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# On a Theorem of T. Gneiting on $\alpha$-Symmetric Multivariate Characteristic Functions

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T. Gneiting (1998, *J. Multivariate Analysis* **64**, 131–147) proved a relation between the primitives of the classes $\Phi_d(2)$ and $\Phi_d(1)$ of 2- and 1-symmetric characteristic functions on $\mathbb{R}^d$, respectively. We will give a straightforward proof of



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## 1. INTRODUCTION

The characteristic function of a $d$-dimensional distribution is called α-*symmetric*, $\alpha > 0$, if it can be written in the form

$$\varphi((|v_1|^\alpha + \cdots + |v_d|^\alpha)^{1/\alpha}), \qquad v = (v_1, ..., v_d) \in \mathbb{R}^d, \tag{1}$$

where $\varphi \in C(\overline{\mathbb{R}}_+)$. The set of all these functions will be denoted by $\Phi_d(\alpha)$.

Functions of type (1) are called *radial with respect to the* $l_\alpha$-*norm*, or α-*radial* for short. Due to Bochner's famous theorem α-symmetric characteristic functions can be interpreted as positive definite α-radial functions.

A basic notion in this context is the so-called *scale mixture* of a function $f$ on $\mathbb{R}^d$, i.e., a function $g$ on $\mathbb{R}^d$ for which there exists a distribution function $G$ on $\overline{\mathbb{R}}_+$ such that

$$g(x) = \int_0^\infty f(x\tau)\, dG(\tau), \qquad x \in \mathbb{R}^d.$$

The classes we are interested in consist of all functions which can be represented as scale mixtures of a specific function, which we will call the *primitive* generating the class.



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In 1938 Schoenberg [9] characterized the class $\Phi_d(2)$ as the set of all scale mixtures of the function $\Omega_d$; i.e., as the set of all $\varphi$ satisfying

$$\varphi(t)=\int_0^\infty \Omega_d(tu)\,dF(u), \qquad t\in\mathbb{R}_+, \tag{2}$$

where the primitive $\Omega_d$ is defined by

$$\Omega_d(u)=\Gamma\left(\frac{d}{2}\right)\left(\frac{u}{2}\right)^{-(d-2)/2} J_{(d-2)/2}(u), \qquad u\in\mathbb{R}_+. \tag{3}$$

$J_\nu(u)$ is the *Bessel function of the first kind of order* $\nu$, $\nu\in\mathbb{R}$.

For the class $\Phi_d(1)$, Cambanis *et al.* [4] gave a similar characterization: a function $\varphi$ is an element of $\Phi_d(1)$ if and only if $\varphi$ can be expressed in the form

$$\varphi(t)=\int_0^\infty \omega_d(tu)\,dF(u), \qquad t\in\mathbb{R}_+, \tag{4}$$

$F$, again, being a distribution function on $\overline{\mathbb{R}}_+$. The primitive of the class $\Phi_d(1)$ is given as a Hankel transform by

$$\omega_d(u)=\frac{2^{d/2}\Gamma^2\left(\frac{d}{2}\right)}{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{d-1}{2}\right)}u^{-(d-2)/2}\int_1^\infty(\tau^2-1)^{(d-3)/2}\tau^{-(3d-4)/2} \times J_{(d-2)/2}(u\tau)\,d\tau, \qquad u\in\mathbb{R}_+. \tag{5}$$

In this connection, Gneiting [5] recently proved an interesting relation between the two primitives.

