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On the reduction of a non-torsion point of a Drinfeld module

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Abstract

Let ρ be a Drinfeld $\mathbb{F}_q[T]$-module defined over a global function field K. Let $\mathbf{z} \in K$ be a non-torsion point. We prove that for almost all monic elements $\mathfrak{n} \in \mathbb{F}_q[T]$ there exists a place $\wp$ of K such that $\mathfrak{n}$ is the "order" of the reduction of $\mathbf{z}$ modulo $\wp$.

1. Introduction

Let a be a fixed rational number other than 0 or ± 1. Let p be a prime number such that a is a p-unit. Denote the multiplicative order of a modulo p by n_p. It was proved by several authors [1,2,20] that there are only finitely many positive integers which are not an n_p for some p. This theorem was generalized later to any number field by Postnikova and Schinzel [14] and in strengthened form by Schinzel [15]. More precisely, let K be a number field and let a be an element of K which is not a root of unity. Then almost all positive integers N occur as the order of a modulo $\mathfrak{P}$ for some prime ideal $\mathfrak{P}$ of K.

Replacing the multiplicative group $\mathbb{Q}^*$ by an elliptic curve, Silverman [16] showed that the same phenomenon exists for elliptic curves. Namely, given elliptic curve E defined over $\mathbb{Q}$ and a point $Q \in E(\mathbb{Q})$ of infinite order, then for all but finitely many positive integers n there is a prime p such that n is the order of the point Q modulo p with respect to the group law of the

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reduced curve at prime p. This result was later extended also to elliptic curves defined over any number field K [5].

In this paper, we study an analogous phenomenon in the setting of Drinfeld $\mathbb{F}_q[T]$-modules. Throughout the paper, by a global function field we mean a function field which is of transcendence degree 1 over a finite field. Let K be a global function field together with a fixed ring homomorphism $\iota : \mathbb{F}_q[T] \to K$. Let ρ be a Drinfeld $\mathbb{F}_q[T]$-module defined over K. The additive group of K together with the $\mathbb{F}_q[T]$-action via ρ gives rise to an $\mathbb{F}_q[T]$-module, denoted by ${}^{\rho}K$. Let $\mathbf{z} \in {}^{\rho}K$ be a point which is not a torsion under the given $\mathbb{F}_q[T]$-action. Let $\mathfrak{n} \in \mathbb{F}_q[T]$ be a monic polynomial. We can ask whether or not $\mathfrak{n}$ can occur as the "order" of $\mathbf{z}$ modulo $\wp$ for some place $\wp$ of K (see Section 2 for a more precise definition and formulations). In the setting of Drinfeld modules, we give an affirmative answer to this question (Theorem 2.3).

The proof of our main result follows ideas in [14,15]. However, in order to apply the ideas in [14,15] to our situation, we need to study the dynamics at each place of K associated to the Drinfeld module in question. We organize our paper as follows. In Section 2 we recall the definition of a Drinfeld module and some elementary properties of Drinfeld modules. After some preliminaries, we formulate an analogue of the above classical theorem in the setting of Drinfeld modules.

Let $\wp$ be a place of K. In Sections 3 and 4 we study, with respect to $\wp$-adic topology, how close to 0 the $\mathbb{F}_q[T]$-orbit of a non-torsion point can be. This problem is treated separately in the two sections depending on whether $\iota(T)$ is $\wp$-integral (finite places of K) or not (infinite places of K). We first treat the case where $\wp$ is a finite place of K in Section 3. In this section, only elementary non-archimedean analysis over $K_\wp$ is involved. We show that there exists a $\wp$-adic disc $\mathcal{N}_0$ of the origin in $K_\wp$ and a $\wp$-adic unbounded region $\mathcal{N}_\infty$ which are stable under the $\mathbb{F}_q[T]$-action (Propositions 3.1 and 3.3). Moreover, we are able to give a quantitative description of the $\mathbb{F}_q[T]$-orbit of a point in question (Corollaries 3.2 and 3.4).

