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Quark mass and field anomalous dimensions to $\mathcal{O}(\alpha_s^5)$

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ABSTRACT: We present the results of the first complete analytic calculation of the quark mass and field anomalous dimensions to $\mathcal{O}(\alpha_s^5)$ in QCD.

KEYWORDS: Renormalization Group, QCD, Quark Masses and SM Parameters

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Contents

1	Introduction	1
2	Technical preliminaries	2
3	Results	3
4	Discussion	4
5	Applications	6
5.1	RGI mass	6
5.2	Higgs decay into quarks	7
6	Conclusions	8

1 Introduction

The quark masses depend on a renormalization scale. The dependence is usually referred to as “running” and is governed by the quark mass anomalous dimension, γ_m , defined as:

$$\mu^2 \frac{d}{d\mu^2} m|_{g^0, m^0} = m\gamma_m(a_s) \equiv -m \sum_{i \geq 0} \gamma_i a_s^{i+1}, \tag{1.1}$$

where $a_s = \alpha_s/\pi = g^2/(4\pi^2)$, g is the renormalized strong coupling constant and μ is the normalization scale in the customarily used $\overline{\text{MS}}$ renormalization scheme. Up to and including four loop level the anomalous dimension is known since long [1–5]. In this paper we will describe the results of calculation of γ_m and a related quantity — the quark field anomalous dimension — in the five-loop order.

The evaluation of the quark mass anomalous dimension with five-loop accuracy has important implications. The Higgs boson decay rate into charm and bottom quarks is proportional to the square of the respective quark mass at the scale of m_H and the uncertainty from the presently unknown 5-loop terms in the running of the quark mass is of order 10^{-3} . This is comparable to the precision advocated for experiments e.g. at TLEP [6]. Similarly, the issue of Yukawa unification is affected by precise predictions for the anomalous quark mass dimension.

The paper is organized as follows. The next section deals with the overall set-up of the calculations. Then we present our results (section 3), and a brief discussion (section 4) as well as a couple of selected applications (section 5). Our short conclusions are given in section 6.

2 Technical preliminaries

To calculate γ_m one needs to find the so-called quark mass renormalization constant, Z_m , which is defined as the ratio of the bare and renormalized quark masses, viz.

$$Z_m = \frac{m^0}{m} = 1 + \sum_{i,j}^{0 < j \leq i} (Z_m)_{ij} \frac{a_s^i}{\epsilon^j}. \quad (2.1)$$

Within the $\overline{\text{MS}}$ scheme [7, 8] the coefficients $(Z_m)_{ij}$ are just numbers [9]; $\epsilon \equiv 2 - D/2$ and D stands for the space-time dimension. Combining eqs. (1.1), (2.1) and using the RG-invariance of m^0 , one arrives at the following formula for γ_m :

$$\gamma_m = \sum_{i \geq 0} (Z_m)_{i1} i a_s^i. \quad (2.2)$$

To find Z_m one should compute the vector and scalar parts of the quark self-energy $\Sigma_V(p^2)$ and $\Sigma_S(p^2)$. In our convention, the bare quark propagator is proportional to $[\not{p}(1 + \Sigma_V^0(p^2)) - m_q^0(1 - \Sigma_S^0(p^2))]^{-1}$. Requiring the finiteness of the renormalized quark propagator and keeping only massless and terms linear in m_q , one arrives at the following recursive equations to find Z_m

$$Z_m Z_2 = 1 + K_\epsilon \{Z_m Z_2 \Sigma_S^0(p^2)\}, \quad Z_2 = 1 - K_\epsilon \{Z_2 \Sigma_V^0(p^2)\}, \quad (2.3)$$

where $K_\epsilon \{f(\epsilon)\}$ stands for the singular part of the Laurent expansion of $f(\epsilon)$ in ϵ near $\epsilon = 0$ and Z_2 is the quark wave function renormalization constant. Eqs. (2.3) express Z_m through massless propagator-type (that is dependent on one external momentum only) Feynman integrals (FI), denoted as *p-integrals* below.

Eqs. (2.3) require the calculation of a large number¹ of the *five-loop* p-integrals to find Z_m and Z_2 to $\mathcal{O}(\alpha_s^5)$.