THEOREM (Gneiting). *For the primitive* $\omega_d$ *of the class* $\Phi_d(1)$ *and the primitive* $\Omega_{2d-1}$ *of the class* $\Phi_{2d-1}(2)$ *we have*

$$\omega_d(t)=\frac{\Gamma^2\left(\frac{d}{2}\right)}{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{2d-1}{2}\right)}I^{d-1}\Omega_{2d-1}(t), \qquad t\in\mathbb{R}_+, \tag{6}$$

*where* $If(t)=\int_t^\infty f(u)\,du$, $t\in\mathbb{R}_+$, *defined for all functions* $f\colon\overline{\mathbb{R}}_+\to\mathbb{R}$ *for which* $If(0)$ *exists.*

Gneiting proved the theorem by using arguments similar to those of Cambanis. His proof seems to hide the simple structure of the relation (6). Therefore, Gneiting himself asked in [6] on page 24 for a more direct verification just using integration and/or differentiation arguments.

In [2] Berens and the current author gave a representation of the primitives of both classes in terms of *generalized hypergeometric functions*. The advantage of using special functions lies in the fact that there is a powerful calculus available to handle operations like integration and differentiation, as well as integral transforms. We want to use this calculus to give a straightforward proof of the theorem.

## 2. SOME SPECIAL FUNCTION THEORY

In connection with summability of the inverse Fourier integral on $\mathbb{R}^d$, Berens and Xu [1] reestablished among others Cambanis' characterization of $\Phi_d(1)$ (cf. [4, Theorem 3.1]); except for the factor $2^d\Gamma^2(d/2)$ they identified the primitive $\omega_d$ as the Fourier integral of the $(d-1)$st B-spline $M_{d-1}(u \mid x_1^2, ..., x_d^2)$, where $x=(x_1, ..., x_d)$ is a point in $\mathbb{R}^d$ and $u$ is a parameter in $\mathbb{R}_+$. To be precise, independent of Cambanis they proved

$$M_{d-1}(u \mid (\cdot)_1^2, ..., (\cdot)_d^2)^\wedge\ (v)=\frac{(\sqrt{u})^{d-2}}{\Gamma\left(\frac{1}{2}\right)\Gamma^2\left(\frac{d}{2}\right)}\,\omega_d(\sqrt{u}\,|v|_1), \qquad v\in\mathbb{R}^d. \tag{7}$$

In [2] the authors picked up the paper [1] and reformulated and reproved the basic results using the calculus of generalized hypergeometric functions. Indeed, the primitives $\Omega_d$ and $\omega_d$ can be represented as ${}_pF_q$-symbols. We postpone a detailed discussion of formula (7) for the appendix, since it is not of immediate relevance for the understanding of the following proof.

Using the *Pochhammer symbol* $(a)_\nu=a\cdot(a+1)\cdots(a+\nu-1)$, $\nu\in\mathbb{N}$, $(a)_0:=1$, $a\in\mathbb{C}$, the ${}_pF_q$*-functions* are defined for some non-negative integers $p$ and $q$ as

$${}_pF_q\left[\begin{matrix}a_1, ..., a_p\\ b_1, ..., b_q\end{matrix}\,\middle|\, z\right]=\sum_{k=0}^{\infty}\frac{(a_1)_k\cdots(a_p)_k}{(b_1)_k\cdots(b_q)_k}\cdot\frac{z^k}{k!}, \qquad z\in\mathbb{C},$$

where $p$ denotes the number of numerator parameters and $q$ the number of denominator parameters; both groups of parameters can be chosen in the complex plane. In this context the notion *symbol* is used if the hypergeometric series is handled in a formal sense. For a general treatment of the theory, especially for questions of convergence, asymptotic behavior, etc., the reader is referred to the book by Luke [7].

One example, which will be needed in the following proof, is given by

$$J_\nu(z)=\frac{\left(\frac{z}{2}\right)^\nu}{\Gamma(\nu+1)}\,{}_0F_1\left(\begin{matrix} \\ \nu+1\end{matrix}\,\middle|\,-\frac{z^2}{4}\right) \tag{8}$$

[7, Eq. 6.2.7(1)].