We consider the case where $\wp$ is an infinite place of K in Section 4. In this case, $\iota(T)$ is not $\wp$-integral and there is no neighborhood of the origin in $K_\wp$ which is stable under the action of the $\mathbb{F}_q[T]$-action. From the viewpoint of ($\wp$-adic) dynamical systems, this is due to the fact that the origin is a repelling fixed point so that any of its neighborhood will not be stable under the $\mathbb{F}_q[T]$-action. Let J_ρ be the $\wp$-adic closure of the set of torsion points of ρ. Then J_ρ is a closed subset of $\mathbb{C}_\wp$ (the completed algebraic closure of $K_\wp$) and J_ρ is stable under the $\mathbb{F}_q[T]$-action via ρ. We show that if $z \notin J_\rho$ then its orbit $\{\rho_\mathfrak{n}(z) \mid \mathfrak{n} \in \mathbb{F}_q[T]\}$ is unbounded with respect to the $\wp$-adic topology as $\deg(\mathfrak{n}) \to \infty$ (Corollary 4.5). If $z \in J_\rho$, then its orbit remains bounded and $\rho_\mathfrak{n}(z)$ can be close to the origin for suitable choices of $\mathfrak{n}$. Assume that z is non-torsion of ρ and that $z \in \bar{K} \cap J_\rho$, then for any given positive ϵ we show that there exists a positive constant c_ϵ depending only on ρ and z such that $\deg(\rho_\mathfrak{n}(z)) \geqslant -c_\epsilon(\deg \mathfrak{n})^{1+\epsilon}$ for all $\mathfrak{n} \in \mathbb{F}_q[T] \setminus \mathbb{F}_q$ (Theorem 4.6). Here deg denotes the extension to $\mathbb{C}_\wp$ of the degree function on $\mathbb{F}_q[T]$. The key ingredient of the proof of Theorem 4.6 is the analogue of Baker's theorem on linear forms in logarithms for Drinfeld modules which is established by Yu [18,19] for qualitative version and is obtained by Bosser [3] for a quantitative result. For a more detailed discussion, see Section 4.

In Section 5 we prove our main result. We treat the case where K is generic $\mathbb{F}_q[T]$-characteristic in Section 5.1 and then the case of finite characteristic in Section 5.2. A key tool for combining the local information obtained in the previous sections is the *canonical height* $\hat{h}_\rho$ associated to the Drinfeld module ρ in question (see Section 5 for definition). The canonical height associated to a Drinfeld module was introduced in [6] which is an analogue of the Néron–Tate height on Abelian varieties (see [13] for the definition and properties). For $\mathbf{z} \in {}^{\rho}K$, its canonical height $\hat{h}_\rho(\mathbf{z})$ can be thought of as a measurement of the arithmetic complexity of the orbit of $\mathbf{z}$

under the action of $\mathbb{F}_q[T]$. If $\mathbf{z}$ is not a torsion, then its canonical height $\hat{h}_\rho(\mathbf{z})$ is positive. The idea of the proof is to evaluate $A_\mathfrak{n} := \sum n_\wp v_\wp(\Phi_{\rho,\mathfrak{n}}(\mathbf{z}))$ where $\Phi_{\rho,\mathfrak{n}}(X)$ is a an analogue of the cyclotomic polynomial for monic $\mathfrak{n} \in \mathbb{F}_q[T]$ in the setting of Drinfeld modules and the sum is taken over all places $\wp$ of K with $n_\wp, v_\wp$ the local degree and the normalized valuation at $\wp$, respectively. If $\mathfrak{n}$ is not the order of $\mathbf{z}$ modulo $\wp$ for any place $\wp$ of K, then we show that $A_\mathfrak{n}$ has an upper bound of the form $\varphi_r(\mathfrak{n})(-\hat{h}_\rho(\mathbf{z}) + O(q^{-\gamma \deg \mathfrak{n}}))$ for some fixed positive constant γ. Here, φ_r is a positive function of non-zero elements in $\mathbb{F}_q[T]$ and the implied constant in the minor term is independent of $\mathfrak{n}$ (see Section 5 for details). As $\hat{h}_\rho(\mathbf{z})$ is positive, we see that the upper bound is negative if the degree of such monic polynomial $\mathfrak{n}$ is large enough. On the other hand, we have $A_\mathfrak{n} = 0$ by the product formula. From this, our main result follows.

2. Preliminaries

In this section, we gather some basic facts of the theory of Drinfeld modules and fix some notations which will be needed in this paper. To ease the notations, we use $\mathbb{A}$ to denote the ring of polynomials $\mathbb{F}_q[T]$ and k the rational function field $\mathbb{F}_q(T)$ in the variable T over the finite field $\mathbb{F}_q$, where q is the cardinality of $\mathbb{F}_q$. As usual, the degree function on polynomials is denoted by $\deg(\cdot)$ and its extension to k is also denoted by $\deg(\cdot)$.

