Journal of Computational and Applied Mathematics 31 (1990) 57–62
North-Holland

Applications of uniformly distributed functions and sequences in statistic ergodic measuring techniques

Michael DRMOTA
Institut für Algebra und Discrete Mathematik, Technical University of Vienna, Wiedner Hauptstraße 8–10/118, A-1040 Vienna, Austria

Received 2 December 1988

Abstract: The stochastic ergodic measuring technique is used to measure voltages and mean values by a stochastic principle. After describing this principle it is shown how to estimate the measuring error deterministically with the discrepancy of functions and sequences.

Keywords: Simulation, uniformly distributed sequences, uniformly distributed functions, discrepancy, statistic ergodic measuring techniques.

1. Introduction

The stochastic ergodic measuring technique (SEM) has been developed for measuring some significant values of electronic signals, e.g., the mean value, the root-mean-square-voltage, or the correlation functions. The advantage of the SEM is that it is possible to measure the voltage of some constant signal, sums, products, the square, the square root, or the mean value with the same simple principle without using nonlinear elements or complicated electronics. Today this advantage is not as important as it has been in the early 70s because digital processing has become cheaper and faster. But the same principle can be used now for reducing costs and complexity in digital-analog-converters (DAC) in a way that can be called stochastic interpolation.

2. The stochastic ergodic measuring principle

Let $e(t) = E \in [0, 1]$ be a constant (normalized) input voltage to be measured. The first step is to compare $e(t)$ with a reference signal $r(t)$ to get an output impulse function

$$z(t) = \mathbf{1}_{[0,e(t)]}(r(t)) = \begin{cases} 1 & \text{for } r(t) \leqslant e(t), \\ 0 & \text{for } r(t) > e(t), \end{cases} \tag{1}$$

where $\mathbf{1}_A(x)$ denotes the characteristic function of the set A. (Compare with Fig. 1.)

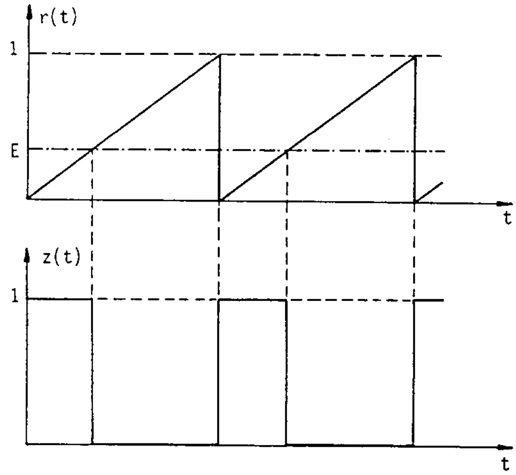

Fig. 1.

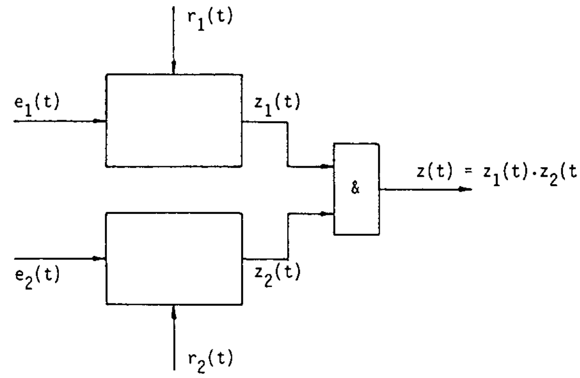

Fig. 2.

The mean value of $z(t)$,

$$M(T)=\frac{1}{T}\int_0^T z(t)\,\mathrm{d}t \tag{2}$$

is an approximation for E if $r(t)$ is uniformly distributed. This is easy to see, since $r(t)$ is uniformly distributed if and only if

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T \mathbf{1}_{[0,E]}(r(t)) = \lim_{T\to\infty} M(T)=E \tag{3}$$

for all $E\in[0, 1]$. $M(T)$ can easily be approximated by sampling $z(t)$ to get a 0,1-sequence $z_n=z(nT_0)$ and by counting

$$M_N=\frac{1}{N}\sum_{n=1}^{N} z_n=\frac{1}{N}\sum_{n=1}^{N}\mathbf{1}_{[0,E]}(r(nT_0)). \tag{4}$$

If the sequence $r_n=r(nT_0)$ is uniformly distributed, then

$$\lim_{N\to\infty} M_N=E \tag{5}$$

for all $E\in[0, 1]$.

