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Large Solution of a Semilinear Elliptic Problem

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Abstract—We show that large positive solutions exist for the following equation

$$\Delta u + |\nabla u|^q = p(x) f(u) \qquad (p+)$$

in $\Omega \subseteq R^N (N \geq 3)$ in which the domain Ω is either bounded or equal to R^N. The nonnegative function p is continuous and may vanish on large parts of Ω. If $\Omega = R^N$, then p must satisfy a decay condition

$$\int_0^\infty r\varphi(r)dr < \infty, \qquad \text{where } \varphi(r) = \max_{|x|=r} p(x) \quad \text{as } |x| \to \infty.$$

Furthermore, we show that the given conditions on p are nearly optimal for equation $(p+)$.

Keywords—Entire large solution, Large solution, Elliptic equation, Existence of solution, Semilinear elliptic equation.

1. INTRODUCTION

We consider the existence of large solutions of the equation

$$\nabla u + |\nabla u|^q = p(x) f(u), \qquad (p+)$$

where q is a positive constant, the function f is continuous and nondecreasing on $[0, \infty)$ with $f(0) = 0$ and $f(s) > 0$ if $s > 0$ while the function p is nonnegative and continuous on $\bar{\Omega}$, and the domain Ω is either bounded with smooth boundary or equal to R^N. A solution $u(x)$ of $(p+)$ is called a large solution if $u \to \infty$ as $x \to \partial\Omega$. If $\Omega = R^N$, then $x \to \partial\Omega$ implies $|x| \to \infty$ and such a solution is called an entire large solution. Equation $(p+)$ arises from many branches of

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mathematics and physics. Almost all such studies have dealt with the equation of the form

$$\nabla u = g(x, u), \tag{1.1}$$

in which the function g takes various forms (see [1–9] and references therein).

Lazer and McKenna, Diaz and Letelier showed that equation (1.1) has a unique solution $u \in C^2(\Omega)$, such that $|u(x) + 2\ln d(x)|$ is bounded on Ω provided $g(x,u) = e^u$, and

$$C_1[d(x)]^{2/(p-1)} \le u(x) \le C_2[d(x)]^{2/(p-1)}, \qquad \forall x \in \Omega,$$

provided $g(x,u) = u^p$ ($p > 1$), where C_1, C_2 are positive constants and $d(x) = \operatorname{dist}(x, \partial\Omega)$ (see [7–9]). In this case, (1.1) arises in the study of the electric potential in a golwing hollow metal body and high speed diffusion, and plays an important role in the theory of the Riemannian surfaces of constant negative curvature and in the theory of automorphic function.

When p satisfies the following condition

(C) for any $z \in \Omega$ satisfying $p(z) = 0$, there exists a domain D_z, such that $z \in D_z$, $\bar{D}_z \subseteq \Omega$, and $p(x) > 0$, for all $x \in \partial D_z$.

Lair [10] showed that a necessary and sufficient condition for the equation

$$\nabla u = p(x)f(u), \tag{1.2}$$

to have a nonnegative large solution on a bounded domain Ω is that the function f satisfies

$$\int_1^\infty \left[\int_0^s f(t)\,dt\right]^{-1/2} ds < \infty. \tag{1.3}$$

Moveover, Lair also showed that if p is a nonnegative $C(R^N)$ function which satisfies Condition (C) with $\Omega = R^N$ and

$$\int_0^\infty r\phi(r)\,dr < \infty, \tag{1.4}$$

where $\phi(r) \equiv \max_{|x|=r} p(x)$. Then (1.2) has a positive entire large solution provided f satisfies condition (1.3). Obviously, both of the special nonlinear function $f = e^u$ and u^p with $p > 1$ satisfy condition (1.3).

In [11], Lair and Wood showed that large positive solutions exist for the equation

$$\nabla u + |\nabla u|^q = p(x)u^\gamma, \tag{1.5}$$

in $\Omega \subseteq R^N$ ($N \ge 3$) for an appropriate choice of $\gamma > 1$, $q > 0$ in which the domain Ω is either bounded or equal to R^N. Furthermore, they showed that the given conditions on γ and p are nearly optimal for equation (1.5) in the sense that no large solution exist if either $\gamma \le 1$ or the function p has compact support in Ω.