A field F is called an $\mathbb{A}$-field if it is equipped with a structural ring homomorphism $\iota_F: \mathbb{A} \to F$. If F is a global function field, the following notations will be used.

Notations:

M_F	the set of places of F,
$F_\wp$	the completion of F at the place $\wp \in M_F$,
$\mathcal{O}_\wp$	the ring of integers of $F_\wp$,
$\pi_\wp$	a uniformizer at the place $\wp$,
$v_\wp$	the normalized valuation at the place $\wp \in M_F$ such that $v_\wp(\pi_\wp) = 1$,
$\mathbb{F}_\wp$	the residue field at the place $\wp$,
$n_\wp$	$= \dim_{\mathbb{F}_q} \mathbb{F}_\wp$, the degree of $\mathbb{F}_\wp$ over $\mathbb{F}_q$,
$\lvert\mathfrak{n}\rvert$	$= q^{\deg \mathfrak{n}}$, the norm of element $\mathfrak{n} \in k$,
$\mathbb{A}_+$	the subset of monic polynomial of $\mathbb{A}$.

2.1. *Drinfeld modules*

Let F be an $\mathbb{A}$-field and let $\tau = (x \mapsto x^q)$ be the qth power Frobenius endomorphism of $\mathbb{G}_a$. Denote by $F\{\tau\}$ the twisted polynomial ring which is generated by F and τ subjected to the relation that $\tau\alpha = \alpha^q\tau$ for all $\alpha \in F$. A Drinfeld ($\mathbb{A}$-)module over F is a ring homomorphism

$$\rho: \mathbb{A} \to F\{\tau\}$$

such that

(i) $\rho_T \neq \iota_F(T)$ where ρ_a denotes the image of $a \in \mathbb{A}$, and
(ii) $\rho_T = g_0\tau^0 + g_1\tau + \cdots + g_r\tau^r$, for some positive integer r, called the *rank* of ρ and $g_i \in F$ $(0 \leqslant i \leqslant r)$, such that $g_0 = \iota(T)$ and $g_r \neq 0$.

Let $\mathcal{P} = \ker(\iota_F)$. If $\mathcal{P} = (0)$ then we say that F is of *generic $\mathbb{A}$-characteristic*; otherwise ρ is of *finite $\mathbb{A}$-characteristic*. To ease the notation, if F is of generic characteristic we will identify $\mathbb{A}$

with its image $\iota_F(\mathbb{A}) \subset F$ and regard F as an extension of k. If F is of finite characteristic, let $\mathfrak{p}$ be the monic generator of $\mathcal{P}$ and call $\mathfrak{p}$ the $\mathbb{A}$-characteristic of F. The additive group $\mathbb{G}_a(F)$ of F is equipped with an $\mathbb{A}$-module structure via ρ and will be denoted by ${}^{\rho}F$. If no confusion arises, we will also call ρ the Drinfeld module to mean the additive group $\mathbb{G}_a$ together with the $\mathbb{A}$-action induced by ρ. Note that

$$\rho_T(x) = g_0 x + g_1 x^q + \cdots + g_r x^{q^r}$$

is the image of $x \in \bar{F}$ under ρ_T where $\bar{F}$ is a fixed algebraic closure of F.

The leading coefficient of $\rho_{\mathfrak{n}}$ in τ will be denoted by $\Delta_{\mathfrak{n}}$ for any $\mathfrak{n} \in \mathbb{A}$ where $\Delta_0 = 0$ by convention. By definition we have $\Delta_T = g_r$ and for non-zero $\mathfrak{n} \in \mathbb{A}$

$$\Delta_{\mathfrak{n}} = \alpha_{\mathfrak{n}} \Delta_T^{\gamma_{\mathfrak{n}}} \quad \text{where } \alpha_{\mathfrak{n}} \text{ is the leading coefficient of } \mathfrak{n} \text{ and } \gamma_{\mathfrak{n}} = \big(|\mathfrak{n}|^r - 1\big)/\big(q^r - 1\big). \tag{1}$$

A point $\alpha \in {}^{\rho}\bar{F}$ is called a torsion point of ρ if there exists a non-zero $\mathfrak{n} \in \mathbb{A}$ such that $\rho_{\mathfrak{n}}(\alpha) = 0$. We will denote the set of $\mathfrak{n}$-torsion of ρ by $\rho[\mathfrak{n}]$ and the submodule of torsion points of ${}^{\rho}F$ by $\rho_{\mathrm{tor}}(F)$.

