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Periodic solutions of superlinear autonomous Hamiltonian systems with prescribed period

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Abstract

In this paper we prove an existence theorem of nonconstant periodic solution of superlinear autonomous Hamiltonian system $\dot{x}(t) = J\nabla H(x(t))$ with prescribed period under an assumption weaker than Ambrosetti–Rabinowitz-type condition:

$$0 < \mu H(x) \leqslant \langle \nabla H(x), x \rangle, \quad \mu > 2, \ |x| \geqslant R > 0.$$

Our result extends the pioneering work of Rabinowitz of 1978.

Keywords: Periodic solution; Ambrosetti–Rabinowitz-type condition; Hamiltonian system; Critical point

1. Introduction

This paper deals with the periodic solutions of the following autonomous Hamiltonian systems with prescribed period:

$$\dot{x}(t) = J\nabla H\big(x(t)\big), \tag{1.1}$$

where $J = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}$ is the standard symplectic matrix on $\mathbf{R}^{2n}$. Denote the inner product and norm of $\mathbf{R}^{2n}$ by $\langle\cdot,\cdot\rangle$ and $|\cdot|$, respectively. In his pioneering paper [6], Rabinowitz proved the following theorem (see also [4,7]):

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Theorem 1.1. *Suppose* $H \in C^1(\mathbf{R}^{2n}, \mathbf{R}^1)$ *and satisfies*

(H_1) $H \geqslant 0$,
(H_2) $H(x) = \circ(|x|^2)$ *as* $|x| \to 0$, *and*
(H_3) *there exist* $\mu > 2$ *and* $R > 0$ *such that for* $|x| \geqslant R$,

$$0 < \mu H(x) \leqslant \langle \nabla H(x), x \rangle. \tag{1.2}$$

Then for any $T > 0$, (1.1) *has a nonconstant* T*-periodic solution.*

The condition (H_3) is called Ambrosetti–Rabinowitz-type condition, it appears frequently in the studying of existence and multiplicity of solutions of various superlinear differential equations. There are some works which improved this condition for certain equations, see, for example, [5]. Our goal of this paper is to prove that Theorem 1.1 still holds if (H_3) is replaced by a weaker condition. The idea of this paper is related to our early papers [1–3] in which the periodic solutions of Hamiltonian systems with prescribed energy were considered.

Definition 1.2. A vector field V defined on $\mathbf{R}^{2n}$ is called positive if $\langle V(x), x\rangle > 0$ for $x \in \mathbf{R}^n \setminus \{0\}$. We call V a normalized positive vector field if V is positive, linear and satisfies the following conditions:

(V_1) $JV = VJ$,
(V_2) $\langle V(x), x\rangle = \langle x, x\rangle$ for $x \in \mathbf{R}^{2n}$.

The main result of this paper is as follows.

Theorem 1.3. *Suppose* $H \in C^1(\mathbf{R}^{2n}, \mathbf{R}^1)$ *satisfies* (H_1), (H_2) *and*

(H_4) *there exist normalized positive vector field* V, *constants* $\mu > 2$ *and* $R > 0$ *such that for* $|x| \geqslant R$,

$$0 < \mu H(x) \leqslant \langle \nabla H(x), V(x) \rangle. \tag{1.3}$$

Then for any $T > 0$, (1.1) *has a nonconstant* T*-periodic solution.*

It is obvious that if $V(x) = x$, then (H_4) becomes (H_3). Example 1.4 below shows that (H_4) is weaker than (H_3) essentially. Therefore Theorem 1.3 is a substantial improvement of Theorem 1.1.

