On $q^2/4$-sets of type $(0, q/4, q/2)$ in projective planes of order $q \equiv 0 \pmod{4}$

A. Maschietti and G. Migliori

Dipartimento di Matematica, Università 'La Sapienza', P.le A. Moro, 00185, Roma, Italy

Received 1 August 1991
Revised 14 November 1991

Abstract

In this paper we investigate $q^2/4$-sets of type $(0, q/4, q/2)$ in projective planes of order $q \equiv 0 \pmod{4}$. These sets arise in the investigation of regular triples with respect to a hyperoval. Combinatorial properties of these sets are given and examples in Desarguesian projective planes are constructed.

1. Introduction

The aim of this paper is to study $q^2/4$-sets of type $(0, q/4, q/2)$ in projective planes whose order is divisible by four. These sets arise in the investigation of regular triples with respect to a hyperoval.

If $\Omega$ is a hyperoval of a projective plane of order $q$, $\Pi_q$ (i.e. $\Omega$ is a set of $q + 2$ points no three of which are collinear), then a nonordered triple $\{X, Y, Z\}$ of distinct points not in $\Omega$ is said to be regular with respect to $\Omega$ if there is no point $P$ of $\Pi_q$ such that the lines $PX$, $PY$ and $PZ$ are exterior (see [2]).

If we define, for every point $P$ not in $\Omega$,

$$E(P) = \{Q \in \Pi_q \mid Q \neq P \text{ and the line } PQ \text{ is exterior to } \Omega\},$$

then the triple $\{X, Y, Z\}$ is regular if and only if $E(X) \cap E(Y) \cap E(Z) = \emptyset$.

In [2, Theorem 1.2] is proved that the existence of a regular triple implies that of $q^2/4$-sets $W$ of type $(0, q/4, q/2)$, namely, the sets $E(X) \cap E(Y)$, $E(X) \cap E(Z)$, $E(Y) \cap E(Z)$. Hence, the order $q$ of the plane must be divisible by four. Furthermore, if $\{X, Y, Z\}$ is regular, then $X$, $Y$ and $Z$ are on the same secant line, $\ell$ of $\Omega$ [3] and if $W$ is...
one of the $q^2/4$-sets determined by \{X, Y, Z\}, then X, Y and Z are the unique points of 
\ell on which there are $0$-secant and $q/2$-secant lines of $W$.

An interesting problem is, in our opinion, to prove the converse.

**Problem.** Given a $q^2/4$-set of type $(0, q/4, q/2)$, are there three collinear points on
which there are only $0$-secant and $q/2$-secant lines of $W$?

A partial answer to the problem is Proposition 1.1: if there exists a $q^2/4$-set $W$
of type $(0, q/4, q/2)$, then there are at most three collinear points, on which
there are only $0$-secant and $q/2$-secant lines of $W$. This is proved using only
combinatorial methods.

Sections 2 and 3 are devoted to the construction of $q^2/4$-sets of type $(0, q/4, q/2)$.
In $PG(2, q)$, such sets always exist (Theorems 2.1 and 3.1) and are at least of
two different types. One type of these $q^2/4$-sets gives examples not related to
regular triples with respect to a complete conic (i.e. a conic plus its nucleus). The
sets of the other type result from the union of $q/4$ irreducible conics pairwise tangent
at the same point $O$ and having the same nucleus, $N$, minus the common points $O$
and arise from regular triples with respect to a complete conic. In every case, for
such sets there exist three collinear points, on which there are only $0$-secant or
$q/2$-secant lines.

2. Preliminaries

Let $\Pi_q$ be a projective plane of order $q$. A $c$-set $\mathcal{C}$ is a subset of $c$
distinct points of $\Pi_q$. The character of index $s$ of $\mathcal{C}$ is the number $t_s$ of lines of $\Pi_q$
meeting $\mathcal{C}$ in exactly $s$ distinct points, $0 \leq s \leq q + 1$. A line $\ell$ meeting $\mathcal{C}$ in $s$
points is said to be an $s$-secant line.