2.2. *The reduction of a Drinfeld module*

In this subsection, we recall the notion of the reduction of a Drinfeld. Assume that F is a local field complete with respect to a discrete valuation v. Let $\mathcal{O}$ be the ring of integers and let $\mathcal{M} = (\pi)$ be the maximal ideal of F where π is a uniformizer of F. Assume further that the residue field $\mathbb{F}_v := \mathcal{O}/\mathcal{M}$ is a finite field containing $\mathbb{F}_q$. Let $n_v = \dim_{\mathbb{F}_q} \mathbb{F}_v$. Let ρ be a Drinfeld module defined over F such that

$$\rho_T = g_0 \tau^0 + g_1 \tau + \cdots + g_r \tau^r, \quad g_i \in \mathcal{O},\ \forall i = 0, \ldots, r.$$

In this case we say that ρ is defined over $\mathcal{O}$. Note that our assumption on $g_0 \in \mathcal{O}$ implies that $\iota_F(\mathbb{A}) \subset \mathcal{O}$. Thus, the residue field $\mathbb{F}_v$ has an $\mathbb{A}$-field structure induced from that of F and the characteristic of $\mathbb{F}_v$ is finite. Let $\mathfrak{p}$ be the characteristic of $\mathbb{F}_v$. In particular, we have $\iota_F(\mathfrak{p}) \in \mathcal{M}$. The reduction $\bar{\rho}$ of ρ is well defined and determined by

$$\bar{\rho}_T = \bar{g}_0 \tau^0 + \bar{g}_1 \tau + \cdots + \bar{g}_r \tau^r$$

where the bar denotes the reductions of g_i modulo the maximal ideal $\mathcal{M}$. The Drinfeld module ρ is said to have *good reduction* if ρ is defined over $\mathcal{O}$ and $\bar{g}_r \neq 0$; otherwise it has *bad reduction.* We see that $\bar{\rho}$ induces a Drinfeld module structure of rank $l \leqslant r$ over $\mathbb{F}_v$. Moreover, $l = r$ if and only if ρ has good reduction.

Assume that ρ has good reduction and let $\mathfrak{q}$ be a monic irreducible polynomial which is different from $\mathfrak{p}$. The Tate module

$$T_{\mathfrak{q}}(\bar{\rho}) = \varprojlim_{\ell} \bar{\rho}\big[\mathfrak{q}^{\ell}\big]$$

gives rise to a $\mathfrak{q}$-adic representation of $\mathrm{End}(\bar{\rho})$. The geometric Frobenius $\mathrm{Frob}_v := \tau^{n_v}$ of $\mathbb{G}_a$ acts on the Tate module $T_{\mathfrak{q}}(\bar{\rho})$. Let $L_{\mathfrak{p}}(X)$ be the characteristic polynomial associated to Frob_v. Then $L_{\mathfrak{p}}(X)$ is a monic polynomial of degree r with coefficients in $\mathbb{A}$ which is independent of $\mathfrak{q}$. Let

$$L_{\mathfrak{p}}(X) = X^r - \mathfrak{a}_1 X^{r-1} + \cdots + (-1)^r \mathfrak{a}_r, \quad \mathfrak{a}_i \in \mathbb{A}.$$

The following properties of $L_{\mathfrak{p}}(X)$ are well known (see [8] or [10, §4.12]).

Proposition 2.1.

(a) *We have* $\deg(\mathfrak{a}_i) \leqslant in_v/r$ *for* $i = 1, \dots, r$ *and* $\deg(\mathfrak{a}_r) = n_v$.
(b) *The ideal generated by* $L_{\mathfrak{p}}(1)$ *annihilates the finite* $\mathbb{A}$*-module* ${}^{\rho}\mathbb{F}_v$.

For any positive integer i, the quotient $F_i := \mathcal{O}_{\wp}/\mathcal{M}^i$ is a finite $\mathbb{A}$-module via the action given by ρ modulo $\mathcal{M}^i$.

