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Note

# On a zeta function associated with automata and codes

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**Abstract**

The zeta function of a finite automaton $\mathcal{A}$ is $\exp\{\sum_{n=1}^{\infty} a_n \frac{z^n}{n}\}$, where $a_n$ is the number of bi-infinite paths in $\mathcal{A}$ labelled by a bi-infinite word of period $n$. It reflects the properties of $\mathcal{A}$: aperiodicity, nil-simplicity, existence of a zero. The results are applied to codes.


*Keywords:* Automata; Codes; Zeta function; Noncommutative series; Rational aperiodic

## 1. Introduction

In this Note, we introduce a generating function associated with automata: its zeta function. Although zeta functions of languages and sofic systems have been previously considered, this zeta function seems to be new: it may not be obtained by associating with the automaton some language and then its zeta function. It may however be obtained by specializing the multivariate Boyle zeta function associated with homomorphisms between dynamical systems after having associated such an homomorphism with the given automaton (see Section 8). We thank Dominique Perrin for having brought to our attention the article of Mike Boyle.

The interest of this zeta function is that, using it, one may characterize several properties of the automaton. For several of them, the proofs are quite easy, or analogous to standard proofs. The main result (Section 6) requires a more involved proof: it characterizes aperiodic automata. It is interesting in view of Schützenberger's theorem on aperiodic automata and languages; aperiodicity of the automaton is characterized by the following equality: the inverse of the determinant of the automaton is equal to the zeta function.

We consider here not only deterministic automata, but also unambiguous ones. This allows us to give applications to codes and characterize several classes of them (Section 7).

## 2. Stable rank and periodic bi-infinite paths in unambiguous automata

We consider in the whole Note only finite unambiguous automata. Let $\mathcal{A}$ be such automaton, over the alphabet $A$, and let $w \in A^*$. In [2], the *rank* of $w$ is defined as the smallest $r$ such that the relation induced by $w$ on the set of

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states $Q$ of $\mathcal{A}$ is the unambiguous product $c\ell$, where $c$ is a relation $Q \to [r]$ and $\ell$ a relation $[r] \to Q$ (relations are composed from left to right). When $\mathcal{A}$ is deterministic, the rank of $w$ is the cardinality of the image of the function induced by $w$ on $Q$. We denote the rank of $w$ by $rk(w)$. One has $rk(w^{n+1}) \leq rk(w^n)$ and we may therefore define the *stable rank*, which is $\lim_{n\to\infty} rk(w^n)$. Notation is $strk(w)$.

If the relation (induced by) $w$ is idempotent, then $strk(w) = rk(w) = |Fix(w)|$, where $Fix(w)$ is the number of fixpoints of $w$: see [2] Prop. IV.4.3. For general $w$, $w^n$ is idempotent for some $n \geq 1$, and we have therefore $strk(w) = rk(w^n) = |Fix(w^n)|$.

A *bi-infinite word* is an element of $A^{\mathbb{Z}}$. In particular, for $w$ nonempty word, we denote ${}^{\infty}w^{\infty}$ the periodic bi-infinite word $(x_i)_{i\in\mathbb{Z}}$, where $x_i = a_r$, if $r \in \{0, 1, \ldots, n-1\}$ is the remainder of the division of $i$ by $n$, with $w = a_0a_1 \ldots a_{n-1}$. A *bi-infinite path* in $\mathcal{A}$ is a mapping of $\mathbb{Z}$ into the set of edges of $\mathcal{A}$, which is compatible with the graph underlying $\mathcal{A}$. The label of a bi-infinite path is the canonically associated bi-infinite word.

**Proposition.** *The stable rank of $w$ is equal to the number of bi-infinite paths in $\mathcal{A}$ labeled ${}^{\infty}w^{\infty}$.*

**Proof.** We claim that:

(i) if $p, q \in Fix(w)$ and $p \xrightarrow{w} q$, then $p = q$,

(ii) if $w$ is idempotent and $p \xrightarrow{w} q \xrightarrow{w} r$, then $q \in Fix(w)$.

