2(M) \to M$ such that the canonical sequence*,

$$0 \to \Gamma^2(M) \overset{\gamma_M}{\to} M \overset{\delta_M}{\to} \Delta^2(M) \to 0$$

is exact, with $\mathrm{Im}(\gamma_M) = \mathrm{Rej}_U(M)$;

(c) *The functors Δ and Γ define a pair of dualities $\mathscr{Y} \underset{\Delta}{\overset{\Delta}{\rightleftarrows}} \mathscr{Y}$ and $\mathscr{X} \underset{\Gamma}{\overset{\Gamma}{\rightleftarrows}} \mathscr{X}$. Moreover, $\Delta \restriction \mathscr{X} = 0 = \Gamma \restriction \mathscr{Y}$, $\Delta \restriction \mathscr{Y}$ is an exact functor, and $\Gamma \restriction \mathscr{X}$ is a left exact functor, which is exact if ${}_SU_R$ is a cotilting bimodule.*

Proof. (a) and (b) follow from Lemma 5 and Theorem 10. To prove (c), we first observe that $\mathscr{X} \subseteq \mathrm{Ker}\,\Delta$, $\mathscr{Y} \subseteq \mathrm{Ker}\,\Gamma$, $\Gamma(\mathscr{X}) \subseteq \mathscr{X}$, and $\Delta(\mathscr{Y}) \subseteq \mathscr{Y}$. Moreover, for any $M \in \mathscr{Y}$ ($M \in \mathscr{X}$, respectively), $M \cong \Delta^2(M)$, ($M \cong \Gamma^2(M)$) by means of the natural functor $\delta(\gamma)$. Hence, we have dualities as stated.

Finally, as $\Gamma \restriction \mathscr{Y} = 0 = \Delta \restriction \mathscr{X}$, $\Delta \restriction \mathscr{Y}$ is an exact functor and $\Gamma \restriction \mathscr{X}$ is a left exact functor. If ${}_SU_R$ is a cotilting bimodule, since $\mathrm{injdim}(U) \leq 1$ (see [CDT, Proposition 1.7]), $\Gamma \restriction \mathscr{X}$ is also right exact. ∎

Our next goal is to study the structure and some closure properties of the class $\mathscr{C}$. To this aim, we start assuming that ${}_SU_R$ is a quasi-cotilting bimodule.

12. PROPOSITION. *Let ${}_SU_R$ be a quasi-cotilting bimodule. Then, the class $\mathscr{C}$ is closed under finite direct sums, images, kernels, cokernels, pullbacks, and pushouts of morphisms in $\mathscr{C}$. In particular, $\mathscr{C}$ is an abelian category.*

Proof. From Lemma 4(i), it follows that $\mathscr{Y}$ is closed under finite direct sums. Hence, if $C_1 = Y_1/Y_1'$ and $C_2 = Y_2/Y_2'$ with $Y_1, Y_1', Y_2, Y_2' \in \mathscr{Y}$, then $C_1 \oplus C_2 = Y_1 \oplus Y_1'/Y_1' \oplus Y_2' \in \mathscr{C}$. Now let $C_1, C_2 \in \mathscr{C}$ and $f: C_1 \to C_2$.

Then, we get the following commutative diagram with exact rows and columns:

$$\begin{tikzcd}
 & & 0 & 0 \\
0 \arrow[r] & \mathrm{Ker}\, f \arrow[r] & C_1 \arrow[r, "f"] \arrow[u] & C_2 \arrow[u] \\
0 \arrow[r] & \mathrm{Ker}\, f \arrow[r] \arrow[u, equal] & P \arrow[r, "f_1"] \arrow[u] & Y_1 \arrow[u] \\
 & & Y_2 \arrow[r, equal] \arrow[u] & Y_2 \arrow[u] \\
 & & 0 \arrow[u] & 0 \arrow[u]
\end{tikzcd}$$

If we apply Lemma 6(b) to the first column, we get $P \in \mathscr{C}$. Since $\mathrm{Im}\, f_1 \leq Y_1$, then $\mathrm{Im}\, f_1 \in \mathrm{Cogen}(U)$ and therefore, from Lemma 6(d), $\mathrm{Im}\, f_1 \in \mathscr{Y}$. Hence, $\mathrm{Ker}\, f_1 = \mathrm{Ker}\, f \in \mathscr{C}$. Moreover, applying Lemma 6(c) to the exact sequences,

$$0 \to \mathrm{Ker}\, f \to C_1 \to \mathrm{Im}\, f \to 0 \qquad \text{and} \qquad 0 \to \mathrm{Im}\, f \to C_2 \to \mathrm{Coker}\, f \to 0,$$

we get $\mathrm{Im}\, f$, $\mathrm{Coker}\, f \in \mathscr{C}$.