In this paper, we study equation (p+). At first, we show that equation (p+) has a large solution in a bounded domain Ω and Condition (C) is nearly optimal for (p+). In addition, we obtain the existence of entire large solution for equation (p+). This study generalizes the right-hand side of (1.5) to be the form of (1.2).

The main results of this paper are as follows.

THEOREM 1.1. *Suppose that Ω is a bounded domain in R^N ($N \ge 3$), with smooth boundary and p is a nonnegative continuous function on $\bar{\Omega}$ satisfying Condition (C). Assume that f satisfies (1.3). Then, equation (p+) has a large positive solution in Ω.*

THEOREM 1.2. *Let $\Omega = R^N$. If the same assumptions of f and $p(x)$ as in Theorem 1.1, and condition (1.4) hold as well, then equation (p+) has a positive entire large solution.*

2. THE PROOF OF MAIN RESULTS

Before proving the main results, we need to give some lemmas which will be used later.

LEMMA 2.1. *(See [10, Lemma 1].) Suppose that f satisfies inequality (1.3). Then,*

$$\int_1^\infty \frac{ds}{f(s)} < \infty. \tag{2.1}$$

One point needs to be highlighted. In [10], in order to prove (2.1), Lair first proved the following fact: there exist positive numbers δ and M, such that

$$\frac{f(s)}{s} \geq \delta^2, \qquad \text{for } s \geq M, \tag{2.2}$$

which will be used in the proof of Lemma 2.2 below.

LEMMA 2.2. *Let u_n be a solution of the problem*

$$\begin{aligned} \nabla u_n + |\nabla u_n|^q &= p(x) f(u_n), \qquad && x \in \Omega, \\ u_n(x) &= n, && x \in \partial\Omega, \end{aligned} \tag{2.3}$$

then $0 < u_n \leq n$ on $\bar{\Omega}$. Furthermore, let B_R be a ball of radius R, such that $\bar{B}_R \subseteq \Omega$. Then there exists a constant $M = M(R, q)$, such that $u_n(x) \leq M$ on $\bar{B}_R$ for any n, provided that $0 < m_0 \leq p(x) \leq M_0$ in Ω and (1.3) holds.

PROOF. To prove that $u_n > 0$ in Ω, without loss of generality, let $n = 1$. It is easy to verify that $0 \leq u_1 \leq 1$ by the maximum principle. Furthermore, for any $0 < \varepsilon < 1$, any solution z to the problem (which exists by [12, Theorem 8.3, p. 801])

$$\begin{aligned} \nabla z + |\nabla z|^q &= p(x) f(z), \qquad && x \in \Omega, \\ z &= \varepsilon_0, && x \in \partial\Omega, \end{aligned} \tag{2.4}$$

satisfies $z \leq u_1$ and $0 \leq z \leq \varepsilon_0$. Thus, if we show that $z > 0$ in Ω for some $\varepsilon_0 \in (0,1)$, we will be done. To do this, let $x_0 \in R^N \setminus \bar{\Omega}$. Without loss of generality, assume that $x_0 = 0$. Let $r = |x|$ and choose $R_0 > 0$ large, such that $\bar{\Omega} \subseteq B(0, R_0)$. Choose $M_0 > 0$, such that $p(x) \leq M_0$ on $\bar{\Omega}$. Now, choose $0 < \varepsilon_0 < 1$, such that

$$\frac{M_0 f(\varepsilon_0) R_0^2}{2N} \leq \varepsilon_0. \tag{2.5}$$

Let $v(x) = (M_0 f(\varepsilon_0)/2N) r^2$ for $r = |x| \leq R_0$. Define w on the ball $B(0, R_0)$ as $w(x) = z(x)$ for $x \in \bar{\Omega}$ and $w(x) = \varepsilon_0$ on $\overline{B(0, R_0)} \setminus \bar{\Omega}$. We show that $v \leq w$ in $\overline{B(0, R_0)}$. In fact, if we suppose that $\max(v - w)$ in $\overline{B(0, R_0)}$ is positive, then the point where the maximum occurs must lie in Ω since

$$v(x) = \frac{M_0 f(\varepsilon_0)}{2N} r^2 \leq \frac{M_0 f(\varepsilon_0)}{2N} R_0^2 \leq \varepsilon_0 = w(x), \qquad x \in \overline{B(0, R_0)} \setminus \Omega.$$