Taking the claims for granted, let $w \in A^+$. For some integer $n \geq 1$, $w^n$ is idempotent. Since ${}^{\infty}w^{\infty} = {}^{\infty}(w^n)^{\infty}$, we may assume that $w$ is idempotent.

Then $strk(w) = |Fix(w)|$ and we have to show that there is a bijection $f$ between the set $Fix(w)$ and the set of bi-infinite paths labelled ${}^{\infty}w^{\infty}$. If $q \in Fix(w)$, we have the finite path $q \xrightarrow{w} q$ and therefore we obtain a bi-infinite path labelled ${}^{\infty}w^{\infty}$ (by iterating this finite path), which will be the image of $q$ under $f$. Now, if a bi-infinite path $(e_n)_{n\in\mathbb{Z}}$ is labelled ${}^{\infty}w^{\infty}$, we may decompose it as

$$\ldots\ q_{-2} \xrightarrow{w} q_{-1} \xrightarrow{w} q_0 \xrightarrow{w} q_1 \xrightarrow{w} q_2\ \ldots,$$

where the edge $e_0$ corresponds to the first edge in the path $q_0 \xrightarrow{w} q_1$. By claim (ii), each $q_i \in Fix(w)$; by claim (i), the $q_i$ are all equal. We thus obtain that $f$ is a bijection.

It remain to prove the claims. For (i), we have the paths $p \xrightarrow{w} p \xrightarrow{w} q$ and $p \xrightarrow{w} q \xrightarrow{w} q$ and thus, by unambiguity of the automaton, we must have $p = q$. For (ii), we have the path $p \xrightarrow{w} r$, since $w$ is idempotent. By Prop. IV.3.3 in [2], there exists $q' \in Fix(w)$ such that $p \xrightarrow{w} q' \xrightarrow{w} r$. Then, by unambiguity, we obtain $q' = q$ and $q \in Fix(w)$. $\square$

## 3. Two series associated with an unambiguous automaton

Let $\mathcal{A}$ be such an automaton. We associate with $\mathcal{A}$ the noncommutative formal series

$$S_{\mathcal{A}} = \sum_{w\in A^+} strk(w)\, w.$$

On the other hand, we associate with $\mathcal{A}$ the series, called *zeta function of $A$*

$$\zeta_{\mathcal{A}} = \exp\left\{\sum_{n=1}^{\infty} r_n \frac{z^n}{n}\right\},$$

where $r_n = \sum_{|w|=n} strk(w)$.

**Proposition.** *$S_{\mathcal{A}}$ and $\zeta_{\mathcal{A}}$ are $\mathbb{N}$-rational.*

*Recall that a series $S \in \mathbb{N}\langle\langle X\rangle\rangle$ is called $\mathbb{N}$-rational if it may be obtained from polynomials by sums, products and star operations $T \mapsto T^* = \sum_{n\geq 0} T^n$ (defined if $T$ has no constant term). Equivalently it is $\mathbb{N}$-recognizable, by the Kleene–Schützenberger theorem. See [6] for these notions.*

**Proof.** Denote by $\underline{L}$ the characteristic series of the language $L$ and by $\zeta_L$ the zeta function of $L$, following [4]: $\zeta_L = \exp\{\sum_{n\geq 1} |L \cap A^n| \frac{z^n}{n}\}$.

