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An upper bound for the minimum number of queens covering the $n \times n$ chessboard

A.P. Burger, C.M. Mynhardt *
Department of Mathematics, University of South Africa, P.O. Box 392, 0003 Pretoria, South Africa

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Abstract

We show that the minimum number of queens required to cover the $n \times n$ chessboard is at most $\frac{8}{15}n + \mathcal{O}(1)$. © 2002 Elsevier Science B.V. All rights reserved.

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1. Introduction

We generally follow the notation and terminology pertaining to domination of [12]. We repeat the main concepts for chessboards here. The *queens graph* Q_n has the squares of the $n \times n$ chessboard as its vertices; two squares are *adjacent* if they are in the same row, column or diagonal. A queen on square x of Q_n *covers* a square y if x and y are adjacent. A set D of squares is a *dominating set* of Q_n if every square of Q_n is either in D or adjacent to a square in D, i.e., if a set of queens, one on each square in D, covers the board. If no two squares of the dominating set D are adjacent, then D is an *independent dominating set.* The *domination number* $\gamma(Q_n)$ (*independent domination number* $i(Q_n)$) of Q_n is the minimum size amongst all dominating (independent dominating) sets of Q_n. It is easily seen that $\gamma(Q_n) \leqslant i(Q_n)$ for all n.

Chessboard domination problems initiated the study of dominating sets of graphs, at first rather informally until the topic of domination was given formal mathematical definition with the publication of the books by Berge [2] and Ore [14] in 1962. That even the original chessboard domination problems are astonishingly difficult is apparent in view of the fact that so few of these problems have been solved completely. The unsolved classical problems were important in motivating the revival of the study of

* Corresponding author.
E-mail address: mynhacm@hotmail.com, mynhacm@unisa.ac.za (C.M. Mynhardt).

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Table 1
Known values for $\gamma(Q_n)$ and $i(Q_n)$

n	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	30	31
γ	2	3	3	4	5	5	5	5	6	7	7/8	8/9	8/9	9	9	10	15	16
i	3	3	4	4	5	5	5	5	7	7	8	9	9	9				

Table 2
Known values for $\gamma(Q_n)$ and $i(Q_n)$

k	5	6	7	8	9	10	11	12	13	14	15	17	19
n	21	25	29	33	37	41	45	49	53	57	61	69	77
γ	11	13	15	17	19	21	23	25	27	29	31	35	39
i	11	13		17			23						

dominating sets in graphs in the early 1970s. One of the most interesting—and most difficult—chessboard problems is the *queens domination problem* of determining $\gamma(Q_n)$.

De Jaenisch [7] first considered the problem of determining values of $\gamma(Q_n)$ in 1862 and Ball [1] gave values of $\gamma(Q_n)$ up to $n=8$ in 1892. Much more recently Spencer, as cited in [6,15], proved the lower bound $\gamma(Q_n) \geqslant \frac{1}{2}(n-1)$, $n \geqslant 1$. Several researchers (see [6,8,9,5]) established upper bounds. Most of the known values of $\gamma(Q_n)$ (see Tables 1 and 2) are established by a placement D of queens, where $|D|$ attains some lower bound. For $n<5$ optimal placements are easily discovered by trial. With Spencer's bound, $\gamma(Q_{11})=i(Q_{11})=5$ is established. However, $n=3$ and 11 are the only values known for which this bound holds exactly. If n is even Spencer's bound becomes $\gamma(Q_{2m}) \geqslant m$, and this together with dominating sets of the required size establishes values of $\gamma(Q_n)$ for $n=4,6,10,12,18$ and 30. The value $\gamma(Q_{18})=9$ was determined by McRae (see [9]) and $\gamma(Q_{30})=15$ was obtained in [3]. The exceptional value $\gamma(Q_8)=5$ is claimed by Ball [1], and Weakley [15] proved $\gamma(Q_7)=4$.