Therefore, at the point where $\max(v - u)$ occurs, we have

$$0 \geq \nabla(v - w) = \nabla(v - z) = M_0 f(\varepsilon_0) - p f(z) + |\nabla z|^q > p\left(f(\varepsilon_0) - f(z)\right) \geq 0.$$

That is a contradiction. So, $v \leq w$ in $B(0, R_0)$ which yields $v \leq w$ in Ω or $(M_0 f(\varepsilon_0)/2N) r^2 \leq z(x)$ in Ω. Since $r > 0$ in Ω , we get $z(x) > 0$ in Ω. Hence, $u_1 > 0$ in Ω.

Now, let ε be a sufficiently small positive number so that $B_{R+\varepsilon} \subseteq \Omega$ and let v_n be a solution of

$$\begin{aligned} \nabla v_n &= m_0 f(v_n), \qquad && x \in B_{R+\varepsilon}, \\ v_n &= n, && x \in \partial B_{R+\varepsilon}. \end{aligned} \tag{2.6}$$

A similar argument as above implies that $v_n > 0$ in $B_{R+\varepsilon}$. By the maximum principle, it is clear that $v_n \le v_{n+1}$, $n = 1, 2, \ldots$. It is also easy to show that v_n is a radial solution by [13, Theorem A]. Thus, v_n satisfies

$$\begin{aligned} v_n'' + \frac{N-1}{r} v_n' &= m_0 f(v_n), \qquad x \in B_{R+\varepsilon}, \\ v_n &= n, \qquad x \in \partial B_{R+\varepsilon}. \end{aligned} \tag{2.7}$$

It is clear that $v_n'(0) = 0$ and $v_n'(r) \ge 0$ for any n and r. From (2.7), we have

$$\left(r^{N-1} v_n'(r)\right)' = m_0 r^{N-1} f(v_n(r)), \tag{2.8}$$

integrating from 0 to r, we get

$$\int_0^r \left(s^{N-1} v_n'(s)\right)' ds = m_0 \int_0^r s^{N-1} f(v_n(s))\, ds. \tag{2.9}$$

Thus, we have

$$r^{N-1} v_n'(r) = m_0 \int_0^r s^{N-1} f(v_n(s))\, ds \le m_0 r^N f(v_n(r)), \tag{2.10}$$

which implies that

$$v_n'(r) \le m_0 r f(v_n(r)). \tag{2.11}$$

Let v be a solution of

$$\begin{aligned} \nabla v &= m_0 f(v), \qquad x \in B_{R+\varepsilon}, \\ v|_{\partial B_{R+\varepsilon}} &= \infty. \end{aligned} \tag{2.12}$$

The existence of v is proved by [10, Theorem 1]. By the maximum principle, $v_n \le v$ in $B_{R+\varepsilon}$ for all n. Thus, v_n is bounded above on B_R by a constant which is independent of n. By (2.11), $v_n'(r)$ is also bounded above by a constant independent of n. Let k be an upper bound for both v_n and v_n' on $\bar{B}_R$. If we can find a function w_n which satisfies

$$\begin{aligned} \nabla w_n + |\nabla w_n|^q &\le m_0 f(w_n), \qquad x \in B_{R+\varepsilon} \subseteq \Omega, \\ w_n &= n, \qquad x \in \partial B_{R+\varepsilon}, \\ w_n &\le K_0, \qquad x \in \bar{B}_R, \end{aligned} \tag{2.13}$$

where K_0 is a constant independent of n, then by the maximum principle, we have $u_n \le w_n \le K_0$, and we will be done.