Let $L_i$ be the language $L_i = \{w \in A^* \mid strk(w) = i\}$. Since $\mathcal{A}$ is a finite automaton, $L_i$ is a rational language and therefore $\underline{L_i}$ is an $\mathbb{N}$-rational series. Therefore

$$S_{\mathcal{A}} = \sum_i i\, \underline{L_i}$$

is an $\mathbb{N}$-rational series, since the sum is finite: indeed, $L_i$ is empty for $i$ larger than the number of states of $\mathcal{A}$. A similar calculation shows that

$$\zeta_{\mathcal{A}} = \prod_i \zeta_{L_i}^i.$$

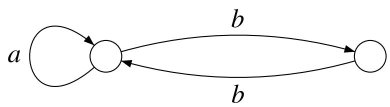
Now, $\zeta_{L_i}$ is $\mathbb{N}$-rational by [9]: indeed, each $L_i$ is a cyclic language, that is, closed under conjugation of words and under taking power and inverse power. Hence $\zeta_{\mathcal{A}}$ is $\mathbb{N}$-rational. □

## 4. An example

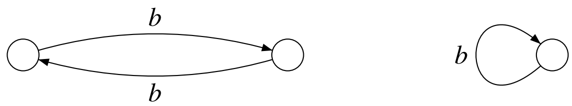
Given an automaton $\mathcal{A}$, the *trace* of $\mathcal{A}$ is the noncommutative series that counts the number of fixpoints of each word. Formally

$$tr(\mathcal{A}) = \sum_{w \in A^+} |Fix(w)|\, w.$$

Let $\mathcal{A}$ be the automaton below:




Following the method of [4] we see that its series $S_{\mathcal{A}}$ is equal to $tr(\mathcal{A}) - tr(\mathcal{A}_1) + 2tr(\mathcal{A}_2)$, where $\mathcal{A}_1$ and $\mathcal{A}_2$ are the automata below




Indeed, the formula is easily seen to be correct when $w = b^n$, since this word has stable rank 2. Let now $w$ be another word. If its stable rank is 1, then it must be of the form

$$w = b^{i_0} a b^{i_1} a \ldots a b^{i_n}, \quad n \geq 1, \tag{1}$$

where $i_1, i_2, \ldots, i_{n-1}$ and $i_0 + i_n$ are even. This implies the formula for $S_{\mathcal{A}}$. From this formula, following [4], we find the zeta function of $\mathcal{A}$: is it equal to

$$\frac{\det(1 - M_1 z)}{\det(1 - Mz)\ \det(1 - M_2 z)^2},$$

where $M,\ M_1,\ M_2$ are the adjacency matrices of the automata $\mathcal{A}$, $\mathcal{A}_1$ and $\mathcal{A}_2$. Thus

$$\zeta_{\mathcal{A}} = \frac{1 - z^2}{(1 - z - z^2)(1 - z)^2} = \frac{1 + z}{(1 - z - z^2)(1 - z)}.$$

Note that the zeta function of the sofic system associated with this automaton is different from $\zeta_{\mathcal{A}}$: it is $\frac{1+z}{1-z-z^2}$.

## 5. Some properties of the zeta function

Let $\mathcal{A}$ be an unambiguous automaton.

*5.1.* $\mathcal{A}$ has finitely many paths if and only if $S_{\mathcal{A}} = 0$ if and only if $\zeta_{\mathcal{A}} = 1$.

*5.2.* The monoid of transitions of $\mathcal{A}$ is nil-simple if and only if $S_{\mathcal{A}} = dA^+$ for some nonnegative integer $d$. In this case, $\zeta_{\mathcal{A}}$ is equal to $(1 - |A|z)^{-d}$.

Recall that a finite monoid is called nil-simple is each element has a power in the minimal ideal. The property follows since the minimal ideal of the monoid of transitions of an unambiguous automaton is characterized by the fact that its elements have minimum rank; see [2] Chapter IV.

*5.3.* The monoid of transitions of $\mathcal{A}$ contains the null relation if and only if the support of $S_{\mathcal{A}}$ is not $A^*$ if and only if $\zeta_{\mathcal{A}}$ converges for $z = \frac{1}{|A|}$.