A detailed description of the SEM with many applications can be found in [5].

3. Error and discrepancy

It is of special interest to know something about the maximal error of this measuring procedure. It is given by

$$\sup_{0\leqslant E\leqslant 1}\left|\frac{1}{T}\int_0^T \mathbf{1}_{[0,E]}(r(t))\,\mathrm{d}t-E\right| = D_T(r(t)), \tag{6}$$

or by

$$\sup_{0\leqslant E\leqslant 1}\left|\frac{1}{N}\sum_{n=1}^{N}\mathbf{1}_{[0,E]}(r_n)-E\right|=D_N(r_n), \tag{7}$$

which is known as the discrepancy $D_T(r(t))$ of the function $r(t)$ or as the discrepancy $D_N(r_n)$ of the sequence r_n.

It should be noticed that the function of a sequence is uniformly distributed if and only if the discrepancy converges to 0 if T or N tends to infinity.

4. Product and mean value

Let $e_1(t)=E_1\in[0,1]$, $e_2(t)=E_2\in[0,1]$ be two constant (normalized) input signals and $r_1(t)$, $r_2(t)$ two reference signals. If $r_1(t)$ and $r_2(t)$ are independently uniformly distributed—this means that

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T\mathbf{1}_{[0,E_1]}(r_1(t))\,\mathbf{1}_{[0,E_2]}(r_2(t))\,\mathrm{d}t=E_1E_2 \tag{8}$$

for all E_1, $E_2\in[0,1]$—then

$$M(T)=\frac{1}{T}\int_0^T z_1(t)z_2(t)\,\mathrm{d}t, \tag{9}$$

where $z_i(t)=\mathbf{1}_{[0,E_i]}(r_i(t))$ $(i=1,2)$ is an approximation for the product E_1E_2. Figure 2 illustrates this method.

The error can be estimated by the two-dimensional discrepancy

$$D_T(r_1(t),r_2(t))=\sup_{0\leqslant E_1,E_2\leqslant 1}\left|\frac{1}{T}\int_0^T\mathbf{1}_{[0,E_1]}(r_1(t))\,\mathbf{1}_{[0,E_2]}(r_2(t))\,\mathrm{d}t-E_1E_2\right|. \tag{10}$$

Again $M(T)$ can be approximated by sampling $z(t)=z_1(t)z_2(t)$ and counting $z_n=z(nT_0)$.

If $r_{1n}=r_1(nT_0)$, $r_{2n}=r_2(nT_0)$ are independently uniformly distributed—this means that the two-dimensional discrepancy

$$D_N(r_{1n},r_{2n})=\sup_{0\leqslant E_1,E_2\leqslant 1}\left|\frac{1}{N}\sum_{n=1}^{N}\mathbf{1}_{[0,E_1]}(r_{1n})\,\mathbf{1}_{[0,E_2]}(r_{2n})-E_1E_2\right| \tag{11}$$

converges to 0 as N tends to infinity—then the measuring error gets arbitrarily small.

In many cases the input signal $e(t)$ is nonconstant but varies due to a noise or deterministically. In both cases it would be useful if $M(T)$ or M_N converges to the mean value of $e(t)$. First consider the case that $e(t)$ is periodic with period T_1. Here

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T\mathbf{1}_{[0,e(t)]}(r(t))\,\mathrm{d}t=\frac{1}{T_1}\int_0^{T_1}e(t)\,\mathrm{d}t \tag{12}$$

if $r(t)$ is uniformly distributed and independent of t/T_1 (mod 1). This can be seen by the following observation. Let $f(x, y) = \mathbf{1}_{[0,e(xT_1)]}(y)$ $(0 \leqslant x,\ y \leqslant 1)$. Then