To verify Eq. (6) we need two more formulae. The $n$th derivative of the symbol ${}_pF_q$ is again a hypergeometric ${}_pF_q$-series (cf. [7, 3.4.(1)]) and is given by

$$\frac{d^n}{dz^n}\,{}_pF_q\left[\begin{matrix}a_1,...,a_p\\ b_1,...,b_q\end{matrix}\,\middle|\,z\right] =\frac{(a_1)_n\cdots(a_p)_n}{(b_1)_n\cdots(b_q)_n}\,{}_pF_q\left[\begin{matrix}a_1+n,...,a_p+n\\ b_1+n,...,b_q+n\end{matrix}\,\middle|\,z\right]. \tag{9}$$

A second formula gives the derivative of a product of a ${}_pF_q$-symbol and a power of the argument (cf. [7, 3.4(2)]):

$$\frac{d^n}{dz^n}\left\{z^\delta\,{}_pF_q\left[\begin{matrix}a_1,...,a_p\\ b_1,...,b_q\end{matrix}\,\middle|\,z\right]\right\} =(\delta-n+1)_n\,z^{\delta-n}\,{}_{p+1}F_{q+1}\left[\begin{matrix}\delta+1,a_1,...,a_p\\ \delta+1-n,b_1,...,b_q\end{matrix}\,\middle|\,z\right]. \tag{10}$$

Finally, we need the representation of the primitive $\omega_d$ in terms of hypergeometric functions as given in [2]

$$\omega_d(\xi)={}_1F_2\left[\begin{matrix}-\frac{d-2}{2}\\ \frac{1}{2},\frac{d}{2}\end{matrix}\,\middle|\,\frac{-\xi^2}{4}\right] -\frac{\Gamma^2\left(\frac{d}{2}\right)}{\Gamma\left(\frac{d+1}{2}\right)\Gamma\left(\frac{d-1}{2}\right)}\,\xi\cdot{}_1F_2\left[\begin{matrix}-\frac{d-3}{2}\\ \frac{3}{2},\frac{d+1}{2}\end{matrix}\,\middle|\,-\frac{\xi^2}{4}\right],\qquad \xi\in\mathbb{R}_+. \tag{11}$$

## 3. PROOF OF THE THEOREM

As stated in [5] the operator $I$ can be inverted using differentiation. Instead of Eq. (6) we will therefore verify the equivalent statement

$$\Omega_{2d-1}(t)=(-1)^{d-1}\frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{2d-1}{2}\right)}{\Gamma^2\left(\frac{d}{2}\right)}\omega_d^{(d-1)}(t), \qquad t\in\mathbb{R}_+. \tag{12}$$

One of the two terms in (11) is a polynomial of degree $d-2$; for $d$ even it's the first term, for $d$ odd it's the second one. The $(d-1)$th derivative of the polynomial term therefore vanishes.

Let us first consider the case when $d$ is odd. Just for brevity we will use the notation

$$h(\xi)={}_1F_2\left[\begin{matrix}-\frac{d-2}{2}\\ \frac{1}{2},\frac{d}{2}\end{matrix}\,\middle|\,-\frac{\xi^2}{4}\right]$$

for the nonpolynomial term of the function $\omega_d$. The case $d=1$ is quickly verified and omitted. For $d>1$ formula (9) gives

$$\frac{d}{d\xi}h(\xi)=\frac{d-2}{d}\cdot 2(-1)\,i\cdot\left(-\frac{\xi^2}{4}\right)^{1/2}\cdot{}_1F_2\left[\begin{matrix}-\frac{d-2}{2}+1\\ \frac{3}{2},\frac{d}{2}+1\end{matrix}\,\middle|\,-\frac{\xi^2}{4}\right],$$

$i$ being the complex unit. Using (10) we continue differentiating to get

$$\frac{d^2}{d\xi^2}h(\xi)=\frac{d-2}{d}\left(\frac{\xi}{2}\right)^{-1}\frac{\xi}{2}\cdot{}_1F_2\left[\begin{matrix}-\frac{d-2}{2}+1\\ \frac{1}{2},\frac{d}{2}+1\end{matrix}\,\middle|\,-\frac{\xi^2}{4}\right].$$

We remark that the resulting series has a numerator and a denominator parameter in common. The series therefore reduces in order to the above ${}_1F_2$-function.