The first equivalence is clear, since the rank of the null relation is 0. If the support of $S_{\mathcal{A}}$ is $A^*$, then $S_{\mathcal{A}} \geq A^*$ coefficientwise. Hence

$$\zeta_{\mathcal{A}} \geq \exp\left\{\sum_{n\geq 1} |A|^n \frac{z^n}{n}\right\} = \frac{1}{|1 - |A|z|}$$

and therefore $\zeta_{\mathcal{A}}$ diverges for $z = |A|^{-1}$. Suppose now that the support of $S_{\mathcal{A}}$ is not $A^*$. Then it is included in the complement of some principal ideal of $A^*$: $supp(S_{\mathcal{A}}) \subseteq A^* \setminus A^*wA^*$. The same calculations as in [2], Proof of Prop. I.5.6, then show that $a_n \leq c\, r^n$, for some constant $c$ and some $r$ strictly smaller that $|A|$. It is then easy to verify that $\zeta_{\mathcal{A}} \leq \left(\frac{1}{1-rz}\right)^c$, and therefore $\zeta_{\mathcal{A}}(|A|^{-1})$ converges.

*5.4.* The zeta function has an infinite product expansion: $\zeta_{\mathcal{A}} = \prod_{\ell} \left(\frac{1}{1-z^{|\ell|}}\right)^{\alpha_\ell}$, where the product is over all Lyndon words $\ell$ and where $\alpha_\ell$ is the stable rank of $\ell$.

This is proved by taking logarithmic derivative, following the lines of the proof of the similar Prop. 1 in [4].

## 6. Aperiodicity

Recall that a monoid is *aperiodic* if his subgroups are all trivial (note that the identity element of the monoid and of a subgroup may differ). A finite monoid $M$ is aperiodic if and only if for any $x \in M$, there is an integer $n \geq 0$ such that $x^n = x^{n+1}$. An automaton is aperiodic if his monoid of transitions is. The determinant of an automaton is the determinant of the matrix $1 - zM$, where $M$ is the adjacency matrix. Note that a multivariate version of this determinant has been considered by Perrin [7].

**Proposition.** *The three following conditions are equivalent:*

(1) *$\mathcal{A}$ is an aperiodic automaton,*
(2) *$S_{\mathcal{A}} = tr(\mathcal{A})$,*
(3) *$\zeta_{\mathcal{A}}$ is the inverse of the determinant of $\mathcal{A}$.*

To prove this, we use the following lemma.

**Lemma.** *Let the word $w$ act on the states of an unambiguous automaton. Then the number of fixpoints of $w$ is smaller or equal to the stable rank of $w$, with equality if and only if $w$ is aperiodic, that is, the submonoid generated by the relation induced by $w$ is aperiodic.*

**Proof.** 1. Each fixpoint of $w$ is also a fixpoint of $w^N$, for any $N$. We may choose $N$ such that $w^N$ is idempotent. Then we know by Proposition IV.4.3 of [2] that the number of fixpoints of $w^N$ is the rank of $w^N$, hence the stable rank of $w$. Hence, the inequality of the lemma is established.

2. Suppose that $w$ is aperiodic. Then $w^k = w^{k+1} = w^{k+2} = \ldots$ for $k$ large enough. Since all the powers of $w^N$ are idempotent, we may assume by increasing $N$ that $w^N = w^{N+1}$. Let $q$ be a fixpoint of $w^N$. Then, we have a path $\pi$ with label $w^N$ : $q = q_0 \xrightarrow{w} q_1 \xrightarrow{w} q_2 \xrightarrow{w} \cdots \xrightarrow{w} q_{N-1} \xrightarrow{w} q_0$ . We have therefore also a path $\pi'$ : $q_0 \xrightarrow{w^{N+1}} q_0$ . If $\pi$ is not a prefix of $\pi'$, we obtain two distinct paths $\pi^{N+1}$ and $\pi'^N$ with label $w^{N(N+1)}$ and with

same initial and final states. This contradicts non-ambiguity. Hence, $\pi$ is a prefix of $\pi'$ and there is a path $q_0 \xrightarrow{w} q_0$. Thus $q$ is a fixpoint of $w$ and we have equality in the lemma.