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T \mathbf{1}_{[0,e(t)]}(r(t))\,\mathrm{d}t = \lim_{T\to\infty} \frac{1}{T}\int_0^T f\left(\frac{t}{T_1},\ r(t)\right)\mathrm{d}t$$
$$= \int_0^1\int_0^1 f(x,\ y)\,\mathrm{d}x\,\mathrm{d}y = \frac{1}{T_1}\int_0^{T_1} e(t)\,\mathrm{d}t. \tag{13}$$

If $e(t)$ is of bounded variation $V_0^{T_1}(e) = \int_0^{T_1} |\mathrm{d}e(t)|$, then the measuring error can be estimated by using an inequality of Niederreiter and Wills [4]:

$$\left|\frac{1}{T}\int_0^T \mathbf{1}_{[0,e(t)]}(r(t))\,\mathrm{d}t - \frac{1}{T_1}\int_0^{T_1} e(t)\,\mathrm{d}t\right|$$
$$\leqslant C\left(1 + V_0^{T_1}(e) + N_1\right) D_T^{1/2}\left(\frac{t}{T_1},\ r(t)\right), \tag{14}$$

where C is an absolute constant and N_1 is the number of points of discontinuity of $e(t)$, $t \in [0,\ T_1]$. A similar statement holds for sequences:

$$\left|\frac{1}{N}\sum_{n=1}^{N} \mathbf{1}_{[0,e(nT_0)]}(r(nT_0)) - \frac{1}{T_1}\int_0^{T_1} e(t)\,\mathrm{d}t\right|$$
$$\leqslant C\left(1 + V_0^{T_1}(e) + N_1\right) D_N^{1/2}\left(n\frac{T_0}{T_1},\ r(nT_0)\right). \tag{15}$$

Combining the methods to measure the product and the mean value, it is also possible to get an approximation for the mean-square-voltage:

$$\left|\frac{1}{T}\int_0^T \mathbf{1}_{[0,e(t)]}(r_1(t))\ \mathbf{1}_{[0,e(t)]}(r_2(t))\,\mathrm{d}t - \frac{1}{T_1}\int_0^{T_1} e(t)^2\,\mathrm{d}t\right|$$
$$\leqslant C\left(1 + V_0^{T_1}(e) + N_1\right) D_T^{1/3}\left(\frac{t}{T_1},\ r_1(t),\ r_2(t)\right) \tag{16}$$

and

$$\left|\frac{1}{N}\sum_{n=1}^{N} \mathbf{1}_{[0,e(nT_0)]}(r_1(nT_0))\ \mathbf{1}_{[0,e(nT_0)]}(r_2(nT_0)) - \frac{1}{T_1}\int_0^{T_1} e(t)^2\,\mathrm{d}t\right|$$
$$\leqslant C\left(1 + V_0^{T_1}(e) + N_1\right) D_N^{1/3}\left(n\frac{T_0}{T_1},\ r_1(nT_0),\ r_2(nT_0)\right). \tag{17}$$

Another method for estimating the error (14) and (16) can be found in [2]. The easiest way to generate a uniformly distributed reference function is to use a periodic linear function $r(t) = \alpha t$ (mod 1). The discrepancy satisfies

$$D_T(\alpha t) = \mathrm{O}\left(\frac{1}{\alpha T}\right). \tag{18}$$

The sampled sequence $r_n = \alpha T_0 n$ (mod 1) is uniformly distributed if and only if αT_0 is irrational. For example, if the partial quotients of αT_0 are bounded by K, then we have (see [3])

$$D_N(\alpha T_0 n) = \mathrm{O}\left(\frac{K}{\log K}\,\frac{\log N}{N}\right). \tag{19}$$

The two linear functions $r_1(t) = \alpha_1 t$ (mod 1), $r_2(t) = \alpha_2 t$ (mod 1) are independently uniformly distributed if the quotient α_1/α_2 is irrational. If α_1/α_2 is of approximation type < 2, then it can be shown that (see [1])

$$D_T(\alpha_1 t, \alpha_2 t) = O\left(\frac{1}{T}\right). \tag{20}$$

The sampled sequences $r_{n1} = \alpha_1 T_0 n$ (mod 1), $r_{n2} = \alpha_2 T_0 n$ (mod 1) are independently uniformly distributed if and only if 1, $\alpha_1 T_0$ and $\alpha_2 T_0$ are linear independent over the rationals. Estimates for the discrepancy can be obtained but again they depend on the approximation type of the vector $(1, \alpha_1 T_0, \alpha_2 T_0)$ (see [3]).