We say that $\mathcal{C}$ is of class $[m_1, m_2, \ldots, m_d]$ if $t_m = 0$ for every $m \notin \{m_1, \ldots, m_d\}$,
where $m_1, \ldots, m_d$ are integers such that $0 \leq m_1 < m_2 < \cdots < m_d \leq q + 1$. $\mathcal{C}$ is said to be of type
$(m_1, m_2, \ldots, m_d)$ when it is of class $[m_1, m_2, \ldots, m_d]$ and $t_{m_j} \neq 0$ for every $j = 1, 2, \ldots, d$.

For a $c$-set the following character equations hold:

$$
\sum_{s=0}^{q+1} \binom{s}{i} t_s = \lambda_i \binom{c}{i}, \quad i = 0, 1, 2, (1.1)
$$

where $\lambda_0 = q^2 + q + 1$, $\lambda_1 = q + 1$, $\lambda_2 = 1$ (for a general reference, see [1, 5, 6]).

From now on we assume that $\Pi_q$ is a projective plane of order $q \equiv 0 \pmod{4}$ and that
$W$ is a $q^2/4$-set of type $(0, q/4, q/2)$. For such a set the characters are

$$
t_0 = \frac{3q}{2} + 1, \quad t_{q/4} = q(q - 2), \quad t_{q/2} = \frac{3q}{2}. (1.2)
$$
Furthermore, if \( v_s(P) \) denotes the number of \( s \)-secant lines on a point \( P \not\in W \) and \( v_s(P) \) the number of \( s \)-secant lines on a point \( P \in W \), then

\[
\begin{align*}
  v_0 &= 0, & v_{q/4} &= q - 2, & v_{q/2} &= 3, \\
  u_0 &= u_{q/2} + 1, & u_{q/4} &= q - 2u_{q/2}.
\end{align*}
\]

Therefore, \( u_{q/2}(P) \leq q/2 \) and \( 1 \leq u_0(P) \leq (q/2) + 1 \), for every point \( P \in \Pi_q \setminus W \).

**Proposition 1.1.** Let \( W \) be a \( q^2/4 \)-set of type \((0, q/4, q/2)\). Then there are at most three necessarily collinear points for which \( u_0 = (q/2) + 1 \).

**Proof.** Let \( \mathcal{L} \) be the set of 0-secant lines of \( W \). In the dual plane, \( \Pi_q^* \), \( \mathcal{L} \) is a \(((3q/2) + 1)\)-set of class \([0, 1, 2, \ldots, (q/2) + 1]\), whose characters \( \theta_i \) satisfy the equations

\[
\begin{align*}
  \sum_{i=0}^{(q/2)+1} \theta_i &= q^2 + q + 1, \\
  \sum_{i=1}^{(q/2)+1} i\theta_i &= \left(\frac{3q}{2} + 1\right)(q + 1), \\
  \sum_{i=2}^{(q/2)+1} i(i-1)\theta_i &= \frac{3q}{2} \left(\frac{3q}{2} + 1\right).
\end{align*}
\]

Since \( \theta_0 = q^2/4 \), eliminating \( \theta_1 \) and \( \theta_2 \) we obtain

\[
\sum_{i=3}^{q/2} (i-2)(i-1)\theta_i = \frac{q}{2} \left(\frac{q}{2} - 1\right)(3 - \theta_{q/2+1}),
\]

which is nonnegative. Therefore, \( \theta_{(q/2)+1} \leq 3 \) and equality holds if and only if \( \theta_1 = 0 \), for every \( i = 3, \ldots, q/2 \).

If \( \theta_{(q/2)+1} = 3 \), then \( \mathcal{L} \) is of class \([0, 1, 2, (q/2) + 1]\). Equalities (1.3) and (1.3a) imply that the set \( \mathcal{K} \) of \( q/4 \)-secant lines to \( W \) is a \( q(q-2) \)-set of class \([0, q-2, q]\) in \( \Pi_q^* \), whose characters are \( \sigma_0 = 3, \sigma_{q-2} = q^2, \sigma_q = q - 2 \). We show that the exterior lines of \( \mathcal{K} \) are concurrent. By way of contradiction, if they form a triangle then the number of \((q-2)\)-secant lines to \( \mathcal{K} \) is \( q/2 \) or \( q \). Hence, this set would be a \( q^2 \)-set of type \((q/2, q)\), which cannot exist, as follows from its character equations. Thus, the exterior lines of \( \mathcal{K} \) are concurrent in the dual plane. Hence, by duality, the three corresponding points of \( \Pi \) are collinear, as claimed. \( \square \)

### 3. \( q^2/4 \)-sets of type \((0, q/4, q/2)\) in \( \text{PG}(2, q) \)

In this section we prove the existence of \( q^2/4 \)-sets of type \((0, q/4, q/2)\) in \( \text{PG}(2, q) \), the Desarguesian projective plane over the Galois field \( GF(q) \), \( q = 2^h, h \geq 3 \).