Since $\mathscr{C}$ is closed under finite direct sums, it follows that it is closed under pullbacks and pushouts. ∎

13. Proposition. *Let ${}_S U_R$ be a quasi-cotilting bimodule. Then, $(\mathscr{X}, \mathscr{Y})$ is a torsion theory in the abelian category $\mathscr{C}$, in the sense that for $M, N \in \mathscr{C}$:*

(1) $\mathrm{Hom}(N, M) = 0$ *for all* $N \in \mathscr{X}$ *and for all* $M \in \mathscr{Y}$,

(2) *if* $\mathrm{Hom}(N, M) = 0$ *for all* $M \in \mathscr{Y}$, *then* $N \in \mathscr{X}$,

(3) *if* $\mathrm{Hom}(N, M) = 0$ *for all* $N \in \mathscr{X}$, *then* $M \in \mathscr{Y}$.

Proof. (1) is clear since $(\mathrm{Ker}\, \Delta, \mathrm{Cogen}\, U)$ is a torsion theory in Mod-R. (2) and (3) follow immediately from Theorem 11(b) (see [S, Chap. VI] for a discussion on torsion theories). ∎

The previous proposition allows us to read the cotilting theorem as natural generalization of Morita duality obtained by relaxing the assumptions on the representing bimodule. In this case, a non-trivial torsion theory takes the place of the whole category.

The class $\mathrm{Ker}\, \Delta$ is obviously closed under direct summands, since Δ is an additive functor. When ${}_S U_R$ is a quasi-cotilting bimodule, this is again the case for $\mathscr{X} \subseteq \mathrm{Ker}\, \Delta$ and $\mathscr{C}$.

14. Lemma. *Let ${}_S U_R$ be a quasi-cotilting bimodule. Then, $\mathscr{X}$ is closed under direct summands.*

Proof. Let $M \in \mathscr{X}$ and $M = N \oplus L$. Then, N and $L \in \operatorname{Ker}\Delta$. From the exact sequences,

$$0 \to L \xrightarrow{i} M \xrightarrow{\pi} N \to 0 \qquad \text{and} \qquad 0 \to Y_1 \to Y_2 \xrightarrow{g} M \to 0,$$

where $Y_1, Y_2 \in \mathscr{Y}$, defining $g_1 = \pi \circ g$, we get the exact sequence,

$$0 \to \operatorname{Ker} g_1 \to Y_2 \xrightarrow{g_1} N \to 0, \tag{$*$}$$

where $\operatorname{Ker} g_1 \in \operatorname{Ker}\Gamma$. To get the thesis, we prove that $\operatorname{Ker} g_1 \in \mathscr{Y}$. Applying the functor Δ to $(*)$, we obtain the exact sequence,

$$0 \to \Delta(Y_2) \to \Delta(\operatorname{Ker} g_1) \to \Gamma(N) \to 0.$$

As Γ is an additive functor, $\Gamma(M) = \Gamma(N) \oplus \Gamma(L)$. Since $M \in \mathscr{C}$, then $\Gamma(M) \in \operatorname{Ker}\Delta$, and so $\Gamma(N) \in \operatorname{Ker}\Delta$. Hence we get the exact commutative diagram defining α,

$$\begin{tikzcd}
0 \arrow[r] & \Delta^2(\operatorname{Ker} g_1) \arrow[r] & \Delta^2(Y_2) \arrow[r] & \Gamma^2(N) \arrow[r] & 0 \\
0 \arrow[r] & \operatorname{Ker} g_1 \arrow[r] \arrow[u, "\delta_{\operatorname{Ker} g_1}"] & Y_2 \arrow[r] \arrow[u, "\delta_{Y_2}", "\cong"'] & N \arrow[r] \arrow[u, "\alpha"] & 0
\end{tikzcd} \tag{$**$}$$

where $\delta_{\operatorname{Ker} g_1}$ is monic. If α is an isomorphism, it follows that $\delta_{\operatorname{Ker} g_1}$ is an isomorphism, i.e., $\operatorname{Ker} g_1 \in \mathscr{Y}$. Therefore, to conclude the proof, we just have to prove that $\gamma_{M|_N}^{-1}\colon N \to \Gamma^2(N)$ is a well-defined isomorphism and that $\alpha = \gamma_{M|_N}^{-1}$. So, let us consider the following diagram,