Let $w_n = C v_n^\lambda$, where v_n is a solution of (2.7), the constants C ($C > 1$) and λ ($\lambda > 1$), both independent of n, are determined later. Since

$$\begin{aligned} &\nabla w_n + |\nabla w_n|^q - m_0 f(w_n) \\ &= C\lambda v_n^{\lambda-1} \nabla v_n + C\lambda(\lambda-1) v_n^{\lambda-2} |\nabla v_n|^2 + C^q \lambda^q v_n^{(\lambda-1)q} |\nabla v_n|^q - m_0 f\left(C v_n^\lambda\right) \\ &= m_0 C\lambda v_n^{\lambda-1} f(v_n) + C\lambda(\lambda-1) v_n^{\lambda-2} |\nabla v_n|^2 + C^q \lambda^q v_n^{(\lambda-1)q} |\nabla v_n|^q - m_0 f\left(C v_n^\lambda\right). \end{aligned}$$

By (2.2), if we let $\delta = \sqrt{C}$ (C is defined as above), then there is M, such that $f(s)/s \ge C$ as $s \ge M$. Since $C > 1$ and $\lambda > 1$, then $C v_n^\lambda \ge M$ as $v_n \ge M$. Thus, $f(C v_n^\lambda)/C v_n^\lambda \ge C$, which implies $f(C v_n^\lambda) \ge C^2 v_n^\lambda$. On the other hand, $v_n(r) \le k$, $v_n'(r) \le m_0(R+\varepsilon) f(k)$, thus we have

$$\begin{aligned} &m_0 C\lambda v_n^{\lambda-1} f(v_n) + C\lambda(\lambda-1) v_n^{\lambda-2} |\nabla v_n|^2 + C^q \lambda^q v_n^{(\lambda-1)q} |\nabla v_n|^q - m_0 f\left(C v_n^\lambda\right) \\ &\le m_0 C\lambda v_n^{\lambda-1} f(k) + C\lambda(\lambda-1) v_n^{\lambda-2} |v_n'|^2 + C^q \lambda^q v_n^{(\lambda-1)q} |v_n'|^q - m_0 C^2 v_n^\lambda \\ &\le m_0 C\lambda v_n^{\lambda-1} f(k) + C\lambda(\lambda-1) v_n^{\lambda-2} m_0^2 (R+\varepsilon)^2 f^2(k) + C^q \lambda^q v^{(\lambda-1)q} m_0^q (R+\varepsilon)^q f^q(k) - m_0 C^2 v_n^\lambda \\ &= m_0 C v_n^{\lambda-2} \Big[\lambda f(k) v_n + m_0 \lambda(\lambda-1)(R+\varepsilon)^2 f^2(k) + m_0^{q-1} C^{q-1} \lambda^q v_n^{(\lambda-1)q+2-\lambda} (R+\varepsilon)^q f^q(k) - C v_n^2\Big]. \end{aligned}$$

To complete the proof, it suffices to find C ($C > 1$) and λ ($\lambda > 1$), such that

$$\lambda f(k)v_n + m_0\lambda(\lambda-1)(R+\varepsilon)^2 f^2(k) + m_0^{q-1}C^{q-1}\lambda^q v_n^{(\lambda-1)q+2-\lambda}(R+\varepsilon)^q f^q(k) - Cv_n^2 \le 0.$$

We can choose λ ($\lambda > 1$) so that $2 > (\lambda-1)q + 2 - \lambda$. In fact, it is clear that if $q = 1$. If $q > 1$, then $\lambda < q/(q-1)$, we can choose $1 < \lambda < q/(q-1)$. If $q < 1$, then $\lambda > q/(q-1)$, we can choose $\lambda > 1$. For the choice of λ, let C ($C > 1$) be large so that

$$\lambda f(k)s + m_0\lambda(\lambda-1)(R+\varepsilon)^2 f^2(k) + m_0^{q-1}C^{q-1}\lambda^q s^{(\lambda-1)q+2-\lambda}(R+\varepsilon)^q f^q(k) - Cs^2 < 0, \quad (2.14)$$

for $s \ge 2$. Since $0 < v_1 \le v_2 \le \cdots \le v_n \le v_{n+1} \le \dots$, in $B^{R+\varepsilon}$, we may find $\beta > 0$, such that $v_n(r) \ge \beta$, for any n and r. For the above choice of λ, choose the constant $C \ge 1$ so that the following inequality holds:

$$2\lambda f(k) + m_0\lambda(\lambda-1)(R+\varepsilon)^2 f^2(k) + m_0^{q-1}C^{q-1}\lambda^q 2^{(\lambda-1)q+2-\lambda}(R+\varepsilon)^q f^q(k) - C\beta^2 < 0.$$