Let \( \omega \) be a primitive element of \( GF(q) \) and \( T_0 \) (resp. \( T_1 \)) the set of elements in \( GF(q) \) of trace 0 (resp. 1). Briefly, we recall that an element \( d \in GF(q) \) is of trace 0 if the second
degree equation \( x^2 + x + d = 0 \) has solution in \( GF(q) \). Furthermore, \( T_0 \) is an additive subgroup of order \( q/2 \) in \( GF(q) \) and the sum of elements of different traces is an element of trace 1 (for a general reference, see [1, 4]).

**Lemma 2.1.** For every \( \omega \in GF(q) \setminus \{0, 1\} \), let \( H' = \{ x \in T_0 | x \omega \in T_0 \} \) and \( H'' = \{ \beta \in T_1 | \beta \omega \in T_1 \} \). Then

\[
|H'| = |H''| = \frac{q}{4}.
\] (2.1)

**Proof.** We put \( |H'| = x \) and \( |H''| = y \). Then, \( x = y \), since \( x + (q/2) - y = q/2 \). Now, by way of contradiction, suppose \( x < q/4 \). Hence \( |T_0 \setminus H'| > q/4 \). If we denote by \( u_0, u_1, \ldots, u_{q/4}, \ldots \), the elements of \( T_0 \setminus H' \), then each sum \( u_0 + u_i, i = 1, 2, \ldots, q/4 \), belongs to \( H' \), which contradicts \( x < q/4 \).

To prove that \( x = q/4 \), it will suffice to show that \( x < q/2 \), since \( H' \) is a subgroup of \( T_0 \). Suppose that \( x = q/2 \). Then

\[
\omega^k x \in T_0 \text{ if and only if } x \in T_0 \text{ for every } k \in \mathbb{Z}.
\] (2.2)

Let \( \langle \omega \rangle \) be the multiplicative cyclic subgroup of \( GF(q) \setminus \{0\} \) generated by \( \omega \). Each class of \( GF(q) \setminus \{0\} \) (mod \( \langle \omega \rangle \)) is either a subset of \( T_0 \setminus \{0\} \) or \( T_1 \). In fact, \( x \) is congruent to \( \beta \) (mod \( \langle \omega \rangle \)) if and only if \( x^{-1} \beta = \omega^k \). Thus, by (2.2), \( x \) and \( \beta \) belong either to \( T_0 \setminus \{0\} \) or to \( T_1 \). Hence the order of \( \langle \omega \rangle \) divides both \( (q/2) - 1 \) and \( q/2 \), which is a contradiction, since \( \omega^k \neq 1 \). \( \square \)

**Theorem 2.2.** Let \( (x_0, x_1, x_2) \) denote homogeneous coordinates in \( PG(2, q) \). Then each of the sets \( W_{ij} = \{(x, y, z) | x \in T_i, y \in T_j \} \), for every \( i, j \in \{0, 1\} \), is a \( q^2/4 \)-set of type \( (0, q/4, q/2) \).