Weakley [15] refined the lower bound by proving $\gamma(Q_{4k+1}) \geqslant 2k+1$ for all $k \geqslant 0$. He also showed that $\gamma(Q_{4k+1})=2k+1$ for $k=3,4,5,6$ and 8 by constructing dominating sets of order $2k+1$. Burger et al. [4] added the cases $k=9,12,13$ and 15 to the list and Gibbons and Webb [10] filled in the gaps by finding dominating sets for $k=7,10,11$ and 14 so that $\gamma(Q_{4k+1})=2k+1$ for all k with $0 \leqslant k \leqslant 15$. Recently, Burger [3] showed that this equality also holds for $k=17$ and 19. The values $\gamma(Q_{19})=10$ and $\gamma(Q_{31})=16$ are also obtained in [3].

Thus, we have ample evidence that at least for small values of n, $\gamma(Q_n) \approx n/2$. General upper bounds that are close to $n/2$ seem to be harder to find. The previously best upper bound is given in [5], where it is shown that if $n=108m-37$, then $\gamma(Q_n) \leqslant 62m-23=\frac{31}{54}n-\frac{95}{54}$. Compared with the lower bound of $\gamma(Q_n) \geqslant \frac{1}{2}n+\mathcal{O}(1)$, this leaves a gap of $\frac{2}{27}n+\mathcal{O}(1)$ between the lower and upper bound. In this paper we narrow the gap by more than half to $\frac{1}{30}n+\mathcal{O}(1)$ by showing that if $n=60m-11$, then $\gamma(Q_n) \leqslant 32m-6=\frac{8}{15}n-\frac{2}{15}$.

Note that there are restrictions on n, but the set of admissible values of n is an arithmetic progression and for all other values of n we can create a dominating set by adding queens to a dominating set on a largest admissible board of size less than n. At most one queen is needed for each new row and column. Therefore, the number of queens added is never more than a constant. This will show that $\gamma(Q_n) \leqslant \frac{8}{15}n + \mathcal{O}(1)$.

Observe that the pattern formed by the queens to obtain this bound is different from the so-called Z-pattern used in [5,11]. The results presented here were obtained as part of the doctoral thesis [3]. For surveys on the queens domination problem and upper bound results we refer the reader to [6,8,9,11] and especially the recent survey [13].

2. Dominating sets on Q_{4k+1}

Since $n = 60m - 11 \equiv 1 \pmod 4$, we number the squares as illustrated in Fig. 1. A row or column is called *even* or *odd* according to the parity of its label, while a square is called *even–even* , *odd–odd*, *even–odd* or *odd–even* according to the parity of its coordinates. We consider placements of queens with at least one queen in each even row and column. Thus, all the squares in these rows and columns are dominated and the only squares that need to be considered are the odd–odd squares (shaded in Fig. 1), which must be dominated either diagonally or by placing a queen in its row or column.

We can simplify the representation by drawing only the odd-odd squares as illustrated in Fig. 2. Imagine that the even rows and even columns are squeezed to be only lines. Place the simplified board on the (x, y)-plane with the centre of the board at coordinates $(0, 0)$ and the lines formed by the even rows and columns at unit lengths from each other. The coordinates of the queens are then the coordinates of the points (in the plane) on which they are placed. Notice that a queen with coordinates $(2x, 2y)$ on the normal chessboard has coordinates (x, y) on the simplified representation. Henceforth, when we refer to coordinates, it will be of the simplified representation. As in the case of squares of the chessboard, a queen in the simplified representation is called *even–even, even–odd, odd–even or odd–odd* according to the parity of her coordinates.