Lemma 2.2. *The finite* $\mathbb{A}$*-module* F_i *is annihilated by the ideal generated by* $\mathfrak{p}^{i-1}L_{\mathfrak{p}}(1)$.

Proof. To ease the notation, let $\chi_i = \mathfrak{p}^{i-1}L_{\mathfrak{p}}(1)$. Let $\alpha \in \mathcal{O}$ be arbitrary. We need to show that $\rho_{\chi_i}(\alpha) \in \mathcal{M}^i$. By Proposition 2.1(b), we have $\rho_{\chi_1}(\alpha) \in \mathcal{M}$.

We notice that $\rho_{\chi_i}(\alpha) = \rho_{\mathfrak{p}}(\rho_{\chi_{i-1}}(\alpha))$ for $i > 1$. On the other hand, we have

$$\rho_{\mathfrak{p}}(x) = \iota_F(\mathfrak{p})x + \text{higher terms in } x.$$

Thus, $v(\rho_{\mathfrak{p}}(\alpha)) \geqslant v(\alpha) + 1$ if $v(\alpha) > 0$. The lemma now follows by induction on i. □

2.3. Statement of the main result

From now on, we will reserve the notation K for a fixed global function field which is assumed to be an $\mathbb{A}$-field with structural homomorphism $\iota : \mathbb{A} \to K$. We use the symbol ∞ to denote the unique place of k where T has a pole. A place $\wp$ of K is said to be a finite place if $\iota(T)$ does not have a pole at $\wp$; otherwise it is a infinite place of K. Note that K has infinite places only if K is of generic $\mathbb{A}$-characteristic. In this case, the set of infinite places of K is denoted by M_K^{∞}. In the case of finite characteristic $\mathfrak{p}$, the Drinfeld module ρ is called supersingular if $\rho_{\mathfrak{p}} = \Delta_{\mathfrak{p}}\tau^{rd}$ where $d = \deg(\mathfrak{p})$; otherwise it is called ordinary.

Given a non-torsion point $\mathbf{z} \in {}^{\rho}K$ of ρ, let $\mathcal{S}_{\rho,\mathbf{z}}$ be the subset of places $\wp$ of K where ρ has bad reduction at $\wp$ or $\mathbf{z}$ is not $\wp$-integral. Let $\wp \notin M_K \setminus \mathcal{S}_{\rho,\mathbf{z}}$ and let $\bar{\mathbf{z}}_{\wp}$ denote the reduction of $\mathbf{z}$ modulo $\wp$. The annihilator of $\bar{\mathbf{z}}_{\wp} \in {}^{\rho}\mathbb{F}_{\wp}$ is a non-zero ideal of $\mathbb{A}$ and its monic generator is denoted by $\mathfrak{d}_{\wp}$ which we call the order of $\mathbf{z}$ modulo $\wp$. Analogous to the case of the multiplicative group over a number field, our main result is the following.[2]

Theorem 2.3. *Let* $\mathbf{z} \in {}^{\rho}K$ *be a non-torsion point of* ρ. *Let* $B = \mathbb{A}_{+,\mathfrak{p}}$, *the set of monic polynomials which are prime to* $\mathfrak{p}$, *if* K *is of finite* $\mathbb{A}$*-characteristic* $\mathfrak{p}$ *and* ρ *is supersingular*; *otherwise* $B = \mathbb{A}_+$. *Then, for all but finitely many* $\mathfrak{n} \in B$ *there is a place* $\wp$ *of* K *such that* $\mathfrak{n}$ *is the order* $\mathfrak{d}_{\wp}$ *of* $\mathbf{z}$ *modulo* $\wp$.

3. $\mathbb{A}$-orbits over $K_{\wp}$ for $\wp$ a finite place of K

In this section, we study the orbits of non-torsion points of a Drinfeld module ρ over a complete local field. Let $\wp$ be a fixed finite place of K. The complete local field $K_{\wp}$ has an $\mathbb{A}$-field

[2] The author was informed that the same result was obtained by Ghioca and Tucker [9] for the generic characteristic case using their equidistribution theorem for Drinfeld modules.

structure induced from that of K via the embedding $K \hookrightarrow K_\wp$. Recall that ρ is a Drinfeld module of rank r defined over K given by

$$\rho_T = g_0\tau^0 + g_1\tau + \cdots + g_r\tau^r, \quad g_0 = \iota(T),\ g_r \neq 0.$$