Thus, whether $v_n(r) \le 2$ or $v_n(r) \ge 2$, we get

$$\lambda f(k)v_n + m_0\lambda(\lambda-1)(R+\varepsilon)^2 f^2(k) + m_0^{q-1}C^{q-1}\lambda^q v_n^{(\lambda-1)q+2-\lambda}(R+\varepsilon)^q f^q(k) \le Cv_n^2.$$

Hence, $\nabla w_n + |\nabla w_n|^q \le m_0 f(w_n)$, where $K_0 = Ck^\lambda$.

Proof of Theorem 1.1. By [12, Theorem 8.3, p. 301], it is easy to prove that, for each $k \in N$, the boundary value problem

$$\begin{aligned} \nabla v_k + |\nabla v_k|^q &= p(x) f(v_k), \qquad && x \in \Omega, \\ v_k(x) &= k, && x \in \partial\Omega, \end{aligned} \qquad (2.15)$$

has a unique positive classical solution. It can be shown that $v_k \le v_{k+1}$, $k \ge 1$ in Ω by the maximum principle. Indeed, suppose that there is a point x_0, such that $v \equiv v_{k+1} - v_k$ is negative at x_0. Let $x_0 \in R^N \setminus \bar{\Omega}$, we assume, without loss of generality, that $x_0 = 0$. Let $r = |x|$, then for some small $\varepsilon > 0$, $v + \varepsilon/(1+r)$ has a negative minimum in Ω. At that minimal point, we have

$$\begin{aligned} 0 &\le \nabla\left(v + \frac{\varepsilon}{1+r}\right) = p\left[v_{k+1}^\gamma - v_k^\gamma\right] - |\nabla v_{k+1}|^q + |\nabla v_k|^q + \varepsilon\left[\frac{2}{(1+r)^3} - \frac{N-1}{r(1+r)^2}\right] \\ &\le 0 - \varepsilon\frac{N-1}{r(1+r)^3} < 0. \end{aligned}$$

It is a contradiction. Hence, $v_k \le v_{k+1}$, for $k = 1, 2, \dots$. Furthermore, by Lemma 2.2, $v_1 > 0$ in Ω.

Of course, it is understood that the maximum principle is applied as above, where the factor $\varepsilon/(1+r)$ is used whenever the function p is not strictly positive. To complete the proof, it suffices to show the following facts:

(C_1) $\forall x_0 \in \Omega$, there exists M (depending on x_0 but independent of k), such that $v_k(x) \le M$ for any x near x_0,

(C_2) $\lim_{x\to\partial\Omega} v(x) = \infty$, where $v(x) = \lim_{k\to\infty} v_k(x)$ for $x \in \Omega$,

(C_3) v is classical solution of (p+).

To prove (C_1), we consider the following two cases.

Case (a). $p(x_0) > 0$. Since p is continuous, there exists a ball $B(x_0, r)$, such that $p(x) \ge m_0$ in $\overline{B(x_0,r)}$ for some $m_0 > 0$, then (C_1) follows easily from Lemma 2.2.

Case (b). $p(x_0) = 0$. By Condition (C), there exists a domain $\Omega_0 \subseteq \Omega$, such that $x_0 \in \Omega_0$ and $p(x) > 0$ for any $x \in \partial\Omega_0$. From the above Case (a), we know that for any $x \in \partial\Omega_0$ there exists a ball $B(x, r_x)$ and a positive constant M_x, such that $v_k \le M_x$ on $B(x, r_x/2)$. Since Ω

is bounded, $\partial\Omega_0$ is compact. Thus, there exists a finite number of such balls that cover $\partial\Omega_0$. Let $M = \max\{M_{x_1}, \dots, M_{x_k}\}$, where the balls $B(x_i, r_{x_i}/2)$, $i = 1, 2, \dots, k$, cover $\partial\Omega_0$. Clearly, $v_k \le M$ on $\partial\Omega_0$. By the maximum principle again, we obtain $v_k \le M$ on Ω_0.