The diagonals (of squares) that rise from left to right correspond to the straight lines with equations $y = x + d$, where $d \in \{-(2k-1), \ldots, -1, 0, 1, \ldots, 2k-1\}$. These lines are called *d-diagonals* and are labelled $d = -(2k-1), \ldots, d = -1, d = 0, d = 1, \ldots, d = 2k-1$ according to their intersections with the y-axis. Similarly, the *s-diagonals* fall from left to right, and correspond to the straight lines with equations $y = -x + s$, $s \in \{-(2k-1), \ldots, -1, 0, 1, \ldots, 2k-1\}$ and are also labelled according to their intersections with the y-axis. An *even* (*odd*) *diagonal* is a diagonal with an even (odd) intersection with the y-axis. Notice that a queen (in an even row and column of the unreduced board) that lies on an odd (even) d-diagonal, also lies on an odd (even) s-diagonal and *vice versa*. Queens on odd (even) diagonals are also referred to as *odd* (*even*) *queens*. By a *point on a d-diagonal* (or *s-diagonal*) we mean an intersection point of its corresponding line and a line formed by an even row or column. Notice that the difference between the y and x coordinates of any point on a d-diagonal is equal to its label. Similarly,

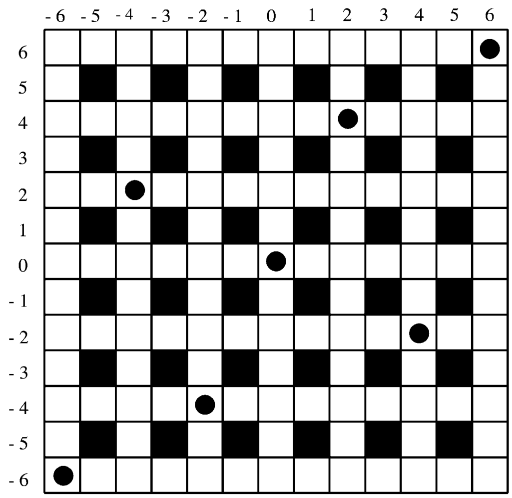

Fig. 1. A dominating set of Q_{13}.

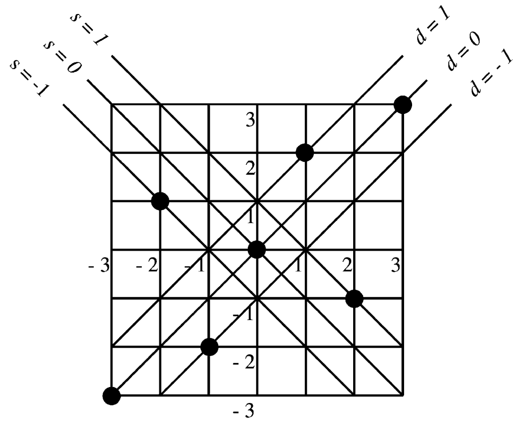

Fig. 2. Simplified representation of Q_{13}.

the sum of the coordinates of any point on the s-diagonals is equal to its label. Fig. 3 illustrates these concepts for a general board.

The following result is easily seen to be true (Fig. 3) and is stated without proof.

Lemma 1. *If an s-diagonal $s = l \geqslant 0$ intersects a row $y = p$ or a column $x = q$, then any s-diagonal $s = l'$ with $0 \leqslant l' \leqslant l$ intersects all rows $y = p'$ with $p' \geqslant p$ and all columns $x = q'$ with $q' \geqslant q$. A symmetric statement holds for negative s-diagonals and similar results are true for d-diagonals.*

A line (row, column, diagonal) which does not contain a queen is called an *empty line* (*row*, *column*, *diagonal*). A set of $2k + 1$ queens in every second row and column