**Proof.** We can consider the affine plane \( AG(2, q) \), having as line at infinity the line \( x_2 = 0 \). For every line \( \ell \) of \( AG(2, q) \), consider the numbers \( x_{ij} = \ell \cap W_{ij} \). We show that \( x_{ij} \in \{0, q/4, q/2\} \). Since the group \( GF(q) \times GF(q) \) acts transitively on the lines of \( AG(2, q) \) and permutes the numbers \( x_{ij} \), it suffices to study lines of equation \( y = mx \) or \( x = 0 \). If \( m = 0 \), then the line \( y = 0 \) meets \( W_{i0} \) in \( q/2 \) points and \( W_{ij}, j \neq 0 \) in \( 0 \) points. When \( m = 1 \), the line \( y = x \) has \( q/2 \) (resp. \( 0 \)) points in common with \( W_{0i} \) (resp. \( W_{ij}, i \neq j \)). The line \( x = 0 \) has \( q/2 \) points on \( W_{0j} \) and \( 0 \) points on \( W_{ij}, i \neq 0 \). Finally, let \( m \neq 0, 1 \). It is easily seen that \( x_{00} + x_{01} + x_{10} + x_{11} = q \) and that \( x_{01} + x_{11} = x_{00} + x_{10} \). To achieve the proof, it suffices to show that \( x_{00} = x_{11} \) and \( x_{01} + x_{10} = q/2 \). These equalities follow from Lemma 2.1. \( \square \)

**Remark.** Observe that for each of the \( q^2/4 \)-sets \( W_{ij} \) there exist three collinear points, on which there are only 0-secant or \( q/2 \)-secant lines of \( W_{ij} \). They are the points \( (0, 1, 0) \), \( (0, 0, 1) \) and \( (0, 1, 1) \).
4. On $q^2/4$-sets of type $(0, q/4, q/2)$, which are union of conics of $PG(2, q)$

Let $\Gamma$ be a conic of $\Pi_q = PG(2, q)$, $q = 2^n$, and $N$ its nucleus, i.e. the meet of the tangents of $\Gamma$. For every $P \notin \Gamma \cup \{N\}$, we define

$$E(P) = \{Q \in \pi_q | Q \neq P \text{ and the line } PQ \text{ is exterior to } \Gamma\}.$$

**Theorem 3.1.** Let $X$ and $Y$ be two distinct points other than $N$ on a tangent line $s$ of $\Gamma$, $X, Y \notin \Gamma$. The set $W = E(X) \cap E(Y)$ is a $q^2/4$-set of type $(0, q/4, q/2)$. Furthermore, $W$ is union of $q/4$ irreducible conics pairwise tangent at the same point $O$ and having the same nucleus $N$, minus the point $O$.

**Proof.** If we adopt a convenient frame in $PG(2, q)$, we may suppose that $\Gamma$ has equation $x_0x_2 = x_1^2$ and that $s$ is the line of equation $x_2 = 0$, which is tangent to $\Gamma$ in the point $O(1, 0, 0)$. We directly construct $W$ from two distinct points $X(1, 0', 0)$ and $Y(1, 0'', 0)$, where $\omega$ is a primitive element of $GF(q)$.

The line on $X$,

$$x_2 = \omega^j(x_1 + \omega^j x_0),$$

is exterior to $\Gamma$ if and only if $\omega^j \omega = T_1$.

Similarly, the line on $Y$,

$$x_2 = \omega^j(x_1 + \omega^j x_0),$$

is exterior to $\Gamma$ if and only if $\omega^j \omega = T_1$.

Therefore,

$$W = E(X) \cap E(Y) = \{\omega^j, \omega^j \omega^j + \omega^j \omega^j, \omega^j \omega^j \omega^j = T_1\}.$$
Because of this remark, it will suffice to consider only the intersections of $W$ with one line of the pencil of center $P(1, \omega^k, 0)$.

The line $t$ of equation
\[ x_1 = \omega^k x_0, \]
\[ \omega^k \in GF(q), \] is on $P$ and is a secant line of $\Gamma$. If $Q$ is a point of $W$, then $Q \in t$ if and only if
\[ \omega^i \omega^{-1} = d \quad \text{and} \quad \omega^j \omega^{-1} = d^* \]
are in $T_1$. Equality (3.1) implies
\[ d^*(\omega^k + \omega^j) = d(\omega^k + \omega^j). \] (3.2)
If $\omega^k = \omega^i$ or $\omega^k = \omega^j$, then $|t \cap W| = 0$. Then, let $\omega^k \neq \omega^i$ and $\omega^k \neq \omega^j$.

Equality (3.2) can be written as
\[ d^* = d\omega^j, \quad \text{where} \quad \omega^j = \frac{\omega^k \omega^j + \omega^k}{\omega^j (\omega^k + \omega^k)}. \]

Now, $\omega^j = 1$ if and only if $\omega^k = \omega^i + \omega^j$. In this case $|t \cap W| = q/2$.