In the next step we use (9) again and continue differentiating. Let us remark that in each odd step the derivative of the argument $\{-(\cdot)^2/4\}$ of the hypergeometric function yields the complex factor which is needed to use formula (10). The negative sign is obtained by inserting the complex unit; this factor vanishes again in the even steps. Because of $d$ being odd, differentiation ends in an even step, which yields, after another reduction of order,

$$\left(\frac{d}{d\xi}\right)^{d-1} h(\xi) = C \cdot {}_1F_2\left[\begin{matrix} -\frac{d-2}{2}+\frac{d-1}{2} \\ \frac{1}{2}, \frac{d}{2}+\frac{d-1}{2} \end{matrix} \middle| -\frac{\xi^2}{4}\right]$$

$$= C \cdot {}_0F_1\left[\begin{matrix} \\ d-\frac{1}{2} \end{matrix} \middle| -\frac{\xi^2}{4}\right], \qquad \xi \in \mathbb{R}_+,$$

where $C$ denotes a constant which will be determined in the following paragraph. Setting $\nu = d - \frac{3}{2}$ in the representation (8) of the Bessel function, the ${}_0F_1$-series turns out to be the function $\Omega_{2d-1}$.

It remains to take a closer look at the constant $C$. After $d-1$ differentiations we have

$$C = \frac{(d-2)(d-4)\cdots(d-(d-1))}{d\cdot(d+2)\cdots(d+(d-3))} = \frac{2^{(d-1)/2}\Gamma\left(\frac{d}{2}\right)\cdot\Gamma\left(\frac{d}{2}\right)}{\Gamma\left(\frac{1}{2}\right)\cdot 2^{(d-1)/2}\Gamma\left(\frac{2d-1}{2}\right)}.$$

Observing that $(-1)^{d-1} = 1$, $C$ will be the right constant in (12).

The case of $d$ being even is done analogously. Using both differentiation formulae alternatively, the derivatives can again be expressed in terms of hypergeometric functions. Abbreviating again the nonpolynomial term of the function $\omega_d$ by

$$h(\xi) = \frac{(-1)\,\Gamma^2\left(\frac{d}{2}\right)}{\Gamma\left(\frac{d+1}{2}\right)\Gamma\left(\frac{d-1}{2}\right)}\,\xi\cdot {}_1F_2\left[\begin{matrix} -\frac{d-3}{2} \\ \frac{3}{2}, \frac{d+1}{2} \end{matrix} \middle| -\frac{\xi^2}{4}\right],$$

we get

$$\left(\frac{d}{d\xi}\right)^{d-1} h(\xi)$$

$$= C \cdot \frac{(-1)\,\Gamma^2\left(\frac{d}{2}\right)}{\Gamma\left(\frac{d+1}{2}\right)\Gamma\left(\frac{d-1}{2}\right)}\;{}_1F_2\left[\begin{matrix} -\frac{d-3}{2}+\frac{d-2}{2} \\ \frac{1}{2}, \frac{d+1}{2}+\frac{d-2}{2}\end{matrix}\,\middle|\, -\frac{\xi^2}{4}\right], \quad \xi \in \mathbb{R}_+.$$

Reduction of order then yields the function $\Omega_{2d-1}$. Here, the constant $C$ has the form

$$C = \frac{(d-3)(d-5)\cdots(d-(d-2)-1)}{(d+1)(d+3)\cdots(d+(d-2)-1)} = \frac{2^{(d-2)/2}\Gamma\left(\frac{d-1}{2}\right)\cdot\Gamma\left(\frac{d+1}{2}\right)}{\Gamma\left(\frac{1}{2}\right)\cdot 2^{(d-2)/2}\Gamma\left(\frac{2d-1}{2}\right)}.$$