At present there exists no direct way to analytically evaluate five-loop p-integrals. However, according to (2.1) for a given five-loop p-integral we need to know only its *pole* part in ϵ in the limit of $\epsilon \rightarrow 0$. A proper use of this fact can significantly simplify our task. The corresponding method — so-called Infrared Rearrangement (IRR)—first suggested in [11] and elaborated further in [12–14] allows to effectively decrease number of loops to be computed by one.² In its initial version IRR was not really universal; it was not applicable in some (though rather rare) cases of complicated FI's. The problem was solved by elaborating a special technique of subtraction of IR divergences — the R^* -operation [15, 16]. This technique succeeds in expressing the UV counterterm of every L-loop Feynman integral in terms of divergent and finite parts of some (L-1)-loop massless propagators.

In our case $L = 5$ and, using IRR, one arrives at around 10^5 four-loop p-integrals. These can, subsequently, be reduced to 28 four-loop master-integrals, which are known analytically, including their finite parts, from [17, 18] as well as numerically from [19].

¹We have used QGRAF [10] to produce around 10^5 FI's contributing to the quark self-energy at $\mathcal{O}(\alpha_s^5)$.

²With the price that resulting one-loop-less p-integrals should be evaluated up to and *including* their constant part in the small ϵ -expansion.

We need, thus, to compute around 10^5 p-integrals. Their singular parts, in turn, can be algebraically reduced to only 28 master 4-loop p-integrals. The reduction is based on evaluating sufficiently many terms of the $1/D$ expansion [20] of the corresponding coefficient functions [21].

All our calculations have been performed on a SGI ALTIX 24-node IB-interconnected cluster of eight-cores Xeon computers using parallel MPI-based [22] as well as thread-based [23] versions of FORM [24].

3 Results

Our result for the anomalous dimension

$$\gamma_m = - \sum_{i=0}^{\infty} (\gamma_m)_i a_s^{i+1}$$

reads:

$$(\gamma_m)_0 = 1, \quad (\gamma_m)_1 = \frac{1}{16} \left\{ \frac{202}{3} + n_f \left[-\frac{20}{9} \right] \right\}, \quad (3.1)$$

$$(\gamma_m)_2 = \frac{1}{64} \left\{ 1249 + n_f \left[-\frac{2216}{27} - \frac{160}{3} \zeta_3 \right] + n_f^2 \left[-\frac{140}{81} \right] \right\}, \quad (3.2)$$

$$\begin{aligned} (\gamma_m)_3 = \frac{1}{256} \left\{ \frac{4603055}{162} + \frac{135680}{27} \zeta_3 - 8800 \zeta_5 \right. \\ \left. + n_f \left[-\frac{91723}{27} - \frac{34192}{9} \zeta_3 + 880 \zeta_4 + \frac{18400}{9} \zeta_5 \right] \right. \\ \left. + n_f^2 \left[\frac{5242}{243} + \frac{800}{9} \zeta_3 - \frac{160}{3} \zeta_4 \right] + n_f^3 \left[-\frac{332}{243} + \frac{64}{27} \zeta_3 \right] \right\}, \quad (3.3) \end{aligned}$$

$$\begin{aligned} (\gamma_m)_4 = \frac{1}{4^5} \left\{ \frac{99512327}{162} + \frac{46402466}{243} \zeta_3 + 96800 \zeta_3^2 - \frac{698126}{9} \zeta_4 \right. \\ \left. - \frac{231757160}{243} \zeta_5 + 242000 \zeta_6 + 412720 \zeta_7 \right. \\ \left. + n_f \left[-\frac{150736283}{1458} - \frac{12538016}{81} \zeta_3 - \frac{75680}{9} \zeta_3^2 + \frac{2038742}{27} \zeta_4 \right. \right. \\ \left. \left. + \frac{49876180}{243} \zeta_5 - \frac{638000}{9} \zeta_6 - \frac{1820000}{27} \zeta_7 \right] \right. \\ \left. + n_f^2 \left[\frac{1320742}{729} + \frac{2010824}{243} \zeta_3 + \frac{46400}{27} \zeta_3^2 - \frac{166300}{27} \zeta_4 - \frac{264040}{81} \zeta_5 + \frac{92000}{27} \zeta_6 \right] \right. \\ \left. + n_f^3 \left[\frac{91865}{1458} + \frac{12848}{81} \zeta_3 + \frac{448}{9} \zeta_4 - \frac{5120}{27} \zeta_5 \right] + n_f^4 \left[-\frac{260}{243} - \frac{320}{243} \zeta_3 + \frac{64}{27} \zeta_4 \right] \right\}. \quad (3.4) \end{aligned}$$

Here ζ is the Riemann zeta-function ($\zeta_3 = 1.202056903\dots$, $\zeta_4 = \pi^4/90$, $\zeta_5 = 1.036927755\dots$, $\zeta_6 = 1.017343062\dots$ and $\zeta_7 = 1.008349277\dots$). Note that in four-loop order we exactly³ reproduce well-known results obtained in [4, 5]. The n_f^3 and n_f^4

³This agreement can be also considered as an important check of all our setup which is completely different from the ones utilized at the four-loop calculations.

terms in (3.4) are in full agreement with the results derived previously on the basis of the $1/n_f$ method in [25–27].