In the other cases, i.e. when $\omega^k \notin \{\omega^i, \omega^j, \omega^k \pm 1\}$, $\omega^i \neq 0, 1$. Thus, by Lemma 2.1, $|t \cap W| = q/4$.

To prove that $W$ results from the union of $q/4$ irreducible conics, we remark that the group $G$ has $q/4$ orbits on $W$. We show that each $G$-orbit, together with the tangency-point of $s$ to $\Gamma$, is an irreducible conic. In fact, if
\[ P(\omega^i + \omega^j, \omega^i \omega^j + \omega^j \omega^i, \omega^i \omega^k \omega^j (\omega^i + \omega^j)), \]
where $\omega^{-1} \omega^i, \omega^{-1} \omega^j \omega^k \in T_1$, is a point of $W$, then the conic $\mathcal{C}$ of equation
\[ x_0 x_2 = x_1^2 + \delta x_2^3, \]
where $\delta$ is determined by $P$, contains $P$. If $g \in G$ is represented by
\[
\begin{pmatrix}
1 & 0 & \gamma^2 \\
0 & 1 & \gamma \\
0 & 0 & 1
\end{pmatrix},
\]
then
\[ P^g(\omega^i + \omega^j + \gamma^2 \omega^i \omega^j + \omega^i \omega^k \omega^j (\omega^i + \omega^j)), \omega^i \omega^k \omega^j (\omega^i + \omega^j), \omega^i \omega^j (\omega^i + \omega^j) \]
belongs to $\mathcal{C}$ and $G$ stabilizes $\mathcal{C}$. Therefore, $\mathcal{C} \setminus \{(1, 0, 0)\}$ coincides with the $G$-orbit containing $P$. \(\square\)

Remark. The $q^2/4$-set $W$ is different from the sets $W_{ij}, i, j \in \{0, 1\}$. In fact, the line $x_0 = x_2$ meets $W$ in $q/4$ points and $W_{ij}, i, j \in \{0, 1\}$ in $0$ or $q/2$ points.

We now show that, in a certain sense, the construction given in the previous theorem may be inverted: given $q/4$ convenient conics $\mathcal{C}_i$, pairwise tangent at the same
point $O$ and having the same nucleus $N$, then $W = \bigcup_{i=1}^{q/4} (\mathcal{C}_i \setminus \{O\})$ is a $q^2/4$-set of type $(0, q/4, q/2)$. Moreover, on the line $ON$ there are exactly three points $X, Y$ and $Z$, such that $u_0 = (q/2) + 1$ and $W = E(X) \cap E(Y)$ for $q/4$ conics.

**Lemma 3.2.** Let $W = \bigcup_{i=1}^{q/4} (\mathcal{C}_i \setminus \{O\})$, where $\mathcal{C}_i, i = 1, \ldots, q/4$ are irreducible conics of $PG(2, q)$, pairwise tangent in $O$ and having the same nucleus $N$. Then there exists an automorphism group $G$ which stabilizes $W$ and has on $W$ each $\mathcal{C}_i \setminus \{O\}$ as orbit.

**Proof.** Fix a conic $\mathcal{C}$ different from $\mathcal{C}_i, i = 1, \ldots, q/4$, and tangent to each of them in $O$ and having as nucleus $N$. Adopting a convenient frame in $PG(2, q)$, we can assume $\mathcal{C}$ of equation $x_0 x_2 = x_1^2$ and $\mathcal{C}_i$ of equation $x_0 x_2 = x_1^2 + \delta_i x_2^2, \delta_i \in GF(q), i = 1, \ldots, q/4$. Then the group $G$, introduced in the proof of Theorem 3.1, is sharply transitive on the points of $\mathcal{C} \setminus \{O\}$ and of $\mathcal{C}_i \setminus \{O\}$, for every $i = 1, \ldots, q/4$. □

**Lemma 3.3.** Let $\{\mathcal{C}_i\}_{i=1}^{q/4}$ be a set of $q/4$ conics of $PG(2, q)$, pairwise tangent at the same point $O$ and having the same nucleus $N$. Let $W = \bigcup_{i=1}^{q/4} (\mathcal{C}_i \setminus \{O\})$. Then $W$ is a $q^2/4$-set of type $(0, q/4, q/2)$ if and only if on the line $ON$ there are three distinct points, such that on each of them there are $u_0 = (q/2) + 1$ 0-secant lines and $u_{q/2} = q/2$ $q/2$-secant lines of $W$.