3. Conversely, suppose that we have this equality. Then, each fixpoint of $w^N$ is a fixpoint of $w$. Let $q_0 \xrightarrow{w} q_1 \xrightarrow{w} q_2 \xrightarrow{w} \cdots \xrightarrow{w} q_0$ be a closed path of length $p$ in the graph of the relation on $Q$ induced by $w$. Then $q_0$ is a fixpoint of some power of $w$, hence also of $w^N$ (since $w^N = w^{2N} = \ldots$). Hence $q_0$ is a fixpoint of $w$. Then we have the path $q_0 \xrightarrow{w} q_0 \xrightarrow{w} \cdots \xrightarrow{w} q_0$. Comparing this path of length $p$ to the latter, we obtain by non-ambiguity that $q_0 = q_1 = q_2 = \ldots$. This shows that there are no closed paths in the graph of $w$ except those which are repetitions of loops. If we suppress in this graph the loops, we therefore obtain a new graph without closed path; hence there is some $k$ such that there is no path of length $k$ in this new graph. We show that $w^k = w^{k+1}$. Indeed, a path of length $k$ or $k+1$ in the graph of $w$ has necessarily some loop in it. Hence, by repetition or suppression of the loop, we see that $p \xrightarrow{w^k} q$ is equivalent to $p \xrightarrow{w^{k+1}} q$. Thus $w$ is aperiodic. □

**Proof of the proposition.** (1) ⇒ (2) We use directly the previous lemma.

(2) ⇒ (3) We have $r_n = \sum_{|w|=n} strk(w) = \sum_{|w|=n} |Fix(w)| = tr(M^n)$, where $M$ is the adjacency matrix of the automaton $\mathcal{A}$. Hence

$$\zeta(\mathcal{A}) = \exp\left\{\sum_{n=1}^{\infty} r_n \frac{z^n}{n}\right\} = \exp\left\{\sum_{n=1}^{\infty} tr(M^n)\frac{z^n}{n}\right\}$$
$$= \exp\left\{tr \sum_{n=1}^{\infty} \frac{M^n z^n}{n}\right\} = \exp\left\{tr \ln\left(\frac{1}{1-zM}\right)\right\} = \det(1-zM)^{-1},$$

by Jacobi's formula $\exp \circ tr \circ \ln = \det$.

$(\neg 1) \Rightarrow (\neg 3)$ By the lemma, for some word $w$, we have $|Fix(w)| < strk(w)$. Always, $r_n \geq tr(M^n)$ for any $n$, and for at least $n = |w|$, we have strict inequality. Therefore, $\zeta(\mathcal{A}) = \exp\left\{\sum_{n=1}^{\infty} r_n \frac{z^n}{n}\right\} > \exp\left\{\sum_{n=1}^{\infty} tr(M^n)\frac{z^n}{n}\right\}$ and by the same calculations as above, we deduce that $\zeta(\mathcal{A}) > \det(1-zM)^{-1}$. □

## 7. Applications to finite codes

We consider here a finite code $X \subseteq A^*$. An *$X$-factorization* of a bi-infinite word $(a_n)_{n\in\mathbb{Z}}$ is an infinite subset $F$ of $\mathbb{Z}$ such that for any consecutive $i, j$ in $F$, one has $a_i a_{i+1} \ldots a_{j-1} \in X$. This notion is considered in [11]. He shows that for each nonempty word $w$, the $X$-factorizations of ${}^{\infty}w^{\infty}$ are disjoint and that the minimum number of such factorizations is equal to the *degree* of $X$. Our methods below may be used to prove differently these results.