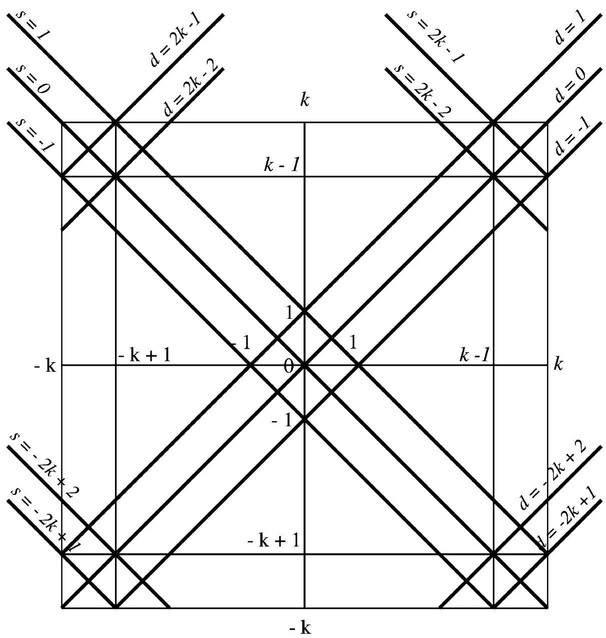

Fig. 3. Numbering of diagonals.

of Q_{4k+1}, where the first empty s-diagonal from the centre is $s = i$ or $s = -i$ and the first empty d-diagonal from the centre is $d = j$ or $d = -j$, is called an (i, j)-*set*, $i, j \geqslant 0$. (See Fig. 2 for a $(1, 1)$-set of Q_{13}.) We need the following result from [4].

Theorem 2. *An* (i, i)-*set of queens which lie on the diagonals*

$$s, d = 0, \pm 1, \pm 2, \ldots, \pm(i-1), \pm(i+1), \pm(i+3), \ldots, \pm(2k-i-1) \quad (1)$$

$(2k-1$ *of each type*) *is dominating.*

Depending on the parity of i, there are either more odd queens or more even queens in a dominating (i, i)-set D of Q_{4k+1} of the type described in Theorem 2. Call the smaller of these sets the *core* of D and the bigger one the *body* of D. The *core diagonals* (respectively *body diagonals*) are the diagonals listed in (1) containing core (respectively body) queens. Note that there can be core (body) queens that are not on the core (body) diagonals, i.e., they lie on diagonals not listed in (1).

3. The upper bound

We now present the new upper bound for the domination number of the queens graph.

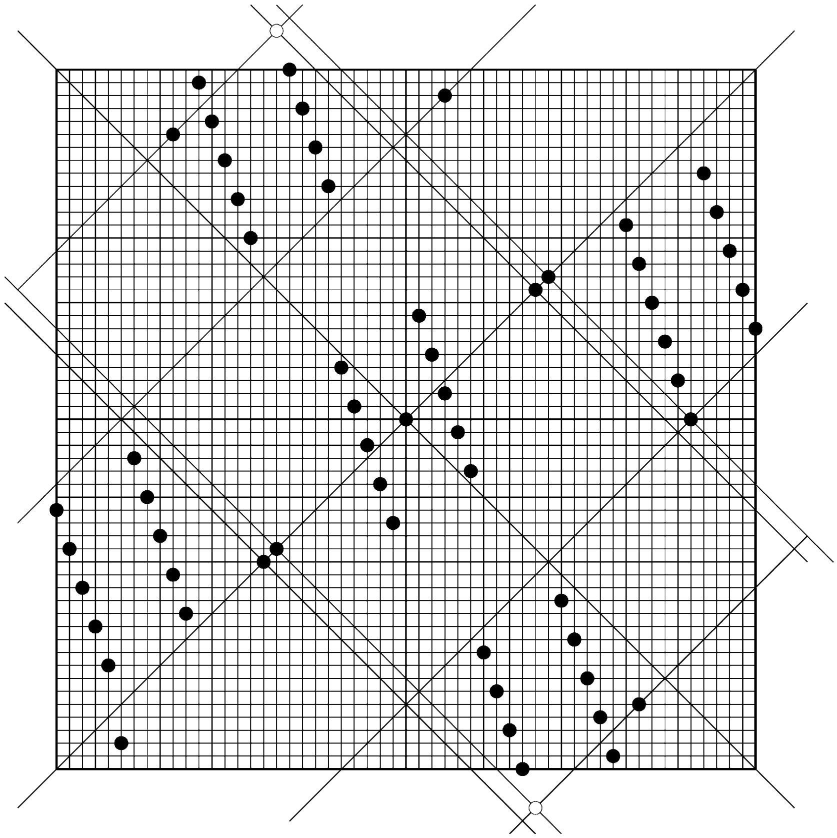

Fig. 4. $m=2$ ($\gamma(Q_{109}) \leqslant 58$).