**Proof.** If $W$ is of type $(0, q/4, q/2)$, then $t_{q/2} = 3q/2$. Since $ON$ is a 0-secant line of $W$, every $q/2$-secant line of $W$ meets $ON$ in only one point, on which there are other $(q/2) - 1$ $q/2$-secant lines, as follows from Lemma 3.2 and Theorem 3.1. Therefore, if $h$ is the number of points of $ON$, on which there are $q/2$-secant lines, then

$$t_{q/2} = h \frac{q}{2}.$$ 

Hence $h = 3$.

Conversely, let $X, Y$ and $Z$ be the three points of $ON$, for which $u_0 = (q/2) + 1$ and $u_{q/2} = q/2$. Then $t_{q/2} = 3q/2$. Furthermore,

$$t_i = t_{(q/2) - i}, \quad 2 \leq i \leq \frac{q}{4} - 2. \quad \text{(*)}$$

In fact, fix a conic $\mathcal{C}$, as in Lemma 3.2. $\mathcal{C}$ admits a group $G$ of elations, having as common axis the line $ON$. This group is sharply transitive on the lines but $ON$ on $O$ or $N$. Thus, every line on $O$ or $N$ has $q/4$ points on $W$. Moreover, $G$ is transitive on the set of exterior (resp. secant but $ON$) lines of $\mathcal{C}$. Therefore, if an exterior line of $\mathcal{C}$ on $P \in ON$ has $x_P$ points on $W$, then every exterior line on $P$ has $x_P$ points on $W$. Hence, every secant line but $ON$ of $\mathcal{C}$ has $\beta_P = (q/2) - x_P$ points on $W$. Since every line of $PG(2, q)$ is on one of the points of $ON$, identity (**) follows.

When $i = 0$ we have

$$t_0 = t_{q/2} + 1. \quad \text{(**)}$$
Using equalities (\(*) and (\(**), the first and third character equations of \(W\) become, respectively,

\[
\begin{align*}
t_{q/4} + 2 \sum_{i=2}^{(q/4)-2} t_i &= q(q-2), \\
\frac{q}{4} \left(\frac{q}{4} - 1\right) t_{q/4} + \sum_{i=2}^{(q/4)-2} i(i-1)t_i + \sum_{i=2}^{(q/4)-2} \left(\frac{q}{4} - 1\right) \left(\frac{q}{2} - i - 1\right) t_{(q/2)-i} \\
&= \frac{q^2}{4} \left(\frac{q^2}{4} - 3q\right) + 2.
\end{align*}
\]

Hence,

\[
\begin{align*}
\frac{q}{4} \left(\frac{q}{4} - 1\right) t_{q/4} + \sum_{i=2}^{(q/4)-2} \left(2q^2 + q^2 - qi - \frac{q}{2}\right) t_i &= \frac{q^2}{4} \left(\frac{q^2}{4} - 3q\right) + 2.
\end{align*}
\]

Elimination of \(t_{q/4}\) between (3.3) and (3.5) gives

\[
\sum_{i=2}^{(q/4)-2} \left(2q^2 + \frac{q^2}{8} - qi\right) t_i = 2 \sum_{i=2}^{(q/4)-2} \left(\frac{q}{4} - \frac{q}{2}\right)^2 t_i = 0.
\]

Hence \(t_i = 0, 2 \leq i \leq (q/4) - 2\). Since

\[
t_i = t_{(q/2)-i}, \quad 2 \leq i \leq (q/4) - 2,
\]

the proof is achieved. \(\square\)

For the next theorem, we introduce the following notation. If \(\omega^k\) is an element of \(GF(q) \setminus \{0,1\}\), where \(\omega\) is a primitive element of the field, set

\[
H_k = \{x \in T_0 \mid x\omega^k \in T_0\}.
\]

We proved in Lemma 2.1 that \(|H_k| = q/4\).