Example 1.4. Let $\theta(x)$ be the argument of $x = (\xi_1, \xi_2) \in \mathbf{R}^2 \setminus \{0\}$ defined by

$$\theta(x) = \begin{cases} \arctan(\xi_2/\xi_1), & \text{if } \xi_1 > 0, \xi_2 \geqslant 0, \\ \frac{\pi}{2}, & \text{if } \xi_1 = 0, \xi_2 > 0, \\ \arctan(\xi_2/\xi_1) + \pi, & \text{if } \xi_1 < 0, \\ \frac{3\pi}{2}, & \text{if } \xi_1 = 0, \xi_2 < 0, \\ \arctan(\xi_2/\xi_1) + 2\pi, & \text{if } \xi_1 > 0, \xi_2 < 0. \end{cases}$$

For any $\mu > 2$, define a function $H \in C^1(\mathbf{R}^2, \mathbf{R}^1)$ as follows:

$$H(x) = \begin{cases} \frac{|x|^\mu}{\exp(\mu \sin 4(\ln|x| + \theta(x)))}, & \text{if } x \neq 0, \\ 0, & \text{if } x = 0. \end{cases} \tag{1.4}$$

The direct computation shows that for $x \neq 0$,

$$H'_{\xi_1}(x) = \frac{\mu|x|^{\mu-2}(\xi_1 - 4(\xi_1 - \xi_2)\cos 4(\ln|x| + \theta(x)))}{\exp(\mu \sin 4(\ln|x| + \theta(x)))},$$

$$H'_{\xi_2}(x) = \frac{\mu|x|^{\mu-2}(\xi_2 - 4(\xi_1 + \xi_2)\cos 4(\ln|x| + \theta(x)))}{\exp(\mu \sin 4(\ln|x| + \theta(x)))}.$$

Define a normalized positive vector field V by

$$V(x) = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} x.$$

Then for $x \neq 0$,

$$\langle \nabla H(x), V(x) \rangle = \frac{\mu|x|^\mu}{\exp(\mu \sin 4(\ln|x| + \theta(x)))} = \mu H(x) > 0.$$

i.e., (H_4) is satisfied. We prove that H does not satisfy (H_3). Note that

$$\langle \nabla H(x), x \rangle = \frac{\mu|x|^\mu(1 - 4\cos 4(\ln|x| + \theta(x)))}{\exp(\mu \sin 4(\ln|x| + \theta(x)))}.$$

Let $x = (1, 0)$, $y = (\sqrt{2}/2, \sqrt{2}/2)$, then

$$\langle \nabla H(x), x \rangle < 0, \qquad \langle \nabla H(y), y \rangle > 0.$$

By continuity, there exists $z \in \mathbf{R}^2 \setminus \{0\}$ such that $\langle \nabla H(z), z \rangle = 0$. Let

$$x_n = e^{n\pi} x, \quad y_n = e^{n\pi} y, \quad z_n = e^{n\pi} z, \quad n = 1, 2, \ldots.$$

One has, $|x_n| \to \infty$, $|y_n| \to \infty$, $|z_n| \to \infty$ as $n \to \infty$, and

$$\langle \nabla H(x_n), x_n \rangle < 0, \quad \langle \nabla H(y_n), y_n \rangle > 0, \quad \langle \nabla H(z_n), z_n \rangle = 0, \quad \forall n.$$

Hence H satisfies (H_4) but does not satisfy (H_3) essentially.

In the rest part of this section we discuss the properties of a modified function H_K which will be used in the proof of Theorem 1.3.

Lemma 1.5. *Suppose $H(x)$ satisfies* (H_1), (H_2) *and* (H_4), *$K > 0$ is a constant. Define*

$$H_K(x) = \chi(|x|)H(x) + (1 - \chi(|x|)), R(K)|x|^4, \tag{1.5}$$

where constant $R(K)$ and function $\chi \in C^\infty(\mathbf{R}^1, \mathbf{R}^1)$ satisfy

$$R(K) = \sup_{K \leqslant |x| \leqslant K+1} \frac{H(x)}{|x|^4}, \qquad \chi(s) = \begin{cases} 1, & \text{if } s \leqslant K, \\ \chi'(s) < 0, & \text{if } K < s < K+1, \\ 0, & \text{if } s \geqslant K+1. \end{cases}$$