$$\begin{tikzcd}
M \arrow[r, "\pi", "\oplus"'] \arrow[d, "\gamma_M^{-1}"] & N \arrow[r] \arrow[d, "\alpha"] & 0 \\
\Gamma^2(M) \arrow[r, "\Gamma^2(\pi)", "\oplus"'] & \Gamma^2(N) \arrow[r] & 0
\end{tikzcd} \tag{d1}$$

where $\Gamma^2(N) \overset{\oplus}{\leq} \Gamma^2(M)$ and $\Gamma^2(\pi) = \pi_{\Gamma^2(N)}$. Using the commutativity of the diagram,

$$\begin{tikzcd}
0 \arrow[r] & \operatorname{Ker} g_1 \arrow[r] & Y_2 \arrow[r, "g_1"] & N \arrow[r] & 0 \\
0 \arrow[r] & Y_1 \arrow[r] \arrow[u] & Y_2 \arrow[r, "g"] \arrow[u, equal] & N \oplus L \arrow[r] \arrow[u] & 0
\end{tikzcd} \tag{d2}$$

and of that obtained applying two times the functor Δ to (d2), it is straightforward to verify that (d1) is commutative. Since $\pi(L) = 0$, from (d1) we get $\gamma_M^{-1}(L) \leq \Gamma^2(L)$ and from an analogous diagram with L in place of N we get $\gamma_M^{-1}(N) \leq \Gamma^2(N)$. As $N \oplus L \overset{\gamma_M^{-1}}{\cong} \Gamma^2(N) \oplus \Gamma^2(L)$, it follows $N \overset{\gamma_{M|_N}^{-1}}{\cong} \Gamma^2(N)$ and $L \overset{\gamma_{M|_L}^{-1}}{\cong} \Gamma^2(L)$. In particular, $\gamma_{M|_N}^{-1}$ is an isomorphism. Now, let $n \in N$: then,

$$\alpha(n) = \alpha(\pi(n)) = \Gamma^2(\pi)\big(\gamma_M^{-1}(n)\big) = \gamma_M^{-1}(n),$$

since $\gamma_M^{-1}(n) \in \Gamma^2(N)$ and $\Gamma^2(\pi) = \pi_{\Gamma^2(N)}$. It follows that $\alpha = \gamma_{M|_N}^{-1}$, which concludes the proof. ∎

15. Proposition. *Let ${}_SU_R$ be a quasi-cotilting bimodule. Then, $\mathscr{C}$ is closed under direct summands.*

Proof. Let $M \in \mathscr{C}$ and let $M = N \oplus L$. Since $\operatorname{Rej}_U(-)$ is a radical, it commutes with finite direct sums. Therefore, we get $\operatorname{Rej}_U(N)$, $\operatorname{Rej}_U(L) \overset{\oplus}{\leq} \operatorname{Rej}_U(M)$ and then, by Lemma 14, they both belong to $\mathscr{X}$. Moreover, $N/\operatorname{Rej}_U(N)$, $L/\operatorname{Rej}_U(L) \overset{\oplus}{\leq} M/\operatorname{Rej}_U(M)$ and so, using Lemma 4, they are both Δ-reflexive. From Lemma 6, it follows that $M/\operatorname{Rej}_U(N) \in \mathscr{C}$. Thus, from the exact sequence,

$$0 \to N/\operatorname{Rej}_U(N) \to M/\operatorname{Rej}_U(N) \to M/N \to 0,$$

we get $L \cong M/N \in \mathscr{C}$. Similarly, one proves that $N \in \mathscr{C}$. ∎

In the following propositions, we further assume that the bimodule ${}_SU_R$ is faithfully balanced.

16. Proposition. *Let ${}_SU_R$ be a faithfully balanced quasi-cotilting bimodule. Then,*

(a) *$U \in \mathscr{Y}$ and $R_R \in \mathscr{Y}_R$, ${}_SS \in {}_S\mathscr{Y}$;*

(b) *if $L \in \operatorname{Ker}\Gamma$ and L is finitely generated, then $L \in \mathscr{Y}$;*

(c) *$\mathscr{C}$ contains all the finitely presented modules.*

Proof. As ${}_SU_R$ is balanced, $\Delta^2(U_R) \cong \Delta({}_SS) \cong U_R$ and $\Delta^2({}_SS) \cong \Delta(U_R) \cong {}_SS$ canonically; the same holds for ${}_SU$ and R_R. For (b) and (c) see [C, Proposition 5]. ∎

The further assumption that ${}_SU_R$ is balanced seems to leave unchanged the structure of the dualities associated to U, except for increasing the size of the involved classes.