The proof of (C_2) is straightforward. For any $L > 0$ and any sequence $x_k \to x \in \partial\Omega$, since $v_{L+1} = L+1$ on $\partial\Omega$ and is continuous, there is some $K > 0$, such that $v_{L+1}(x_k) \ge L$ for $k \ge K$. Note that, since $v \ge v_{L+1}$ in Ω, we have $v(x_k) \ge L$, $k \ge K$. Hence, $v(x_k) \to \infty$ as $k \to \infty$. Thus, we have $v \to \infty$ as $x \to \partial\Omega$.

To prove (C_3), we let $x_0 \in \Omega$ and $B(x_0, r)$ be the ball of radius r centered at x_0, such that it is contained in Ω. Let ψ be a C^∞ function which is equal to 1 on $\overline{B(x_0, r/2)}$ and zero out of $B(x_0, r)$.

Let $g(s) = 1/(1+s)$. Multiplying both sides of equation (2.15) by $\psi^2 g(v_k)$ and integrating over $B(x_0, r)$ yields

$$\int_{B(x_0,r)} \psi^2 g(v_k) \nabla v_k \, dx + \int_{B(x_0,r)} \psi^2 g(v_k) |\nabla v_k|^q \, dx = \int_{B(x_0,r)} \psi^2 g(v_k) p(x) f(v_k) \, dx. \tag{2.16}$$

Integrating by part gets

$$\begin{aligned} &- \int_{B(x_0,r)} \psi^2 g'(v_k) |\nabla v_k|^2 \, dx - \int_{B(x_0,r)} 2\psi \nabla\psi g(v_k) \nabla v_k \, dx \\ &+ \int_{B(x_0,r)} \psi^2 g(v_k) |\nabla v_k|^q \, dx = \int_{B(x_0,r)} \psi^2 g(v_k) p(x) f(v_k) \, dx. \end{aligned} \tag{2.17}$$

Thus, we have

$$\begin{aligned} &\frac{1}{(1+M_r)^2} \int_{B(x_0,r)} \psi^2 |\nabla v_k|^2 \, dx \\ &\le \int_{B(x_0,r)} \frac{\psi^2}{(1+v_k)^2} |\nabla v_k|^2 \, dx + \int_{B(x_0,r)} \psi^2 g(v_k) |\nabla v_k|^q \, dx \\ &= \int_{B(x_0,r)} 2\psi\nabla\psi\nabla v_k \left(\frac{1}{1+v_k}\right) dx + \int_{B(x_0,r)} \psi^2 g(v_k) p(x) f(v_k) \, dx \\ &\le \int_{B(x_0,r)} (\psi\nabla v_k) \left(\frac{2|\nabla\psi|}{1+v_k}\right) dx + M_1, \qquad \left(g(v_k) \le \frac{1}{1+v_1}, \ f(v_k) \le f(M_r)\right) \\ &\le \varepsilon \int_{B(x_0,r)} \psi^2 |\nabla v_k|^2 \, dx + \frac{1}{\varepsilon} \int_{B(x_0,r)} \left(\frac{2|\nabla\psi|}{1+v_1}\right)^2 dx + M_1 \\ &\le \varepsilon \int_{B(x_0,r)} \psi^2 |\nabla v_k|^2 \, dx + M_2, \end{aligned}$$

where M_r is an upper bound for v_k $(k = 1, 2, \dots)$ on $B(x_0, r), \varepsilon$ is any positive number, and the constants M_1 and M_2 are independent of k. Hence, we get

$$\int_{B(x_0,r)} |\psi\nabla v_k|^2 \, dx \le M.$$

That is, the $L^2(B(x_0, r))$-norm of $|\psi\nabla v_k|$ is bounded independently of k. Thus, the $L^2(B(x_0, r/2))$-norm of $|\nabla v_k|$ is bounded independently of k.