Theorem 3. *For each positive integer* m, $\gamma(Q_{60m-11}) \leqslant 32m-6$.

Proof. Let $i=6m-1$ and $k=15m-3$. Note that $\{(0,0), \pm(2,1), \pm(1,-2)\}$ is a minimum dominating set for Q_{11}. The construction in the proof below is based on this pattern — it uses five groups of queens, with group centres the squares in the set $\{(0,0), \pm(2i,i), \pm(i,-2i)\}$, plus a few additional queens to cover the rows, columns and diagonals necessary to give an (i,i)-set satisfying the hypothesis of Theorem 2. This results in a set of queens on Q_{4k+1} with at least one queen in every second row and column and at least one queen on each of the diagonals listed in (1), thus forming a dominating set according to Theorem 2. See Figs. 4 and 5 for the cases $m=2$ and 3 (on the simplified version of the board).

At each point in the construction, one can divide the lines of the chessboard into four types:

(i) Lines that are occupied,

(ii) Lines that are empty, but must be occupied to satisfy the hypothesis of Theorem 2,

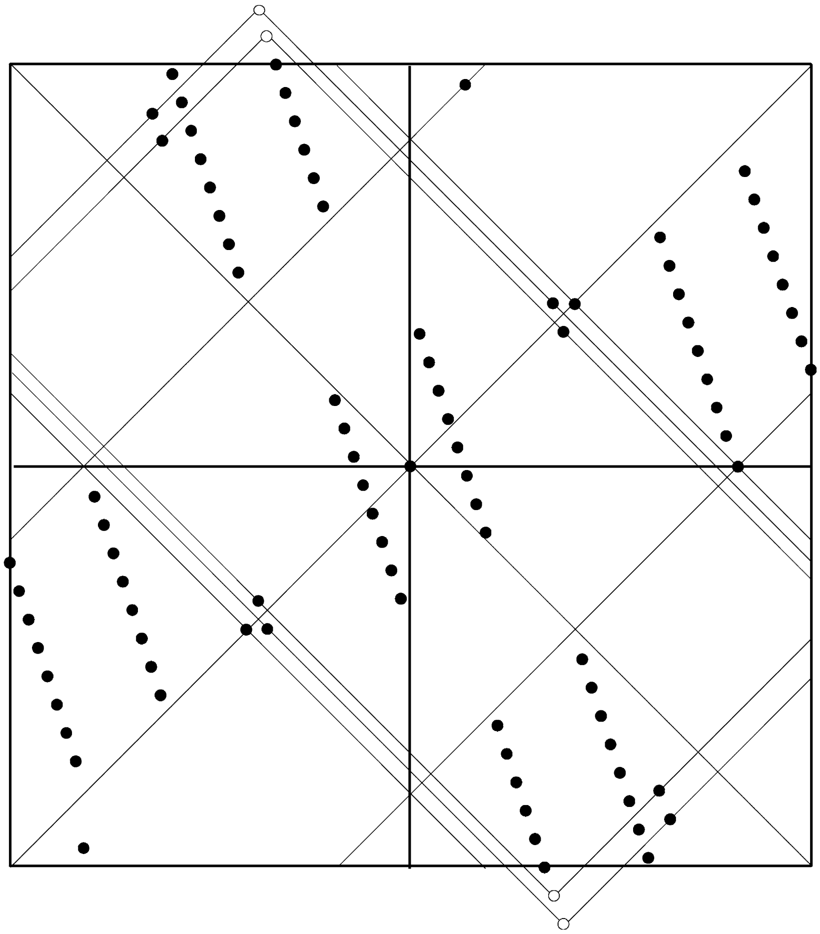

Fig. 5. $m = 3$ ($\gamma(Q_{169}) \leqslant 90$).