Furthermore, if \(\Gamma_j\) is a conic, set, for every \(P \notin \Gamma_j\),

\[
E_j(P) = \{Q \in \Pi_q \mid Q \neq P \text{ and the line } PQ \text{ is exterior to } \Gamma_j\}.
\]

**Theorem 3.4.** Let \(\mathcal{C}_i\) be the conic of \(PG(2,q)\) having equation \(x_0x_2 = x_1^2 + \gamma_i x_2^2\), where

\[
\gamma_i \in H_k, \quad i = 1, \ldots, q/4, \text{ and let } O \text{ be the point } (1,0,0) \text{ of } \mathcal{C}_i. \text{ Then}
\]

(1) the set

\[
W = \bigcup_{i=1}^{q/4} (\mathcal{C}_i \setminus \{O\})
\]

is a \(q^2/4\)-set of type \((0,q/4,q/2)\);

(2) there are \(q/4\) conics \(\Gamma_j, j = 1, \ldots, q/4, \) having the same nucleus \(N\), such that

\[
W = E_j(X) \cap E_j(Y),
\]

for two points \(X\) and \(Y\) other than \(N\) on the tangent line through \(O\).
Proof. $|W| = q^2/4$, since $|H_q| = q/4$.

For abuse of notation, for every $z \in GF(q)$, we denote by $\sqrt{z}$ the unique element of $GF(q)$, whose square is $z$. Consider the points $X(1,1,0), Y(1,\sqrt{\omega^k},0)$ and $Z(1,1+\sqrt{\omega^k},0)$.

We prove that for each of these points $u_{q/2} = q/2$ and $u_0 = (q/2) + 1$.

Proceeding as in Theorem 3.1, consider the line on $X$,

$x_1 = x_0$,

which is a $q/2$-secant line to $W$, since $\gamma \in T_0$.

Similarly, the line on $Y$,

$x_1 = \sqrt{\omega^k}x_0$

meets every conic $C_j$ in two points, since $\omega^k\gamma \in T_0$. Therefore, it is a $q/2$-secant line.

Finally, since $H_q$ is a subgroup of $T_0$, the line on $Z$,

$x_1 = (1 + \sqrt{\omega^k})x_0$,

is a $q/2$-secant line, too.

The proof of statement (1) follows by Lemma 3.3.

(2) Let $P(1, \sqrt{z + \gamma z^2}, z)$ be a point of $W$, $z \in GF(q) \setminus \{0\}$, and let $s$ be the line $PX$, having equation

$(1 + \sqrt{z + \gamma z^2})x_2 = z(x_1 + x_0)$.

If $\Gamma_j$ is the conic of equation

$x_0x_2 = x_1^2 + \delta_jx_2^2$, $\delta_j \in GF(q),$

then a necessary and sufficient condition for $s$ to be an exterior line of $\Gamma_j$ is

$\delta_j + \frac{z + \gamma z^2 + 1}{z^2} \in T_1,$

which is equivalent to

$\delta_j + z^{-1} + z^{-2} + \gamma \in T_1.$

(3.6a)

Since $z^{-1} + z^{-2} + \gamma \in T_0$, we obtain

$\delta_j \in T_1$.

(3.7)

Similarly, consider the line $PY$, whose equation is

$(\sqrt{z + \gamma z^2} + \sqrt{\omega^k})x_2 = x_1 + \sqrt{\omega^k}x_0.$

$PY$ is an exterior line of $\Gamma_j$ if and only if

$\omega^k\delta_j + \omega^kz^{-1} + \omega^k\gamma + \omega^{2k}z^{-2} \in T_1.$
Since
\[ \omega^k z^{-1} + \omega^{2k} z^{-2} \in T_0 \]
and
\[ \omega^k \gamma_i \in T_0, \]
we deduce
\[ \omega^k \delta_j \in T_1. \]  \hspace{1cm} (3.8)

By Lemma 2.1, there are \( q/4 \) values of \( \delta_j \) which satisfy conditions (3.7) and (3.8). This completes the proof. \( \Box \)

**Acknowledgments**

We would like to thank the referees for their useful suggestions and remarks. In particular, we are indebted to one of them, who suggested Theorem 2.1.

**References**