By abuse of the notation, we will still use $\iota : \mathbb{A} \to K_\wp$ to denote the structural homomorphism and consider ρ as a Drinfeld module defined over the complete field $K_\wp$. Note that, as $\wp$ is a finite place of K we must have $\iota(T) \in \mathcal{O}_\wp$ and therefore, $\iota(\mathbb{A}) \subset \mathcal{O}_\wp$. As in Section 2.2, we denote the $\mathbb{A}$-characteristic of ${}^\rho\mathbb{F}_\wp$ by $\mathfrak{p}$ and let d be the degree of $\mathfrak{p}$. For the convenience of our discussion below, we also fix the coefficients of $\rho_\mathfrak{p}$ as follows:

$$\begin{aligned} &\rho_\mathfrak{p} = h_0\tau^0 + h_1\tau + \cdots + h_{rd}\tau^{rd}, \\ &h_0 = \iota(\mathfrak{p}), \quad h_{rd} \neq 0. \end{aligned}$$

Note that since $\mathfrak{p}$ is the characteristic of $\mathbb{F}_\wp$, we must have $v_\wp(\mathfrak{p}) > 0$. Replacing T by $T+1$ as a generator of $\mathbb{A}$ over $\mathbb{F}_q$ if necessary, we will make the assumption that $\mathfrak{p} \neq T$. As a consequence, we have that $v_\wp(g_0) = 0$. As usual, the maximal exponent of $\mathfrak{p}$ dividing $\mathfrak{n}$ is denoted by $\mathrm{ord}_\mathfrak{p}(\mathfrak{n})$.

We first give a quantitative description of the subset of $K_\wp$ consisting of points $z \in K_\wp$ whose $\mathbb{A}$-*orbits* $\{\rho_\mathfrak{n}(z) \mid \text{non-zero } \mathfrak{n} \in \mathbb{A}\}$ are unbounded with respect to $\wp$-adic topology.

Proposition 3.1. *There exists an integer $l_\infty < -v_\wp(\Delta_T)/(q^r - 1)$ so that for $z \in K_\wp$ with $v_\wp(z) \leqslant l_\infty$ we have*

$$\begin{aligned} v_\wp\big(\rho_\mathfrak{n}(z)\big) &= v_\wp\big(\Delta_\mathfrak{n} z^{|\mathfrak{n}|^r}\big) \\ &= |\mathfrak{n}|^r \left\{ v_\wp(z) + \frac{v_\wp(\Delta_T)}{q^r - 1} \right\} - \frac{v_\wp(\Delta_T)}{q^r - 1} \end{aligned} \tag{2}$$

for all non-zero $\mathfrak{n} \in \mathbb{A}$.

Proof. Put

$$\ell = \min\left\{ \frac{v_\wp(g_i) - v_\wp(\Delta_T)}{q^r - q^i} \;\middle|\; g_i \neq 0,\ 0 \leqslant i < r \right\}.$$

Then, $\ell \leqslant (v_\wp(g_0) - v_\wp(\Delta_T))/(q^r - 1) = -v_\wp(\Delta_T)/(q^r - 1)$. Let $z \in K_\wp$ be such that $v_\wp(z) < \ell$. Then $v_\wp(\Delta_T z^{q^r}) < v_\wp(g_i z^{q^i})$ for all $i = 0, \ldots, r-1$. Hence,

$$v_\wp\big(\rho_T(z)\big) = v_\wp\big(\Delta_T z^{q^r}\big) = |T|^r v_\wp(z) + v_\wp(\Delta_T).$$

Note that $v_\wp(\rho_T(z)) < v_\wp(g_0 z) = v_\wp(z) < \ell$. It follows by induction that for integer $n \geqslant 1$,

$$v_\wp\big(\rho_{T^n}(z)\big) < v_\wp\big(\rho_{T^{n-1}}(z)\big) < \cdots < v_\wp\big(\rho_T(z)\big) < v_\wp(z) < \ell, \tag{3}$$

$$v_\wp\big(\rho_{T^n}(z)\big) = \big|T^n\big|^r v_\wp(z) + v_\wp(\Delta_{T^n}). \tag{4}$$

Let $\mathfrak{n} = \sum_{i=0}^{n} \alpha_i T^i$ be a non-zero element of $\mathbb{A}$. It follows from (3) and (4) that

$$v_\wp\big(\rho_\mathfrak{n}(z)\big) = v_\wp\big(\rho_{\alpha_n T^n}(z)\big) = \big|T^n\big|^r v_\wp(z) + v_\wp(\Delta_{T^n}).$$