(iii) Lines that are empty, need not be occupied but will be (by queens placed on lines of Type (ii)),

(iv) Lines that are empty and will remain so.

The core queens are the odd queens and their coordinates are

$$(1, i-3) + j(1, -3) \quad \text{and } (-1, -i+3) + j(-1, 3) \quad \text{for } j = 0, 1, \ldots, 3m-2.$$

These queens are on the diagonals

$$s \text{ (respectively } d) = -i+2, -i+4, \ldots, -1, 1, 3, \ldots, i-2, \tag{2}$$

i.e., all the consecutive odd diagonals required by (1) for $s, d \leqslant i - 2$.

The body consists of four groups of queens, two of which are exact copies of the core. If we regard (0, 0) as the centre of the core queens, then the centres of the exact copies are at $(2i, i)$ and $(-2i, -i)$. First, consider the copy with centre $(2i, i)$. Since it is a copy of the core, it will also cover consecutive diagonals (of the same parity). We can therefore determine the s- and d-diagonals of the copy by adding $i+2i$ and $i-2i$ to (2) respectively. This gives

$$s = 2i+2, 2i+4, \ldots, 3i-1, 3i+1, 3i+3, \ldots, 4i-2 \tag{3}$$

$$d = -2i+2, -2i+4, \ldots, -i-1, -i+1, -i+3, \ldots, -2. \tag{4}$$

Similarly, the copy with centre $(-2i, -i)$ has queens on the diagonals

$$s = -4i+2, -4i+4, \ldots, -3i-1, -3i+1, -3i+3, \ldots, -2i-2 \tag{5}$$

$$d = 2, 4, \ldots, i-1, i+1, i+3, \ldots, 2i-2. \tag{6}$$

The other two groups of queens are also copies of the core with the only difference that if $m \geqslant 2$, then some of the queens do not fit on the board and these, of course, cannot be part of the dominating set. The centres of the groups are at $(i, -2i)$ and $(-i, 2i)$. These queens (including those that do not fit on the board) are on the even diagonals

$$s = -2i+2, -2i+4, \ldots, -2, \tag{7}$$

$$s = 2, 4, \ldots, 2i-2, \tag{8}$$

$$d = -4i+2, -4i+4, \ldots, -2i-2, \tag{9}$$

$$d = 2i+2, 2i+4, \ldots, 4i-2. \tag{10}$$

We see that the diagonals listed in (3)–(10) are all the even diagonals from $s, d = -4i+2 = -(2k-i-1)$ to $s, d = 4i-2 = 2k-i-1$ except $s, d = 0$ and $s, d = \pm 2i$.

We have to consider the queens that do not fit on the board, because they are on the extensions of empty diagonals of Type (ii) which therefore must be occupied in some other way. These queens have y-coordinates bigger than $k = 15m-3$ or smaller than $-k$. Their coordinates are

$$(-i, 2i) + (1, i-3) + j(1, -3) \quad \text{for } j = 0, 1, \ldots, m-2,$$
$$(i, -2i) + (-1, -i+3) + j(-1, +3) \quad \text{for } j = 0, 1, \ldots, m-2,$$

which are the same as (recall $i = 6m-1$)

$$(-6m+2, 18m-6), (-6m+3, 18m-9), \ldots, (-5m, 15m),$$

$$(6m-2, -18m+6), (6m-3, -18m+9), \ldots, (5m, -15m).$$

Therefore these queens are on the diagonals

$$s = \pm(2i-2), \pm(2i-4), \ldots, \quad m-1 \text{ terms},$$
$$d = \pm(4i-4), \pm(4i-8), \ldots, \quad m-1 \text{ terms}.$$