The situation for $(t/T_1, \alpha t)$ is quite similar; $\lim_{T\to\infty} D_T(t/T_1, \alpha t) = 0$ if and only if αT_1 is irrational and $\lim_{N\to\infty} D_N(nT_0/T_1, \alpha T_0 n) = 0$ if and only if 1, T_0/T_1 and αT_0 are linear independent over the rationals.

If $e(t)$ is not periodic, it is also possible to estimate the error:

$$\left| \frac{1}{N} \int_0^T \mathbf{1}_{[0,e(t)]}(r(t))\,\mathrm{d}t - \frac{1}{T} \int_0^T e(t)\,\mathrm{d}t \right| \leqslant C\left(1 + V_0^T(e) + N_1(T)\right) D_T^{1/2}\left(\frac{t}{T}, r(t)\right) \tag{21}$$

and

$$\left| \frac{1}{N} \sum_{n=1}^{N} \mathbf{1}_{[0,e(nT_0)]}(r(nT_0)) - \frac{1}{T} \int_0^T e(t)\,\mathrm{d}t \right| \leqslant C\left(1 + V_0^{NT_0}(e) + N_1(NT_0)\right) D_N^{1/2}\left(\frac{n}{N}, r(nT_0)\right), \tag{22}$$

where $V_0^T(e)$ is the variation and $N_1(T)$ the number of discontinuity points of $e(t)$ in $[0, T]$.

5. Stochastic interpolation

A very important part in signal processing is analog-digital conversion. For example, if one wants to know the first 4 digits of the dyadic expansion (4 bit) one divides the interval [0, 1] into

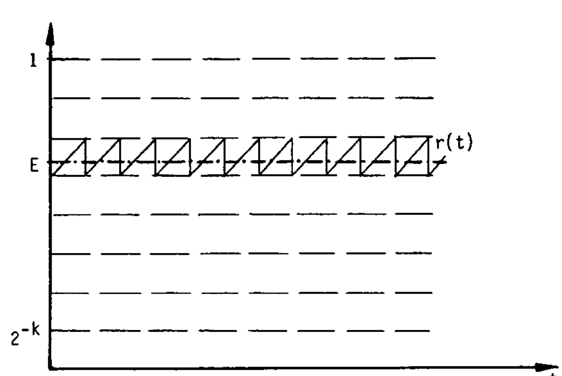

Fig. 3.

16 parts and decides with 4 comparators in which of the 16 subintervals the analog signal is situated. Mostly 4 bits are not enough for digital processing, 8 or 12 bits are the minimum for significant results. But the complexity to determine the dyadic expansion of the analog signal by this method up to 8 or 12 bits is unproportionally large. Therefore it is useful to combine this method with the SEM.

The first step is the same as before, k comparators decide in which of the 2^k subintervals the analog signal is situated. In the second step the SEM is used to determine a more exact value (see Fig. 3).

By this method it is possible to get approximately $3k$ bits. Therefore it is sufficient to use 4 comparators and the SEM to get 12 bits.

References

[1] M. Drmota, Irregularities of continuous distributions, *Ann. Inst. Fourier (Grenoble)* **39** (1989) 501–527.

[2] M. Drmota and R.F. Tichy, Deterministische Approximation stochastisch-ergodischer Signale, *Messen-steuern-regeln* **32** (1989) 109–113.

[3] L. Kuipers and H. Niederreiter, *Uniform Distribution of Sequences* (Wiley, New York, 1974).

[4] H. Niederreiter and J.M. Wills, Diskrepanz und Distanz von Maßen bezüglich konvexer und Jordanscher Mengen, *Math. Z.* **144** (1975) 125–134.

[5] W. Wehrmann, *Einführung in die Stochastisch-Ergodische Impulstechnik* (Oldenbourg, München, 1973).