Let us now suppose ${}_SU_R$ to be a cotilting bimodule. Then, as we already noted in Remark 3, since $(\operatorname{Ker}\Delta, \operatorname{Ker}\Gamma)$ is a torsion theory, two particular results hold: $\operatorname{injdim} U \leq 1$ (see [CDT, Proposition 1.7]), and $L \in \operatorname{Ker}\Gamma$, $\Delta(L) \in \mathscr{Y} \Rightarrow L \in \mathscr{Y}$. These properties are useful to get some additional information about the classes we are studying, like the closure of $\mathscr{C}$ under finitely generated submodules.

17. Proposition. *Let ${}_SU_R$ be a cotilting bimodule. If L is a module which is both a factor of a module $N \in \mathscr{Y}$ and a submodule of a module M such that $\Gamma(M) \in \mathscr{C}$, then $L \in \mathscr{C}$. In particular, $\mathscr{C}$ is closed under finitely generated submodules.*

Proof. Let $L \cong N/K$ with $N \in \mathscr{Y}$. To prove that $L \in \mathscr{C}$, we show that $K \in \mathscr{Y}$. We start from the exact commutative diagram,

$$\begin{tikzcd}
 & & 0 & 0 & & \\
0 \arrow[r] & \mathrm{Rej}_U(L) \arrow[r] & L \arrow[u] \arrow[r] & L/\mathrm{Rej}_U(L) \arrow[u] \arrow[r] & 0 & \\
 & & N \arrow[u] \arrow[r, equal] & N \arrow[u] & & \\
 & 0 \arrow[r] & K \arrow[u] \arrow[r] & T \arrow[u] \arrow[r] & \mathrm{Rej}_U(L) \arrow[r] & 0 \\
 & & 0 \arrow[u] & 0 \arrow[u] & &
\end{tikzcd}$$

where $L/\mathrm{Rej}_U(L) = \mathrm{Cogen}(U) \subseteq \mathrm{Ker}\,\Gamma$. Since the last column belongs to $\mathrm{Ker}\,\Gamma$ and $N \in \mathscr{Y}$, from Lemma 4(iii) we get that both T and $L/\mathrm{Rej}_U(L)$ are Δ-reflexive. From the last row, we get the exact sequence:

$$0 \to \Delta(T) \to \Delta(K) \xrightarrow{f} \Gamma(\mathrm{Rej}_U(L)) = \Gamma(L) \to 0.$$

Since $\mathrm{injdim}\, U \leq 1$, from the exact sequence,

$$0 \to L \to M \to M/L \to 0,$$

we obtain the exact sequence,

$$0 \to \cdots \to \Gamma(M) \xrightarrow{g} \Gamma(L) \to 0.$$

Hence, we get the commutative diagram with exact rows and columns:

$$\begin{tikzcd}
 & & 0 & 0 & \\
0 \arrow[r] & \Delta(T) \arrow[r] & \Delta(K) \arrow[u] \arrow[r, "f"] & \Gamma(L) \arrow[u] \arrow[r] & 0 \\
0 \arrow[r] & \Delta(T) \arrow[u, equal] \arrow[r] & P \arrow[u] \arrow[r] & \Gamma(M) \arrow[u, "g"'] \arrow[r] & 0 \\
 & & Q \arrow[u] \arrow[r, equal] & Q \arrow[u] & \\
 & & 0 \arrow[u] & 0 \arrow[u] &
\end{tikzcd}$$

Since $\Delta(T) \in \mathscr{Y}$ and $\Gamma(M) \in \mathscr{C}$, from Lemma 6, it follows that $P \in \mathscr{C}$. Finally, applying Lemma 6(d) to the first column we get $\Delta(K) \in \mathscr{Y}$ and therefore $K \in \mathscr{Y}$.

Now, let $M \in \mathscr{C}$ and $L \leq M$, $L \cong R^n/K$. Since $R^n \in \mathscr{Y}$ and $\Gamma(M) \in \mathscr{X}$, we conclude that $L \in \mathscr{C}$. ▮

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