Then H_K also satisfies (H_1), (H_2) *and* (H_4). *Furthermore,*

$$|H_K(x)| \leqslant a_1|x|^4 - a_2, \quad a_1, a_2 > 0. \tag{1.6}$$

Proof. It is easy to prove that H_K satisfies (H$_1$), (H$_2$) and (1.6). We only verify (H$_4$). By using (V$_2$) and (1.3), for $|x| \geqslant R > 0$,

$$\begin{aligned}\big\langle \nabla H_K(x), V(x)\big\rangle &= \chi(|x|)\big\langle \nabla H(x), V(x)\big\rangle + |x|^{-1}\big\langle x, V(x)\big\rangle \chi'(|x|)H(x)\\ &\quad + R(K)\big(4\big(1-\chi(|x|)\big)|x|^2\big\langle x, V(x)\big\rangle - \big\langle x, V(x)\big\rangle \chi'(|x|)|x|^3\big)\\ &= \chi(|x|)\big\langle \nabla H(x), V(x)\big\rangle + |x|\chi'(|x|)H(x)\\ &\quad + R(K)\big(4\big(1-\chi(|x|)\big)|x|^4 - \chi'(|x|)|x|^5\big)\\ &\geqslant \mu\chi(|x|)H(x) + 4R(K)\big(1-\chi(|x|)\big)|x|^4\\ &\quad + |x|\chi'(|x|)\big(H(x) - R(K)|x|^4\big).\end{aligned}$$

Since $|x|\chi'(|x|)(H(x) - R(K)|x|^4) \geqslant 0$, one has

$$\big\langle \nabla H_K(x), V(x)\big\rangle \geqslant \nu H_K(x) > 0, \quad \text{for } |x| \geqslant R > 0, \tag{1.7}$$

where $\nu = \min\{\mu, 4\}$. □

Lemma 1.6. *Denote by φ_s the flow of the linear vector field V with property* (V$_2$)*, then*

$$|\varphi_s x| = e^s|x|, \quad \forall s \in \mathbf{R}^1,\ \forall x \in \mathbf{R}^{2n}.$$

Proof. Let $g(s) = |\varphi_s x|^2$, then $g(0) = |x|^2$. By (V$_2$),

$$\frac{d}{ds}g(s) = 2\big\langle V(\varphi_s x), \varphi_s x\big\rangle = 2\langle \varphi_s x, \varphi_s x\rangle = 2g(s).$$

Then $g(s) = e^{2s}|x|^2$ by solving the ordinary differential equation. □

Lemma 1.7. *Let H_K be defined by* (1.5)*, then there exist $a_3, a_4 > 0$ such that*

$$H_K(x) \geqslant a_3|x|^\nu - a_4, \quad \forall x \in \mathbf{R}^{2n}. \tag{1.8}$$

Proof. Denote by $\mathbf{S}^{2n-1}$ the unit sphere in $\mathbf{R}^{2n}$. For any $x \in \mathbf{R}^{2n} \setminus \{0\}$, since

$$\frac{d}{ds}\big(|\varphi_s x|^2\big) = 2\big\langle \varphi_s x, V(\varphi_s x)\big\rangle > 0,$$

$|\varphi_s x|$ is increasing in s. Hence, there exist $s \in \mathbf{R}^1$ and $\xi \in \mathbf{S}^{2n-1}$ such that $x = \varphi_s\xi$ (see [1, Lemma 2.2] for details). Since $|x| = |\varphi_s\xi| = e^s$, by (1.7),

$$\frac{d}{ds}H_K(\varphi_s\xi) = \big\langle \nabla H_K(\varphi_s\xi), V(\varphi_s\xi)\big\rangle \geqslant \nu H_K(\varphi_s\xi) > 0, \quad s \geqslant \ln R. \tag{1.9}$$

Integrating this inequality, for some constant c,

$$\int_{\ln R}^{s} \frac{\frac{d}{ds}H_K(\varphi_s\xi)}{H_K(\varphi_s\xi)}\,ds \geqslant \nu s - c.$$