For completeness we present below the result for the quark field anomalous dimension $\gamma_2 = -\sum_{i=0}^{\infty} (\gamma_2)_i a_s^{i+1}$:

$$\begin{aligned}
 (\gamma_2)_4 = \frac{1}{4^5} & \left\{ \frac{2798900231}{7776} + \frac{17969627}{864} \zeta_3 + \frac{13214911}{648} \zeta_3^2 + \frac{16730765}{864} \zeta_4 - \frac{832567417}{3888} \zeta_5 \right. \\
 & + \frac{40109575}{1296} \zeta_6 + \frac{124597529}{1728} \zeta_7 \\
 & + n_f \left[-\frac{861347053}{11664} - \frac{274621439}{11664} \zeta_3 + \frac{1960337}{972} \zeta_3^2 + \frac{465395}{1296} \zeta_4 \right. \\
 & \left. + \frac{22169149}{5832} \zeta_5 + \frac{1278475}{1944} \zeta_6 + \frac{3443909}{216} \zeta_7 \right] \\
 & + n_f^2 \left[\frac{37300355}{11664} + \frac{1349831}{486} \zeta_3 - \frac{128}{9} \zeta_3^2 - \frac{27415}{54} \zeta_4 - \frac{12079}{27} \zeta_5 - \frac{800}{9} \zeta_6 - \frac{1323}{2} \zeta_7 \right] \\
 & \left. + n_f^3 \left[-\frac{114049}{8748} - \frac{1396}{81} \zeta_3 + \frac{208}{9} \zeta_4 \right] + n_f^4 \left[\frac{332}{729} - \frac{64}{81} \zeta_3 \right] \right\}. \tag{3.5}
 \end{aligned}$$

The above result is presented for the Feynman gauge; the coefficients $(\gamma_2)_i$ with $i \leq 3$ can be found in [28] (for the case of a general covariant gauge and $SU(N)$ gauge group).

4 Discussion

In numerical form γ_m reads

$$\begin{aligned}
 \gamma_m = & -a_s - a_s^2 (4.20833 - 0.138889n_f) \\
 & - a_s^3 (19.5156 - 2.28412n_f - 0.0270062n_f^2) \\
 & - a_s^4 (98.9434 - 19.1075n_f + 0.276163n_f^2 + 0.00579322n_f^3) \\
 & - a_s^5 (559.7069 - 143.6864 n_f + 7.4824 n_f^2 + 0.1083 n_f^3 - 0.000085359 n_f^4) \tag{4.1}
 \end{aligned}$$

and

$$\begin{aligned}
 \gamma_m \Big|_{n_f=3} &= -a_s - 3.79167 a_s^2 - 12.4202 a_s^3 - 44.2629 a_s^4 - 198.907 a_s^5, \\
 \gamma_m \Big|_{n_f=4} &= -a_s - 3.65278 a_s^2 - 9.94704 a_s^3 - 27.3029 a_s^4 - 111.59 a_s^5, \\
 \gamma_m \Big|_{n_f=5} &= -a_s - 3.51389 a_s^2 - 7.41986 a_s^3 - 11.0343 a_s^4 - 41.8205 a_s^5, \\
 \gamma_m \Big|_{n_f=6} &= -a_s - 3.37500 a_s^2 - 4.83867 a_s^3 + 4.50817 a_s^4 + 9.76016 a_s^5. \tag{4.2}
 \end{aligned}$$

Note that significant cancellations between n_f^0 and n_f^1 terms for the values of n_f around 3 or so persist also at five-loop order. As a result we observe a moderate growth of the series in a_s appearing in the quark mass anomalous dimension at various values of active quark flavours (recall that even for scales as small as 2 GeV $a_s \equiv \frac{\alpha_s}{\pi} \approx 0.1$).

n_f	3	4	5	6
$(\gamma_m)_4^{\text{exact}}$	198.899	111.579	41.807	-9.777
$(\gamma_m)_4^{\text{APAP [29]}}$	162.0	67.1	-13.7	-80.0
$(\gamma_m)_4^{\text{APAP [30]}}$	163.0	75.2	12.6	12.2
$(\gamma_m)_4^{\text{APAP [31]}}$	164.0	71.6	-4.8	-64.6

Table 1. The exact results for $(\gamma_m)_4$ together with the predictions made with the help of the original APAP method and its two somewhat modified versions.