By the standard regularity argument (see [1]), we may find a number $r_1 > 0$, such that there is a subsequence of $\{v_k\}_1^\infty$, which we still call $\{v_k\}_1^\infty$, that converges in $C^{1+\alpha}\overline{(B(x_0, r))}$ for some positive number $\alpha < 1$. Let ψ be as before but with r replaced by r_1.

Now, we consider two cases regarding the regularity of the function $p(x)$.

CASE 1. $p(x) \in C^\alpha(\Omega)$. Note that both of $\nabla v_k = pf(v_k) - |\nabla v_k|^q$ and $\nabla(\psi v_k) = 2\nabla\psi\nabla v_k + v_k\nabla\psi + \psi\nabla v_k$ converge in $C^\alpha\overline{(B(x_0, r_1))}$ as $k \to \infty$. By the Schauder theory, $\{\psi v_k\}_1^\infty$ converges in $C^{2+\alpha}\overline{(B(x_0, r_1))}$, and hence, $\{v_k\}_1^\infty$ converges in $C^{2+\alpha}\overline{(B(x_0, r_1/2))}$ when $k \to \infty$. Since x_0 is arbitrary, it follows that $v \in C^{2+\alpha}(\Omega)$ and is a solution of (p+).

CASE 2. $p(x) \in C(\Omega)$. We have $v_k \overset{s-C(B(x_0,r_1))}{\longrightarrow} v$ and, consequently,

$$\nabla v_k = pf(v_k) - |\nabla v_k|^q \overset{s-C(B(x_0,r_1))}{\longrightarrow} p(x)f(v) - |\nabla v|^q \equiv z.$$

That the Laplacian is a closed linear operator implies that $v \in D(\nabla)$ and $\nabla v = z$. Since x_0 is arbitrary, we get that v is a classical solution of (p+).

REMARK 2.1. Theorem 1.1 implies that if the nonnegative function p is such that each of its zero points is enclosed by a bounded surface of nonzero points, then equation (p+) has a large positive solution. The following proposition implies that if the condition does not hold in the sense that p vanishes in an "outer ring" of the domain, then equation (p+) has no positive large solution.

PROPOSITION 2.1. Suppose that $g(x, 0) = 0$ for all $x \in \Omega$. If there exists a domain $D \subseteq \Omega$, such that $\bar{D} \subseteq \Omega$ and $g(x, t) = 0$, $x \in \Omega \setminus D$, $t \geq 0$, then there is no positive large solution of

$$\nabla u + |\nabla u|^q = g(x, u), \qquad x \in \Omega. \tag{2.18}$$

Note that this includes the case $g(x, u) = p(x)f(u)$, and $p(x) = 0$ in $\Omega \setminus D$.

The proof of this result is similar to the proof of Theorem 13 given in [11], so we omit its proof here. From Proposition 2.1, we get that Condition (C) is nearly optimal for (p+).

PROOF OF THEOREM 1.2. By Theorem 1.1, for $k = 1, 2, \ldots$, the boundary blow-up problem

$$\begin{gathered} \nabla v_k + |\nabla v_k|^q = p(x)f(v_k), \qquad |x| < k, \\ v_k(x) \to \infty, \text{ as } |x| \to k. \end{gathered} \tag{2.19}$$

has a classical solution. It is clear that

$$v_1 \geq v_2 \geq \cdots \geq v_k \geq v_{k+1} \geq \cdots, \tag{2.20}$$

in R^N by the maximum principle. Let $v(x) = \lim_{k\to\infty} v_k(x)$, $x \in R^N$. We claim that v is the desired solution. To prove this, we consider the related problem

$$\begin{gathered} \nabla u_k = p(x)f(u_k), \qquad |x| < k, \\ u_k(x) \to \infty, \text{ as } |x| \to k. \end{gathered} \tag{2.21}$$

It is shown in [10] that (2.21) has a unique positive solution for each k, and that

$$u_1 \geq u_2 \geq \cdots \geq u_k \geq u_{k+1} \geq \cdots \geq w > 0, \tag{2.22}$$

for some $w \to \infty$ as $|x| \to \infty$. It follows easily from the maximum principle that $v_k \geq u_k$ for $k = 1, 2, \ldots$. Thus, $v(x) \to \infty$ as $|x| \to \infty$. By a similar argument as (C3) in Theorem 1.1, we have that v is the desired solution.