Let $\mathcal{A}$ be a connected unambiguous automaton, which recognizes $X^*$ with a single final state, equal to its initial state. It is easily seen that the $X$-factorizations of a bi-infinite word are in natural bijection with bi-infinite paths in $\mathcal{A}$ whose label is this word. Hence, if we define $(S_X, w)$ = the number of $X$-factorizations of ${}^{\infty}w^{\infty}$, we see that $S_X = S_{\mathcal{A}}$. Likewise, we define the *zeta function of* $X$ by $\zeta_X = \exp \sum_{n\geq 1} r_n \frac{x^n}{n}$, where $r_n$ is the total number of $X$-factorizations of bi-infinite words of period $n$; thus $r_n = \sum_{|w|=n} (S_X, w)$. Therefore, $\zeta_X = \zeta_{\mathcal{A}}$. Note that the support of $S_X$ is the cyclic closure of $X^*$. From the previous results, we obtain several consequences for codes.

*7.1.* $S_X$ and $\zeta_X$ are $\mathbb{N}$-rational.

*7.2.* $X$ is empty if and only if $S_X = 0$ if and only if $\zeta_X = 1$.

*7.3.* $X$ is complete if and only if the support of $S_X$ is $A^*$ if and only if $\zeta_X$ diverges for $z = \frac{1}{|A|}$.

*7.4.* The code $X$ is a circular code if and only if $S_X$ is a characteristic series. This is merely a reformulation of Prop. VII.1.2 of [2].

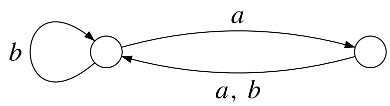
*7.5.* If $X$ is maximal, then $X$ is bifix if and only if $S_X = dA^+$ for some nonnegative integer $d$. In this case, $\zeta_X = (1-|A|z)^{-d}$. This follows directly from 5.2 and Th. V.3.3 in [2].

*7.6.* The submonoid $X^*$ is pure (that is, $u^n \in X^* \Rightarrow u \in X^*$, if $n \geq 1$) if and only if $\zeta_X = (1 - \theta(\underline{X}))^{-1}$, where $\theta$ is the homomorphism $\mathbb{Z}\langle\langle A\rangle\rangle \to \mathbb{Z}[[z]]$ sending each letter onto $z$.

Indeed, $\det(1 - zM)$ is equal to $1 - \theta(\underline{X})$, by a theorem of Schützenberger (see [2] Prop. VIII.2.1), where $M$ is the adjacency matrix of $\mathcal{A}$. Hence the result follows from the proposition in Section 6 and a theorem of Restivo [8]: $X^*$ is pure if and only if the monoid of transitions of $\mathcal{A}$ is aperiodic (see also Exercise VII.2.1 in [2]).

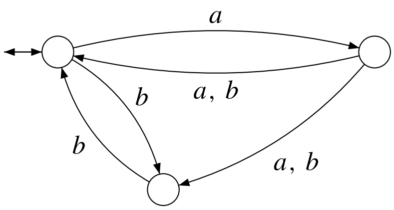
Note that 7.4 and 7.6 together imply a well-known result: compare [5] Prop. 4.6 (see also [10] Prop. 4.7.11).

We close this section by three examples. Let $X = \{aa, ab, b\}$, then we may take as automaton $\mathcal{A}$ below, which is deterministic.



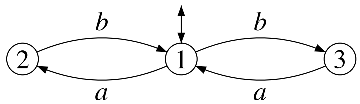

It is easily seen that $strk(a^n) = 2$ and $strk(w) = 1$ for all other words. Hence, $S_X = 2a^+ + a^*bA^*$, $r_n = 2^n + 1$ and $\zeta_X = \frac{1}{(1-2z)(1-z)}$.

Now, let $X = \{aa, ab, aab, abb, bb\}$. As automaton, we take




It is easily seen that $strk(a^n) = strk(b^n) = 2$ and $rk(ab) = rk(ba) = 1$. Since $X$ is complete, $strk(w) = 1$ for each word $w \notin a^* \cup b^*$. Thus $r_n = 2^n + 2$ and $\zeta_X = \frac{1}{(1-2z)(1-z)^2}$.