Denote $a_3 = e^{-c}$, then

$$H_K(x) = H_K(\varphi_s\xi) \geqslant a_3 e^{s\nu} = a_3|x|^\nu, \quad |x| \geqslant R.$$

Since H_K is bounded for $|x| \leqslant R$, there exists a constant a_4 such that

$$H_K(x) \geqslant a_3|x|^\nu - a_4, \quad \forall x \in \mathbf{R}^{2n}. \qquad □$$

2. Proof of Theorem 1.3

In this section we give the proof of Theorem 1.3. At first we state an abstract critical point theorem of [7].

Theorem 2.1. [7, Theorem 5.29] *Let* $\mathbf{E}$ *be a real Hilbert space with* $\mathbf{E} = \mathbf{E}_1 \oplus \mathbf{E}_2$ *and* $\mathbf{E}_2 = \mathbf{E}_1^{\perp}$. *Suppose* $f \in C^1(\mathbf{E}, \mathbf{R}^1)$, *satisfies* (PS) *condition, and*

(f_1) $f(x) = \frac{1}{2}\langle Ax, x\rangle_{\mathbf{E}} + \phi(x)$, *where* $Ax = A_1P_1 + A_2P_2$ *and* $A_i : \mathbf{E}_i \to \mathbf{E}_i$ *is bounded and self-adjoint,* $i = 1, 2$;
(f_2) ϕ' *is compact*; *and*
(f_3) *there exist a subspace* $\tilde{\mathbf{E}} \subset \mathbf{E}$ *and sets* $S \subset \mathbf{E}$, $Q \subset \tilde{\mathbf{E}}$ *and constants* $\alpha > \omega$ *such that*
  (i) $S \subset \mathbf{E}_1$ *and* $f|_S \geqslant \alpha$,
  (ii) Q *is bounded and* $f|_{\partial Q} \leqslant \omega$,
  (iii) S *and* ∂Q *link.*

Then

$$c = \inf_{h\in\Gamma} \sup_{x\in Q} f\big(h(1, x)\big) \tag{2.1}$$

is a critical value of f *and* $c \geqslant \alpha$, *where* Γ *is defined by*

$$\Gamma = \big\{h \in C\big([0, 1] \times \mathbf{E}, \mathbf{E}\big) \bigm| h(0, x) = x,\ h(1, x)|_{\partial Q} = x,\ h(t, x) = e^{\theta(t,x)A} + K(t, x)\big\}.$$

Let $\mathbf{S}_T = \mathbf{R}^1/(T\mathbf{Z})$. Denote $\mathbf{E} = \mathbf{W}^{1/2,2}(\mathbf{S}_T, \mathbf{R}^{2n})$ the Sobolev space consists of all $x(t)$ in $\mathbf{L}^2(\mathbf{S}_T, \mathbf{R}^{2n})$ whose Fourier series

$$x(t) = \sum_{k=-\infty}^{+\infty} \exp\left(\frac{2k\pi tJ}{T}\right)a_k, \quad a_k \in \mathbf{R}^{2n},$$

satisfies

$$\|x\|_{\mathbf{E}}^2 \equiv T|a_0|^2 + T\sum_{k=-\infty}^{\infty} |k| \cdot |a_k|^2 < +\infty.$$

The inner product on $\mathbf{E}$ is defined by

$$\langle x_1, x_2\rangle_{\mathbf{E}} = T\big\langle a_0^1, a_0^2\big\rangle + T\sum_{k=-\infty}^{+\infty} |k|\big\langle a_k^1, a_k^2\big\rangle,$$

where $x_i = \sum_{k=-\infty}^{+\infty} \exp(\frac{2k\pi tJ}{T})a_k^i$, $i = 1, 2$.