Similar behavior shows up for γ_2 :

$$\begin{aligned}
 \gamma_2 = & -0.33333a_s - a_s^2(-1.9583 + 0.08333 n_f) \\
 & -a_s^3(-10.3370 + 1.0877 n_f - 0.01157 n_f^2) \\
 & -a_s^4(-53.0220 + 10.1090 n_f - 0.27703 n_f^2 - 0.0023 n_f^3) \\
 & -a_s^4(-310.0700 + 76.3260 n_f - 4.6339 n_f^2 + 0.0085 n_f^3 + 0.00048 n_f^4) \quad (4.3)
 \end{aligned}$$

and

$$\begin{aligned}
 \gamma_2 \Big|_{n_f=3} &= -0.33333 a_s - 1.7083 a_s^2 - 7.1779 a_s^3 - 25.2480 a_s^4 - 122.5300 a_s^5, \\
 \gamma_2 \Big|_{n_f=4} &= -0.33333 a_s - 1.6250 a_s^2 - 6.1712 a_s^3 - 17.1610 a_s^4 - 78.2430 a_s^5, \\
 \gamma_2 \Big|_{n_f=5} &= -0.33333 a_s - 1.5417 a_s^2 - 5.1877 a_s^3 - 9.6824 a_s^4 - 42.9240 a_s^5, \\
 \gamma_2 \Big|_{n_f=6} &= -0.33333 a_s - 1.4583 a_s^2 - 4.2274 a_s^3 - 2.8251 a_s^4 - 16.4710 a_s^5. \quad (4.4)
 \end{aligned}$$

It is instructive to compare our numerical result for $(\gamma_m)_4$

$$(\gamma_m)_4 = 559.71 - 143.6 n_f + 7.4824 n_f^2 + 0.1083 n_f^3 - 0.00008535 n_f^4 \quad (4.5)$$

with a 15 years old prediction based on the ‘‘Asymptotic Páde Approximants’’ (APAP) method [29] (the n_f^4 term below was used as the input)

$$(\gamma_m)_4^{\text{APAP}} = 530 - 143 n_f + 6.67 n_f^2 + 0.037 n_f^3 - 0.00008535 n_f^4. \quad (4.6)$$

Unfortunately, this impressively good agreement does *not* survive for fixed values of n_f due to severe cancellations between different powers of n_f as one can see from the table 1.

The solution of eq. (1.1) reads:

$$\frac{m(\mu)}{m(\mu_0)} = \frac{c(a_s(\mu))}{c(a_s(\mu_0))}, \quad c(x) = \exp \left\{ \int dx' \frac{\gamma_m(x')}{\beta(x')} \right\}, \quad (4.7)$$

$$\begin{aligned}
 c(x) = & (x)^{\tilde{\gamma}_0} \left\{ 1 + d_1 x + (d_1^2/2 + d_2) x^2 + (d_1^3/6 + d_1 d_2 + d_3) x^3 \right. \\
 & \left. + (d_1^4/24 + d_1^2 d_2/2 + d_2^2/2 + d_1 d_3 + d_4) x^4 + \mathcal{O}(x^5) \right\}, \quad (4.8)
 \end{aligned}$$

$$d_1 = -\bar{\beta}_1 \bar{\gamma}_0 + \bar{\gamma}_1, \tag{4.9}$$

$$d_2 = \bar{\beta}_1^2 \bar{\gamma}_0/2 - \bar{\beta}_2 \bar{\gamma}_0/2 - \bar{\beta}_1 \bar{\gamma}_1/2 + \bar{\gamma}_2/2, \tag{4.10}$$

$$d_3 = -\bar{\beta}_1^3 \bar{\gamma}_0/3 + 2\bar{\beta}_1 \bar{\beta}_2 \bar{\gamma}_0/3 - \bar{\beta}_3 \bar{\gamma}_0/3 + \bar{\beta}_1^2 \bar{\gamma}_1/3 - \bar{\beta}_2 \bar{\gamma}_1/3 - \bar{\beta}_1 \bar{\gamma}_2/3 + \bar{\gamma}_3/3, \tag{4.11}$$