To summarise, the following lines (corresponding to columns, rows and diagonals) of Type (ii) are empty at this stage:

$x = 0, \pm i, \pm 2i$ and $x = \pm(i-1), \pm(i-2), \ldots,$ $m-1$ terms,

$y = 0, \pm i, \pm 2i$ and $y = \pm(k-2), \pm(k-5), \ldots,$ $m-1$ terms,

$d = 0, \pm 2i$ and $d = \pm(4i-4), \pm(4i-8), \ldots,$ $m-1$ terms,

$s = 0, \pm 2i$ and $s = \pm(2i-2), \pm(2i-4), \ldots,$ $m-1$ terms.

Thus, we have $2m+3$ empty rows and columns and $2m+1$ empty s- and d-diagonals of Type (ii). These lines can be occupied by the following procedure:

(1) Place queens on $(0,0)$, (i,i) and $(-i,-i)$. This covers the columns x (rows $y) = 0, \pm i$, s-diagonals $s = 0, \pm 2i$ and d-diagonal $d = 0$, leaving $2m$ empty rows, columns and d-diagonals and $2m-2$ empty s-diagonals.
(2) Place a queen on each of $2m-1$ intersections of an empty row and an empty d-diagonal, leaving one empty row and d-diagonal, say $d = 2i$.
(3) Place a queen on each of $2m-2$ intersections of an empty column and an empty s-diagonal, leaving two empty columns, one of which is $x = -i+1 = -6m+2$, say, and no empty s-diagonals.
(4) Place a queen on the intersection of the empty column $x = -6m+2$ and the remaining empty d-diagonal $d = 2i = 12m-2$, i.e., on the square with coordinates $(-6m+2, 6m)$.
(5) Place a queen on the intersection of the remaining empty row and column.

Steps 2–5 can be executed in a number of ways. Step 2 is always possible since even the shortest positive (negative, respectively) empty d-diagonal $d = 4i-4 = 24m-8$ ($d = -4i+4 = -24m+8$, respectively) intersects all the empty rows in the upper (lower) half of the board. This follows from Lemma 1 since $d = 24m-8$ ($d = -24m+8$, respectively) intersects the empty row $y = 12m+1$ ($y = -12m-1$, respectively) closest to the centre in the upper (lower) half of the board in the point $(-12m+9, 12m+1)$ (respectively $(12m-9, -12m-1)$). Similarly, Step 3 is always possible by Lemma 1 since the shortest empty positive (negative) s -diagonal $s = 12m-4$ ($s = -12m+4$) intersects the empty column $x = 5m$ ($x = -5m$) in the right (left) half of the board closest to the centre (if $m \geqslant 2$) in the point $5m, 7m-4)$ (respectively $(-5m, -7m+4)$). The coordinates of the queens in Steps 4 and 5 depend of course on the placements in Steps 2 and 3; the coordinates chosen to execute Step 4 only serve to illustrate that Step 4 is possible, and Step 5 is trivially possible in all cases. (Note that for Step 4, placements on squares with coordinates different from $(-6m+2, 6m)$, resulting from different executions of Steps 2 and 3, are illustrated in Figs. 4 and 5.)

We need $4m+2$ queens for the procedure described in Steps 1–5. The five original groups (disregarding the few queens not on the board) require $5(6m-2) - 2(m-1)$ queens, giving a total of $32m-6$ queens. □

Remark. In some cases it is possible to improve on the above procedure, because if more than two empty lines cross at one point we need fewer queens to cover those lines. For example, if $m=3$ we can use one fewer queen by placing queens on

$$(0,0), \pm(34,0), \pm(15,17), \pm(16,14), \pm(26,-34), \pm(27,-37), \pm(17,40).$$

The same technique with core patterns that result in more empty lines crossing at the same point would permit us to improve the upper bound even more.

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