Define linear bounded self-adjoint operator A on $\mathbf{E}$ by extending the bilinear form

$$\langle Ax, y\rangle_{\mathbf{E}} = \int_0^T (-J\dot{x}, y)\,dt. \tag{2.2}$$

Clearly, $\ker A = \mathbf{R}^{2n}$. Let $\mathbf{E}^0 = \mathbf{R}^{2n}$,

$$\mathbf{E}^+ = \left\{x \in \mathbf{E} \mid x(t) = \sum_{k>0} \exp\left(\frac{2k\pi t J}{T}\right) a_k\right\},$$

$$\mathbf{E}^- = \left\{x \in \mathbf{E} \mid x(t) = \sum_{k<0} \exp\left(\frac{2k\pi t J}{T}\right) a_k\right\}.$$

Denote by $P^{\pm}$ the projections of $\mathbf{E}$ to $\mathbf{E}^{\pm}$, respectively. Then

$$A = \frac{2\pi}{T} P^+ - \frac{2\pi}{T} P^-. \tag{2.3}$$

Let V be the normalized positive vector field in (H_4) of Theorem 1.3. Then V is an invertible linear operator from $\mathbf{R}^{2n}$ to $\mathbf{R}^{2n}$. Let $a = 1/\|V^{-1}\|$, $b = \|V\|$, where $\|V\|$ and $\|V^{-1}\|$ are operator norms. For any $x \in \mathbf{R}^{2n}$, one has $a|x| \leqslant |Vx| \leqslant b|x|$. Define a vector field $\tilde{V}$ on $\mathbf{E}$ by

$$(\tilde{V}x)(t) = V\big(x(t)\big). \tag{2.4}$$

Using conditions (V_1), (V_2) and the Fourier series, a direct computation shows

Lemma 2.2. [1] *For* $\forall x \in \mathbf{E}$*, there hold*

$$\langle Ax, \tilde{V}x\rangle_{\mathbf{E}} = \langle Ax, x\rangle_{\mathbf{E}}. \tag{2.5}$$

$$a\|x\|_{\mathbf{E}} \leqslant \|\tilde{V}x\|_{\mathbf{E}} \leqslant b\|x\|_{\mathbf{E}}. \tag{2.6}$$

Define $\phi : \mathbf{E} \to \mathbf{R}^1$ by

$$\phi(x) = \int_0^T H_K\big(x(t)\big)\,dt. \tag{2.7}$$

By (1.6) and [7, Proposition B.37], $\phi \in C^1(\mathbf{E}, \mathbf{R}^1)$, $\phi'(x)$ is compact. We consider the critical point of the following functional $f_K \in C^1(\mathbf{E}, \mathbf{R})$:

$$f_K(x) = \frac{1}{2}\langle Ax, x\rangle_{\mathbf{E}} - \phi(x), \quad \forall x \in \mathbf{E}. \tag{2.8}$$

It is easy to see that

$$f_K'(x)y = \langle Ax, y\rangle_{\mathbf{E}} - \int_0^T \big\langle \nabla H_K(x), y\big\rangle\,dt, \quad \forall x, y \in \mathbf{E}. \tag{2.9}$$

It is well known that the critical points of f_K are the T-periodic solutions of

$$\dot{x}(t) = J\nabla H_K\big(x(t)\big). \tag{2.10}$$

Lemma 2.3. *If* f_K *satisfies the* (PS) *condition, i.e., if* $\{x_m\} \subset E$*, with* $f_K'(x_m) \to 0$ *and* $|f_K(x_m)| \leqslant M$ *for some constant* $M > 0$*, then* $\{x_m\}$ *has a convergent subsequence.*