Note that $\Delta_\mathfrak{n} = \alpha_n \Delta_{T^n} = \alpha_n \Delta_T^{\gamma_\mathfrak{n}}$ by (1) and $|\mathfrak{n}| = |T^n|$, we have

$$\begin{aligned} v_\wp\big(\rho_\mathfrak{n}(z)\big) &= |\mathfrak{n}|^r v_\wp(z) + \frac{|\mathfrak{n}|^r - 1}{q^r - 1} v_\wp(\Delta_T) \\ &= |\mathfrak{n}|^r \left\{ v_\wp(z) + \frac{v_\wp(\Delta_T)}{q^r - 1} \right\} - \frac{v_\wp(\Delta_T)}{q^r - 1} \end{aligned}$$

as desired. Take $l_\infty = [\ell]$ if $\ell \notin \mathbb{Z}$; otherwise $l_\infty = \ell - 1$. Then it is clear that l_∞ has the property as claimed. $\square$

Remark 1. (a) It follows from the proof of Proposition 3.1 that the constant l_∞ depends on ρ and $\wp$ only and can be determined effectively. For a finite place $\wp$ of K, we let $\ell_{\infty,\wp}$ be the maximum of integers $l_\infty < -v_\wp(\Delta_T)/(q^r - 1)$ such that the identity (2) in Proposition 3.1 holds. Then, by rewriting (2) as

$$v_\wp\big(\rho_\mathfrak{n}(z)\big) = v_\wp(z) + \big(|\mathfrak{n}|^r - 1\big)\left\{ v_\wp(z) + \frac{v_\wp(\Delta_T)}{q^r - 1} \right\},$$

it follows that for non-zero $\mathfrak{n} \in \mathbb{A}$ we have $v_\wp(\rho_\mathfrak{n}(z)) \leqslant v_\wp(z) \leqslant \ell_{\infty,\wp}$ provided that $v_\wp(z) \leqslant \ell_{\infty,\wp}$.

(b) We will call a non-torsion point $z \in {}^\rho K_\wp$ having ($\wp$-adically) unbounded $\mathbb{A}$-orbit if the orbit $\{\rho_\mathfrak{n}(z) \mid \mathfrak{n} \in \mathbb{A}\}$ of z is unbounded with respect to $\wp$-adic topology; otherwise we say that it has bounded $\mathbb{A}$-orbit. As a consequence of Proposition 3.1, if z has unbounded $\mathbb{A}$-orbit then $v_\wp(\rho_\mathfrak{n}(z)) \leqslant \ell_{\wp,\infty}$ for some $\mathfrak{n} \in \mathbb{A}_+$.

Corollary 3.2. *Let $z \in {}^\rho K_\wp$ be a non-torsion point with unbounded $\mathbb{A}$-orbit. Then, there exists a constant $\delta_{\wp,z} > 0$ depending only on $\wp$ and z such that*

$$v_\wp\big(\rho_\mathfrak{n}(z)\big) = -\delta_{\wp,z}|\mathfrak{n}|^r + O(1)$$

for all non-zero $\mathfrak{n} \in \mathbb{A}$, where the implied constant in $O(1)$ is independent of $\mathfrak{n}$.

Proof. Since z has unbounded orbit we must have $v_\wp(\rho_\mathfrak{a}(z)) \leqslant \ell_{\infty,\wp}$ for some $\mathfrak{a} \in \mathbb{A}_+$. Let $\mathfrak{b}$ be such a polynomial for z whose degree is minimal. Let $\mathfrak{n} \in \mathbb{A}$ be given and let $\mathfrak{c}, \mathfrak{r} \in \mathbb{A}$ be such that $\mathfrak{n} = \mathfrak{c}\mathfrak{b} + \mathfrak{r}$ with $0 \leqslant \deg(\mathfrak{r}) < \deg(\mathfrak{b})$ or $\mathfrak{r} = 0$.