$$d_4 = \bar{\beta}_1^4 \bar{\gamma}_0/4 - 3\bar{\beta}_1^2 \bar{\beta}_2 \bar{\gamma}_0/4 + \bar{\beta}_2^2 \bar{\gamma}_0/4 + \bar{\beta}_1 \bar{\beta}_3 \bar{\gamma}_0/2 - \bar{\beta}_4 \bar{\gamma}_0/4 - \bar{\beta}_1^3 \bar{\gamma}_1/4 + \bar{\beta}_1 \bar{\beta}_2 \bar{\gamma}_1/2 - \bar{\beta}_3 \bar{\gamma}_1/4 + \bar{\beta}_1^2 \bar{\gamma}_2/4 - \bar{\beta}_2 \bar{\gamma}_2/4 - \bar{\beta}_1 \bar{\gamma}_3/4 + \bar{\gamma}_4/4. \tag{4.12}$$

Here $\bar{\gamma}_i = (\gamma_m)_i/\beta_0$, $\bar{\beta}_i = \beta_i/\beta_0$ and

$$\beta(a_s) = -\sum_{i \geq 0} \beta_i a_s^{i+2} = -\beta_0 \left\{ \sum_{i \geq 0} \bar{\beta}_i a_s^{i+2} \right\}$$

is the QCD β -function. Unfortunately, the coefficient d_4 in eq. (4.12) does depend on the yet unknown *five-loop* coefficient β_4 (up to four loops the β -function is known from [14, 32–39]).

Numerically, the c -function reads:

$$c(x) \stackrel{n_f=3}{=} x^{4/9} c_s(x), \quad c(x) \stackrel{n_f=4}{=} x^{12/25} c_c(x), \quad c(x) \stackrel{n_f=5}{=} x^{12/23} c_b(x), \quad c(x) \stackrel{n_f=6}{=} x^{4/7} c_t(x),$$

with

$$\begin{aligned} c_s(x) &= 1 + 0.8950 x + 1.3714 x^2 + 1.9517 x^3 + (15.6982 - 0.11111 \bar{\beta}_4) x^4, \\ c_c(x) &= 1 + 1.0141 x + 1.3892 x^2 + 1.0905 x^3 + (9.1104 - 0.12000 \bar{\beta}_4) x^4, \\ c_b(x) &= 1 + 1.1755 x + 1.5007 x^2 + 0.17248 x^3 + (2.69277 - 0.13046 \bar{\beta}_4) x^4, \\ c_t(x) &= 1 + 1.3980 x + 1.7935 x^2 - 0.68343 x^3 + (-3.5130 - 0.14286 \bar{\beta}_4) x^4. \end{aligned} \tag{4.13}$$

5 Applications

5.1 RGI mass

Eq. (4.7) naturally leads to an important concept: the RGI mass

$$m^{\text{RGI}} \equiv m(\mu_0)/c(a_s(\mu_0)), \tag{5.1}$$

which is often used in the context of lattice calculations. The mass is μ and *scheme* independent; in *any* (mass-independent) scheme

$$\lim_{\mu \rightarrow \infty} a_s(\mu)^{-\bar{\gamma}_0} m(\mu) = m^{\text{RGI}}.$$

The function $c_s(x)$ is used, e.g. by the *ALPHA* lattice collaboration to find the $\overline{\text{MS}}$ mass of the strange quark at a lower scale, say, $m_s(2 \text{ GeV})$ from the m_s^{RGI} mass determined from lattice simulations (see, e.g. [40]). For example, setting $a_s(\mu = 2 \text{ GeV}) = \frac{\alpha_s(\mu)}{\pi} = 0.1$, we arrive at (h counts loops):

$$\begin{aligned} m_s(2 \text{ GeV}) &= m_s^{\text{RGI}} (a_s(2 \text{ GeV}))^{\frac{4}{9}} \left(1 + 0.0895 h^2 + 0.0137 h^3 + 0.00195 h^4 \right. \\ &\quad \left. + (0.00157 - 0.000011 \bar{\beta}_4) h^5 \right) \end{aligned} \tag{5.2}$$

In order to have an idea of effects due the five-loop term in (5.2) one should make a guess about $\bar{\beta}_4$. By inspecting lower orders in

$$\beta(n_f = 3) = - \left(\frac{4}{9} \right) \left(a_s + 1.777 a_s^2 + 4.4711 a_s^3 + 20.990 a_s^4 + \bar{\beta}_4 a_s^5 \right)$$

one can assume a natural estimate of $\bar{\beta}_4$ as laying in the interval 50 – 100. With this choice we conclude that the (apparent) convergence of the above series is quite good even at a rather small energy scale of 2 GeV.