Proof. By Lemmas 1.5 and 2.2, for m large enough,

$$
\begin{aligned}
M+b\|x_m\|_{\mathbf{E}} &\geqslant M+\|\tilde{V}x_m\|_{\mathbf{E}} \geqslant f_K(x_m)-\frac{1}{2}f_K'(x_m)(\tilde{V}x_m)\\
&=\frac{1}{2}\langle Ax,x\rangle_{\mathbf{E}}-\int_0^T H_K(x_m)\,dt-\frac{1}{2}\langle Ax,\tilde{V}x\rangle_{\mathbf{E}}+\frac{1}{2}\int_0^T\big\langle\nabla H_K(x_m),Vx_m\big\rangle\,dt\\
&=\frac{1}{2}\int_0^T\big\langle\nabla H_K(x_m),Vx_m\big\rangle\,dt-\int_0^T H_K(x_m)\,dt\\
&\geqslant\left(\frac{\nu}{2}-1\right)\int_0^T H_K(x_m)\,dt-M_1\\
&\geqslant M_2\|x_m\|_{\mathbf{L}^4}^4-M_3.
\end{aligned}
$$

One has,

$$\|x_m\|_{\mathbf{L}^4}\leqslant M_4\|x_m\|_{\mathbf{E}}^{1/4}+M_5. \tag{2.11}$$

Decompose x_m as

$$u_n=x_m^+ + x_m^- + x_m^0\in\mathbf{E}^+\oplus\mathbf{E}^-\oplus\mathbf{E}^0.$$

By (1.8),

$$
\begin{aligned}
M+b\|x_m\|_{\mathbf{E}} &\geqslant\left(\frac{\nu}{2}-1\right)\int_0^T H_K(x_m)\,dt-M_1\\
&\geqslant M_6\|x_m\|_{\mathbf{L}^\nu}^\nu-M_7\geqslant M_8\|x_m\|_{\mathbf{L}^2}^\nu-M_7\geqslant M_9\big\|x_m^0\big\|_{\mathbf{E}}^\nu-M_7.
\end{aligned}
$$

Hence,

$$\big\|x_m^0\big\|\leqslant M_{10}\big(1+\|x_m\|_{\mathbf{E}}^{1/\nu}\big).$$

On the other hand,

$$
\begin{aligned}
\frac{2\pi}{T}\big\|x_m^+\big\|_{\mathbf{E}}^2 &=\big\langle Ax_m,x_m^+\big\rangle_{\mathbf{E}}=f_K'(x_m)x_m^+ +\int_0^T\big\langle\nabla H_K(x_m),x_m^+\big\rangle\,dt\\
&\leqslant\big\|x_m^+\big\|_{\mathbf{E}}+\left|\int_0^T\big\langle\nabla H_K(x_m),x_m^+\big\rangle\,dt\right|\\
&\leqslant\big\|x_m^+\big\|_{\mathbf{E}}+\left(\int_0^T\big|\nabla H_K(x_m)\big|^{4/3}\,dt\right)^{3/4}\left(\int_0^T\big|x_m(t)\big|^4\,dt\right)^{1/4}\\
&\leqslant\big\|x_m^+\big\|_{\mathbf{E}}+M_{11}\big(\|x_m\|_{\mathbf{L}^4}^3+1\big)\big\|x_m^+\big\|_{\mathbf{L}^4}\\
&\leqslant M_{12}\big(\|x_m\|_{\mathbf{L}^4}^3+1\big)\big\|x_m^+\big\|_{\mathbf{E}}.
\end{aligned}
$$

By (2.11),

$$\big\|x_m^+\big\|_{\mathbf{E}}\leqslant M_{13}\big(\|x_m\|_{\mathbf{L}^4}^3+1\big)\leqslant M_{14}\big(\|x_m\|_{\mathbf{E}}^{3/4}+1\big).$$

In the same fashion,

$$\|x_m^-\|_{\mathbf{E}} \leqslant M_{14}\big(\|x_m\|_{\mathbf{E}}^{3/4} + 1\big).$$

Therefore,

$$\|x_m\|_{\mathbf{E}} \leqslant \|x_m^+\|_{\mathbf{E}} + \|x_m^-\|_{\mathbf{E}} + \|x_m^0\|_{\mathbf{E}} \leqslant M_{15}\big(1 + \|x_m\|_{\mathbf{E}}^{3/4} + \|x_m\|_{\mathbf{E}}^{1/\nu}\big).$$