REMARK 2.2. In [1, Theorem 4], Lair has proved a converse of Theorem 1.2 for the case $f(u) = u^\gamma$ ($\gamma > 1$). We cannot obtain a similar result for general f except for the function p admitting specific decay rates. However, we prove the following result, a "partial converse" to Theorem 1.2.

THEOREM 2.1. *Suppose the function p satisfies the hypothesis of Theorem 1.2 including the inequality (1.4). If equation (p+) has a nonnegative entire large solution, then f satisfies the inequality (2.2), i.e.,* $\int_1^\infty (ds/f(s)) < \infty$.

PROOF. Let u be a nonnegative entire large solution of (p+) and extend f as an odd function on R, let $\{f_k\}$ be a sequence of nondecreasing $C^\infty(R)$ functions which converges uniformly on compact sets to f, satisfy $sf_k(s) > 0$ for $s \neq 0$, $f_k(0) = 0$, and

$$f(s) + f\left(s - \frac{1}{k}\right) \leq 2f_k(s) \leq f(s) + f\left(s + \frac{1}{k}\right). \tag{2.23}$$

Such a sequence is easy to obtain by using mollifiers (see [14, p. 145]). We define nonnegative functions w_k and w on R^n by

$$w_k(x) = \int_0^{u(x)} \frac{ds}{1 + f_k(s)}, \qquad w(x) = \int_0^{u(x)} \frac{ds}{1 + f(s)},$$

and note that $\{w_k\}$ converges uniformly on compact sets to w_0. Let

$$\bar{w}_k(r) \equiv \left(w_n r^{n-1}\right)^{-1} \int_{|x|=r} w_k(x)\, ds \equiv \int_{|x|=r} w_k(x)\, d\sigma,$$

where w_n denotes the surface area of the unit sphere in R^N, and similarly define $\bar{w}$. Then, $\nabla \bar{w}_k(r) = \int_{|x|=r} \Delta w_k(x)\, d\sigma$ (see [15]) and

$$\nabla w_k = \frac{\nabla u}{1 + f_k(u)} - \frac{|\nabla u|^2 f_k'(u)}{[1 + f_k(u)]^2} \leq \frac{\nabla u}{1 + f_k(u)} = \frac{p(x)f(u) - |\nabla u|^q}{1 + f_k(u)} \leq \frac{p(x)f(u)}{1 + f_k(u)} \equiv \hat{p}_k.$$

Thus, $\nabla \bar{w}_k(r) \leq \int_{|x|=r} \hat{p}_k d\sigma$, $r \geq 0$. Integrating this we get

$$\bar{w}_k(r) \leq \bar{w}_k(0) + \int_0^r t^{1-N} \int_0^t s^{N-1} \int_{|x|=s} \hat{p}_k\, d\sigma\, ds\, dt. \tag{2.24}$$

For the sequence $\{f_k\}$ as above, we choose J large such that $f(s - 1/J) \geq -2$ for $s \geq 0$ (because f is extended as an odd function on R). By (2.23), we have

$$2\left[1 + f_J(s)\right] \geq f(s), \qquad s \geq 0, \tag{2.25}$$

thus, $\hat{p}_k \leq 2\phi(r)$. By (2.24), letting $k \to \infty$ in this expression and $\hat{p}_k \leq 2\phi(r)$, we get

$$\begin{aligned} \bar{w}(r) &\leq \bar{w}(0) + 2(N-2)^{-1} \int_0^r s\phi(s)\, ds \qquad (N \geq 3) \\ &\leq \bar{w}(0) + 2\int_0^\infty s\phi(s)\, ds \equiv K. \end{aligned}$$

Thus, $\liminf_{|x|\to\infty} w(x) \leq K$. However, since $\lim_{|x|\to\infty} u(x) = \infty$, we must have

$$K \geq \lim_{|x|\to\infty} \inf w(x) = \lim_{|x|\to\infty} \int_0^{u(x)} \frac{1}{1 + f(s)}\, ds = \int_0^\infty \frac{ds}{1 + f(s)}.$$

Therefore, (2.2) holds, and the proof is complete.

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