Since $(-1)^{d-1} = -1$, this too leads to the constant in formula (12). ∎

## APPENDIX

The *Fourier transform* $\hat{f}$ of a function $f$ in $L(\mathbb{R}^d)$ is defined as $\hat{f}(v) = 1/(2\pi)^d \cdot \int_{\mathbb{R}^d} e^{-iv\cdot x} f(x)\,dx$, $v \in \mathbb{R}^d$. Starting with a function $\varphi$ in $C_0(\overline{\mathbb{R}}_+)$, we want to study nonnegative functions $f \in L(\mathbb{R}^d)$, the Fourier transform of which are positive definite and *α-radial*; i.e., $\hat{f}(v) = \varphi((|v_1|^\alpha + \cdots + |v_d|^\alpha)^{1/\alpha}) = \varphi(|v|_\alpha)$, $v \in \mathbb{R}^d$.

Following Richards [8] we get

$$f(x) = \int_0^\infty \varphi(\rho)\, E^{(\alpha)}_{\rho;\, d}(x)\, d\rho, \qquad x \in \mathbb{R}^d, \tag{13}$$

where the kernel $E^{(\alpha)}_{\rho;\, d}(x)$ is defined as

$$E^{(\alpha)}_{\rho;\, d}(x) = \rho^{d-1} \int_{|v|_\alpha = 1} e^{i\rho x\cdot v}\omega(v),$$

$$\omega(v) = \sum_{j=1}^{d} (-1)^{j-1} v_j\, dv_1 \cdots dv_{j-1}\, dv_{j+1} \cdots dv_d.$$

Note that Richards called the kernels $E^{(\alpha)}_{\rho;\, d}$ *generalized Bessel functions.*

At the first glance, the representation (13) is just a formal one, but for $\alpha=2$ we get the well-known representation

$$E^{(2)}_{\rho;\,d}(x)=(\sqrt{2\pi})^d\,|x|_2^{-(d-2)/2}\,\rho^{d/2}J_{(d-2)/2}(|x|_2\,\rho), \tag{14}$$

while for $\alpha=1$

$$E^{(1)}_{\rho;\,d}(x)=[x_1^2,\,...,\,x_d^2]\,H_{\rho;\,d}(\cdot). \tag{15}$$

The function $H_{\rho;\,d}$ is defined as

$$H_{\rho;\,d}(u)=(-1)^{[(d-1)/2]}\,2^d(\sqrt{u})^{d-1}\begin{cases}\sin\rho\sqrt{u}, & \text{for } d \text{ even},\\ \cos\rho\sqrt{u}, & \text{for } d \text{ odd},\end{cases}\qquad u\in\mathbb{R}_+.$$

The representation (15) for $\alpha=1$ was given by Berens and Xu [1], while formula (14) goes back to Schoenberg [9].

Using the formula of Curry and Schoenberg for the $n$th divided difference of a sufficiently smooth function $g$ on $\mathbb{R}$ with knots $x_0,\,...,\,x_n$, $n\in\mathbb{N}$, not all equal, i.e.,

$$[x_0,\,...,\,x_n]\;g=\int_{\mathbb{R}} g^{(n)}(u)\,M_n(u\,|\,x_0,\,...,\,x_n)\,du, \tag{16}$$

we get a second representation of the kernel $E^{(1)}_{\rho;\,d}$ which enables us to hide the multidimensional structure of $E_{\rho;\,d}$ into the B-spline kernel,

$$E^{(1)}_{\rho;\,d}(x)=\int_0^\infty H^{(d-1)}_{\rho;\,d}(u)\,M_{d-1}(u\,|\,x_1^2,\,...,\,x_d^2)\,du,\qquad x\in\mathbb{R}^d.$$

Compare this with formula (3.6) of [4].