This shows that $\{x_m\}$ is bounded in $\mathbf{E}$. By (2.3),

$$f_K'(x_m) = \frac{2\pi}{T}x_m^+ - \frac{2\pi}{T}x_m^- - \phi'(x_m).$$

Since $f_K'(x_m) \to 0$, ϕ' is compact, it is easy to see that $\{x_m\}$ has a convergent subsequence. $\square$

Lemma 2.4. *f_K has a critical value $c_K > 0$.*

Proof. Let $\mathbf{E}_1 = \mathbf{E}^+$, $\mathbf{E}_2 = \mathbf{E}^- \oplus \mathbf{E}^0$. Then f_K satisfies the conditions (f_1) and (f_2) of Theorem 2.1. We need only to verify (f_3). This can be achieved by the same method used in the proofs of Lemmas 6.16 and 6.20 of [7], here we give the proof for completeness.

By (H_2) and (1.5), for any $\epsilon > 0$, there exists an $M > 0$ such that

$$H_K(x) \leqslant \epsilon|x|^2 + M|x|^4, \quad \forall x \in \mathbf{R}^{2n}.$$

By (2.3) and inequality $\|x\|_{\mathbf{L}^s} \leqslant \alpha_s\|x\|_{\mathbf{E}}$ (see [7, Proposition 6.6]), for $x \in \mathbf{E}_1$,

$$\begin{aligned} f_K(x) &= \frac{1}{2}\cdot\frac{2\pi}{T}\|x\|_{\mathbf{E}}^2 - \int_0^T H_K(x)\,dt \geqslant \frac{\pi}{T}\|x\|_{\mathbf{E}}^2 - \big(\epsilon\|x\|_{\mathbf{L}^2}^2 + M\|x\|_{\mathbf{L}^4}^4\big) \\ &\geqslant \frac{\pi}{T}\|x\|_{\mathbf{E}}^2 - \big(\epsilon\alpha_2 + M\alpha_4\|x\|_{\mathbf{E}}^2\big)\|x\|_{\mathbf{E}}^2. \end{aligned}$$

Choose $\epsilon = \frac{\pi}{3T\alpha_2}$, $\rho^2 = \frac{\pi}{3TM\alpha_4}$. Denote by B_ρ the closed ball in $\mathbf{E}$ with radius ρ centered at origin. Let $S = \partial B_\rho \cap \mathbf{E}_1$, $\alpha = \frac{\pi}{3T}\rho^2$. For $x \in S$,

$$f_K(x) \geqslant \frac{\pi}{3T}\rho^2 = \alpha.$$

Then (i) of (f_3) holds.

Let $e \in \partial B_1 \cap \mathbf{E}_1$, define

$$\tilde{\mathbf{E}} = \operatorname{span}\{e\} \oplus \mathbf{E}_2, \qquad Q = \big\{re \mid r \in [0, r_1]\big\} \oplus (B_{r_2} \cap \mathbf{E}_2),$$

where r_1, r_2 are constants which will be chosen later.

Let $x = x^0 + x^- \in B_{r_2} \cap \mathbf{E}_2$. Then

$$f_K(x + re) = \frac{1}{2}\cdot\frac{2\pi}{T}\big(r^2 - \|x^-\|_{\mathbf{E}}^2\big) - \int_0^T H_K(x + re)\,dt.$$

x^0, x^- and e are mutually orthogonal in $\mathbf{L}^2$, by Lemma 1.7,

$$\int_0^T H_K(x+re)\,dt \geqslant a_3\int_0^T |x+re|^\nu\,dt - Ta_4 \geqslant a_5\left(\int_0^T |x+re|^2\,dt\right)^{\nu/2} - a_6$$
$$= a_5\left(\int_0^T \left(\left|x^0\right|^2 + |x^-|^2 + r^2|e|^2\right)dt\right)^{\nu/2} - a_6 \geqslant a_7\left(\left|x^0\right|^\nu + r^\nu\right) - a_6.$$