On the other hand, the authors of [30] estimate $\bar{\beta}_4$ in the $n_f = 3$ QCD as large as -850! With such a huge and negative value of $\bar{\beta}_4$ the five loop term in (5.2) would amount to 0.01092 and, thus, would significantly exceed the four-loop contribution (0.00195).

5.2 Higgs decay into quarks

The decay width of the Higgs boson into a pair of quarks can be written in the form

$$\Gamma(H \rightarrow \bar{f}f) = \frac{G_F M_H}{4\sqrt{2}\pi} m_f^2(\mu) R^S(s = M_H^2, \mu) \tag{5.3}$$

where μ is the normalization scale and R^S is the spectral density of the scalar correlator, known to α_s^4 from [41]

$$\begin{aligned} R^S(s = M_H^2, \mu = M_H) &= 1 + 5.667 a_s + 29.147 a_s^2 + 41.758 a_s^3 - 825.7 a_s^4 \\ &= 1 + 0.2041 + 0.0379 + 0.0020 - 0.00140 \end{aligned} \tag{5.4}$$

where we set $a_s = \alpha_s/\pi = 0.0360$ (for the Higgs mass value $M_H = 125$ GeV and $\alpha_s(M_Z) = 0.118$).

Expression (5.3) depends on two phenomenological parameters, namely, $\alpha_s(M_H)$ and the quark running mass m_q . In what follows we consider, for definiteness, the dominant decay mode $H \rightarrow \bar{b}b$. To avoid the appearance of large logarithms of the type $\ln \mu^2/M_H^2$ the parameter μ is customarily chosen to be around M_H . However, the starting value of m_b is usually determined at a much smaller scale (typically around 5-10 GeV [42]). The evolution of $m_b(\mu)$ from a lower scale to $\mu = M_H$ is described by a corresponding RG equation which is completely fixed by the quark mass anomalous dimension $\gamma(\alpha_s)$ and the QCD beta function $\beta(\alpha_s)$ (for QCD with $n_f = 5$). In order to match the $\mathcal{O}(\alpha_s^4)$ accuracy of (5.4) one should know *both* RG functions β and γ_m in the five-loop approximation. Let us proceed, assuming conservatively that $0 \leq \bar{\beta}_4^{n_f=5} \leq 200$.

The value of $m_b(\mu = M_H)$ is to be obtained with RG running from $m_b(\mu = 10 \text{ GeV})$ and, thus, depends on β and γ_m . Using the Mathematica package RunDec⁴ [43] and eq. (4.13) we find for the shift from the five-loop term

$$\frac{\delta m_b^2(M_H)}{m_b^2(M_H)} = -1.3 \cdot 10^{-4}(\bar{\beta}_4 = 0) - 4.3 \cdot 10^{-4}(\bar{\beta}_4 = 100) - 7.3 \cdot 10^{-4}(\bar{\beta}_4 = 200)$$

⁴We have extended the package by including the five-loop effects to the running of α_s and quark masses.

If we set $\mu = M_H$, then the combined effect of $\mathcal{O}(\alpha_s^4)$ terms as coming from the five-loop running and four-loop contribution to R^S on

$$\Gamma(H \rightarrow \bar{b}b) = \frac{G_F M_H}{4\sqrt{2}\pi} m_f^2(M_H) R^S(s = M_H^2, M_H) \quad (5.5)$$

is around -2‰ (for $\bar{\beta}_4 = 100$). This should be contrasted to the parametric uncertainties coming from the input parameters $\alpha_s(M_Z) = 0.1185(6)$ [44] and $m_b(m_b) = 4.169(8)$ GeV [45] which correspond to $\pm 1\%$ and $\pm 4\%$ respectively.

We conclude, that the $\mathcal{O}(\alpha_s^4)$ terms in (5.4), (5.5) are of no phenomenological relevancy at present. But, the situation could be different if the project of TLEP [6] is implemented. For instance, the uncertainty in $\alpha_s(M_Z)$ could be reduced to $\pm 2\%$ and Higgs boson branching ratios with precisions in the permille range are advertised.

6 Conclusions

We have analytically computed the anomalous dimensions of the quark mass γ_m and field γ_2 in the five loop approximation. The self-consistent description of the quark mass evolution at five loop requires the knowledge of the QCD β -function to the same number of loops. The corresponding, significantly more complicated calculation is under consideration.

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