The *B-spline* of order $n$, $n\in\mathbb{N}$, with knots $x_0,\,...,\,x_n$ on $\mathbb{R}$ hereby is defined as

$$M_n(u\,|\,x_0,\,...,\,x_n)=[x_0,\,...,\,x_n]\left\{\frac{(\cdot-u)_+^{n-1}}{(n-1)!}\right\},\qquad u\in\mathbb{R},$$

where $(u)_+=u$, for $u>0$, and is 0 otherwise. The spline vanishes outside the largest interval spanned by the knots, is strictly positive inside the interval, and satisfies $\int_{\mathbb{R}} M_n(u\,|\,x_0,\,...,\,x_n)\,du=1/n!$. Note that formula (16) can also be seen as a definition of the B-spline; cf. deBoor's survey [3] for details.

To simplify notation the authors of [1] introduced the functions

$$H_d(u) = (-1)^{[(d-1)/2]}\, u^{d-1} \begin{cases} \sin u, & \text{for } d \text{ even}, \\ \cos u, & \text{for } d \text{ odd}, \end{cases} \qquad \text{and}$$

$$h_d(u) = \left(\frac{1}{u}\frac{d}{du}\right)^{d-1} H_d(u).$$

Then

$$H_{\rho;\, d}(u) = \frac{2^d}{\rho^{d-1}}\, H_d(\rho \sqrt{u}) \qquad \text{and} \qquad H^{(d-1)}_{\rho;\, d}(u) = 2\rho^{d-1} h_d(\rho \sqrt{u}).$$

An obvious transformation of variables then gives

$$f(x) = \int_0^\infty \psi(\sqrt{u})\, M_{d-1}(u \mid x_1^2, ..., x_d^2)\, du, \qquad x \in \mathbb{R}^d, \tag{17}$$

where

$$\psi(u) = 2 \int_0^\infty \rho^{d-1} \varphi(\rho)\, h_d(u\rho)\, d\rho, \qquad u \in \mathbb{R}_+. \tag{18}$$

In [2] the authors studied the integral transform (18) from the point of view of special functions. They derived a representation of the kernel $h_d$ in terms of generalized hypergeometric functions and determined the kernel, say $m_d$, of the inverse transform

$$\varphi(\rho) = 2 \int_0^\infty u^{d-1} \psi(u)\, m_d(\rho u)\, du, \qquad \rho \in \mathbb{R}_+.$$

Formally taking the Fourier transform on both sides of Eq. (17) and changing the order of integration yields (7); i.e.,

$$\varphi(|v|_1) = \int_0^\infty \psi(\sqrt{u}) \left( \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{-iv \cdot x} M_{d-1}(u \mid x_1^2, ..., x_d^2)\, dx \right) du, \qquad v \in \mathbb{R}^d;$$

see [1] and [2] for details.

For $\alpha = 2$ the situation is more transparent. The reason lies in the fact that the Fourier transform of a 2-radial function is itself 2-radial and vice versa. This is not the case for 1-radial functions, as is shown above. Using $\Omega_d(\rho\, |x|_2)$ instead of $E^{(2)}_{\rho;\, d}(x)$, (13) reduces to the classical *Fourier–Bessel* *transform* which is self-inverse. To separate the one-dimensional radial part from the multidimensional component we just have to observe that

$$f(x) = \frac{2\pi^{d/2}}{\Gamma\left(\frac{d}{2}\right)} \int_0^\infty \rho^{d-1} \varphi(\rho)\, \Omega_d(|x|_2\, \rho)\, d\rho, \qquad x \in \mathbb{R}^d.$$

It follows that the transform (17) reduces to the identification $f(x) = \psi(|x|_2)$, $x \in \mathbb{R}^d$.

Theorem 3.1 of Cambanis *et al.* [4] has three equivalent formulations, one in terms of characteristic functions, a second giving a statistical interpretation, and a third using distribution functions. Their central technical lemma (Proposition 3.1) and its proof are purely statistical, but whenever they deal with characteristic functions and/or distribution functions their arguments and the one of Berens and Xu [1] are quite similar.

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