Hence,

$$f_K(x+re) \leqslant \frac{\pi}{T}\left(r^2 - \|x^-\|_{\mathbf{E}}^2\right) - a_7\left(\left|x^0\right|^\nu + r^\nu\right) + a_6$$
$$= \frac{\pi r^2}{T} - a_7 r^\nu + a_6 - \left(\frac{\pi\|x^-\|^2}{T} + a_7\left|x^0\right|^\nu\right).$$

Since $\nu > 2$, we can choose a $r_1 > 0$ such that for $r \geqslant r_1$,

$$\frac{\pi r^2}{T} - a_7 r^\nu + a_6 \leqslant 0.$$

Note that $(\frac{\pi r^2}{T} - a_7 r^\nu + a_6)$ is bounded on $[0, r_1]$ and

$$\lim_{\|x\|\to\infty}\left(\frac{\pi\|x^-\|^2}{T} + a_7\left|x^0\right|^\nu\right) = +\infty \quad \text{uniformly in } \mathbf{E}_2,$$

there exists $r_2 > 0$ such that

$$f_K(x+re) \leqslant M - \left(\frac{\pi\|x^-\|^2}{T} + a_7\left|x^0\right|^\nu\right) \leqslant 0, \quad \text{for } \|x\| \geqslant r_2.$$

It is obvious that $f_K \leqslant 0$ on $\mathbf{E}_2$, then $f_K \leqslant 0 \equiv \omega$ on ∂Q. By Lemma 6.27 of [7], S and ∂Q link. So (ii) and (iii) of (f_3) hold.

According to Theorem 2.1, f_K has a critical value $c_K > 0$. □

Proof of Theorem 1.3. Denote by x_K a critical point of f_K corresponding to critical value c_K, then x_K is a nonconstant T-periodic solution of (2.10) and

$$f_K(x_K) = c_K = \inf_{h\in\Gamma}\sup_{x\in Q} f_K\big(h(1,x)\big).$$

Since $h_0 \in \Gamma$ if $h_0(t,x) \equiv x$, $c_K \leqslant \sup_{x\in Q} f_K(x)$. For $\forall x = re + x^0 + x^- \in Q$,

$$f_K(x) = \frac{\pi}{T}\left(r^2 - \|x^-\|_{\mathbf{E}}^2\right) - \int_0^T H_K(x)\,dt \leqslant \frac{\pi}{T}r_1^2.$$

Therefore $c_K \leqslant \frac{\pi}{T}r_1^2$. Note that the constants r_1 and r_2 in the definition of Q are independent of K.

By (2.5) and (1.7), there exists $M_0 > 0$ such that

$$\frac{\pi}{T}r_1^2 \geqslant c_K = f_K(x_K) - \frac{1}{2}f_K'(x_K)(\tilde{V}x_K)$$
$$= \frac{1}{2}\int_0^T \langle \nabla H_K(x_K), Vx_K\rangle\,dt - \int_0^T H_K(x_K)\,dt$$

$$\geqslant \left(\frac{\nu}{2} - 1\right) \int_0^T H_K\big(x_K(t)\big)\,dt - M_0.$$

Since $H_K(x_K(t))$ is constant, $H_K(x_K(t))$ has a K independent upper bound

$$H_K\big(x_K(t)\big) \leqslant \left(\frac{\nu T}{2} - T\right)^{-1} \left(\frac{\pi}{T} r_1^2 + M_0\right).$$

By using (1.8), $x_K(t)$ has a K independent $\mathbf{L}^\infty$ bound. That is, $\|x_K\|_{\mathbf{L}^\infty} \leqslant K_0$ for some constant K_0.

If $K > K_0$, by (1.5),

$$\nabla H_K(x_K) = \nabla H(x_K).$$

Consequently, x_K is a nonconstant T-periodic solution of (1.1). □

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