Let $X = \{ab, ba\}$. This is a pure code which is not circular (see [2] Example VII.1.3). Its zeta function is, according to 7.6, $\zeta_X = \frac{1}{1-2z^2}$. Note that the sofic system associated with its flower automaton has a different zeta function. It may be computed by the methods of [2] or [5] and is equal to $\frac{1-z^2}{1-2z^2}$.




## 8. Comparison with Boyle's zeta function

In [1], Mike Boyle associates with each homomorphism of dynamical systems a zeta function in several variables. We verify that our construction is a specialization of his'.

Let $\mathcal{A}$ be an unambiguous automaton. One associates with it two dynamical systems. Let $X$ (resp. $Y$) be the set of bi-infinite paths in $\mathcal{A}$ (resp. of bi-infinite words which are labels of such paths). Let $S$ (resp. $T$) be the shift on $X$ (resp. $Y$), defined by translation: $(a_n)_{n\in\mathbb{Z}} \mapsto (a_{n+1})_{n\in\mathbb{Z}}$. The mapping $\phi : X \to Y$ defined by labelling conjugates $S$ and $T$: $\phi \circ S = T \circ \phi$. Hence $\phi$ defines a homomorphism of dynamical systems $(X, S) \to (Y, T)$.

Since $\mathcal{A}$ is unambiguous, $\phi$ is bounded-to-one: this means that the cardinality of $\phi^{-1}(y)$ is bounded for each $y \in Y$; see [3] Prop. 16 and 17.

Let $y \in Y$ have the period $n$. It is then verified that $\phi^{-1}(y)$ is closed under $S^n$. Note that each element in $\phi^{-1}(y)$ is periodic. Now $S^n$ induces a permutation of the finite set $\phi^{-1}(y)$, and we denote by $\lambda(y)$ the partition $\lambda_1, \ldots, \lambda_k$ such that the cycle lengths of this permutation are $\lambda_1, \ldots, \lambda_k$, with multiplicities.

Let $x_\lambda$ be a variable, one for each partition $\lambda$. Since $\phi^{-1}(y)$ is of bounded cardinality, only finitely $\lambda$ will occur.

Fix now $\lambda$ and consider all $y \in Y$ of period $n$ such that $\lambda(y) = \lambda$. Following [1], we denote by $N_n(\lambda)$ their cardinality and define $\zeta_\lambda = \exp\left\{\sum_{n\geq 1} N_n(\lambda)\,\frac{z^n}{n}\right\}$.

Finally, the Boyle zeta function of $\phi$ is defined as $Z_\phi = \prod_\lambda \zeta_\lambda^{x_\lambda}$, where one uses the usual expansion of series of the form $(1 + a_1 z + a_2 z + \cdots)^x$.

According to Boyle, this zeta function specializes to the usual zeta functions of $(X, S)$ and $(Y, T)$: for the second, one maps each $x_\lambda$ onto 1, and for the first one maps each $x_\lambda$ onto the number of parts of $\lambda$ equal to 1 (that is, the number of fixpoints of $S^n$ acting on $\phi^{-1}(y)$, in the above setting). See [1] Eqs. (1.6) and (1.7).

We claim that the specialization $x_\lambda \mapsto |\lambda| = \sum \lambda_i$ will map $Z_\phi$ onto our $\zeta_{\mathcal{A}}$. Indeed, $|\lambda|$ is the number of points of $\phi^{-1}(y)$, in the above setting. Hence $\sum_\lambda N_n(\lambda)\,|\lambda|$ is equal to the number of points of period $n$ in $Y$ which are the label of a point in $X$, that is, $\sum_\lambda N_n(\lambda)\,|\lambda| = r_n$ with the notations of Section 3.

Thus $Z_\phi$ specializes to

$$\prod_\lambda \zeta_\lambda^{|\lambda|} = \prod_\lambda \left(\exp\left\{\sum_{n\geq 1} N_n(\lambda)\,\frac{z^n}{n}\right\}\right)^{|\lambda|} = \exp\left\{\sum_{n\geq 1}\sum_\lambda |\lambda|\,N_n(\lambda)\,\frac{z^n}{n}\right\}$$

$$= \exp\left\{\sum_{n\geq 1} r_n\,\frac{z^n}{n}\right\} = \zeta_{\mathcal{A}}.$$

This specialization of $Z_\phi$ may be defined directly in terms of the dynamical system by the more transparent formula

$$\exp\left\{\sum_{n\geq 1}\sum_{T^n(y)=y} |\phi^{-1}(y)|\,\frac{z^n}{n}\right\}.$$

This function is $\mathbb{N}$-rational, as follows from Section 3. This may be true for more general homomorphisms of dynamical systems (not only those associated with unambiguous automata).

As an example, take the automaton of Section 4. If $y = {}^\infty b^\infty$, then the smallest period of $y$ is 1 and $\phi^{-1}(y)$ has 2 elements. If $n$ is odd, then $S^n$ acts transitively on $\phi^{-1}(y)$ and if $n$ is even there are two orbits. This corresponds respectively to the partitions $\lambda = 2$ and $\lambda = 11$. Since for any other $y$, $\phi^{-1}(y)$ has 0 or 1 element, we obtain: $N_n(2) = 1$ if $n$ is odd, 0 otherwise; $N_n(11) = 1$ if $n$ is even, 0 otherwise. Now, take $y$ such that $\phi^{-1}(y)$ has one element; this correspond exactly to the case where $y = {}^\infty w^\infty$, with $w$ of the form (1). Thus $N_1(1)$ is the number of these words and $\zeta_1$ is the zeta function of $L$, the set of these words.

Now, the characteristic series of $L$ is equal to $tr(\mathcal{A}) - tr(\mathcal{A}_1)$, with the notations of Section 4. Hence, by the methods of [4], $\zeta_1 = \frac{1-z^2}{1-z-z^2}$. Moreover,

$$\zeta_{11} = \exp\left\{\sum_{n\geq 1}\frac{z^{2n}}{2n}\right\} = \exp\left\{\sum_{n\geq 1}\frac{z^{2n}}{n}\right\}^{\frac{1}{2}} = \left(\frac{1}{1-z^2}\right)^{\frac{1}{2}}$$

and

$$\zeta_2 = \exp\left\{\sum_{n\geq 0}\frac{z^{2n+1}}{2n+1}\right\} = \frac{\exp\left\{\sum_{n\geq 1}\frac{z^n}{n}\right\}}{\exp\left\{\sum_{n\geq 1}\frac{z^{2n}}{2n}\right\}} = \frac{(1-z^2)^{\frac{1}{2}}}{1-z} = \left(\frac{1+z}{1-z}\right)^{\frac{1}{2}}.$$

Thus, the Boyle zeta function is

$$Z_\phi = \frac{(1-z^2)^{x_1}\,(1+z)^{\frac{x_2}{2}}}{(1-z-z^2)^{x_1}(1-z^2)^{\frac{x_{11}}{2}}(1-z)^{\frac{x_2}{2}}} = \frac{(1+z)^{\frac{x_2}{2}-\frac{x_{11}}{2}+x_1}}{(1-z-z^2)^{x_1}(1-z)^{\frac{x_{11}}{2}-x_1+\frac{x_2}{2}}}.$$

If we specialize each $x_\lambda$ to $|\lambda|$, that is $x_1 \mapsto 1$, $x_{11}$ and $x_2 \mapsto 2$, this function specializes to $\frac{1+z}{(1-z-z^2)(1-z)}$, as it should be.

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