Notice that if $\mathfrak{c} \neq 0$ then $v_\wp(\rho_{\mathfrak{c}\mathfrak{b}}(z)) = v_\wp(\rho_\mathfrak{c}(\rho_\mathfrak{b}(z))) \leqslant \ell_{\infty,\wp}$ by Remark 1(a) and if $\mathfrak{r} \neq 0$ then $v_\wp(\rho_\mathfrak{r}(z)) > \ell_{\infty,\wp}$ by the choice of $\mathfrak{b}$. Therefore, if $\deg(\mathfrak{n}) \geqslant \deg(\mathfrak{b})$ then

$$\begin{aligned}
v_\wp\bigl(\rho_{\mathfrak{n}}(z)\bigr) &= v_\wp\bigl(\rho_{\mathfrak{c}\mathfrak{b}}(z)\bigr)\\
&= |\mathfrak{c}|^r\left\{v_\wp\bigl(\rho_{\mathfrak{b}}(z)\bigr)+\frac{v_\wp(\Delta_T)}{q^r-1}\right\}-\frac{v_\wp(\Delta_T)}{q^r-1}\quad \text{by Proposition 3.1}\\
&= -\delta_{\wp,z}|\mathfrak{n}|^r-\frac{v_\wp(\Delta_T)}{q^r-1}\quad \text{where } \delta_{\wp,z}=\frac{-1}{|\mathfrak{b}|^r}\left\{v_\wp\bigl(\rho_{\mathfrak{b}}(z)\bigr)+\frac{v_\wp(\Delta_T)}{q^r-1}\right\}>0.
\end{aligned}$$

On the other hand, for non-zero $\mathfrak{r}\in\mathbb{A}$ such that $0\leqslant\deg(\mathfrak{r})<\deg(\mathfrak{b})$, we may write

$$v_\wp\bigl(\rho_{\mathfrak{r}}(z)\bigr)=-\delta_{\wp,z}|\mathfrak{r}|^r+\bigl\{v_\wp\bigl(\rho_{\mathfrak{r}}(z)\bigr)+\delta_{\wp,z}|\mathfrak{r}|^r\bigr\}.$$

Note that there are only finitely many $\mathfrak{r}$ such that $0\leqslant\deg(\mathfrak{r})<\deg(\mathfrak{b})$. Thus we may conclude that $|v_\wp(\rho_{\mathfrak{n}}(z))+\delta_{\wp,z}|\mathfrak{n}|^r|$ is bounded above by a constant independent of non-zero $\mathfrak{n}\in\mathbb{A}$. This completes the proof. □

Now we describe the subset of points of $K_\wp$ whose orbits under the $\mathbb{A}$-action remain ($\wp$-adically) bounded. Notice that if K is of generic $\mathbb{A}$-characteristic, then $K_\wp$ is also of generic $\mathbb{A}$-characteristic. In our discussion below, if K is of generic characteristic then we will identify $\mathbb{A}$ with its image in $K_\wp$ under the structural homomorphism $\iota:\mathbb{A}\to K_\wp$. If K is of finite characteristic $\mathfrak{p}$, then we have $h_0=0$ and in fact

$$\rho_{\mathfrak{p}}=h_{fd}\tau^{fd}+\cdots+h_{rd}\tau^{rd},\quad h_{fd}\neq 0,$$

for some positive integer $f\leqslant r$. To ease the notation, we set $H_\rho:=h_{fd}$ for the case of finite $\mathbb{A}$-characteristic.

Proposition 3.3. *There exists an integer l_0 depending only on ρ and $\wp$ so that for $z\in K_\wp$ with $v_\wp(z)\geqslant l_0$ and for all non-zero $\mathfrak{n}\in\mathbb{A}$ the following hold*:

(a) *if $K_\wp$ is of generic characteristic then*

$$v_\wp\bigl(\rho_{\mathfrak{n}}(z)\bigr)=v_\wp(z)+v_\wp(\mathfrak{n});$$

(b) *if $K_\wp$ is of finite characteristic $\mathfrak{p}$ then*

$$v_\wp\bigl(\rho_{\mathfrak{n}}(z)\bigr)=\bigl|\mathfrak{p}^f\bigr|^{\mathrm{ord}_{\mathfrak{p}}(\mathfrak{n})}v_\wp(z)+\left\{\frac{|\mathfrak{p}^f|^{\mathrm{ord}_{\mathfrak{p}}(\mathfrak{n})}-1}{|\mathfrak{p}^f|-1}\right\}v_\wp(H_\rho).$$

Proof. (a) Assume that $K_\wp$ is of generic characteristic. Set

$$\ell=\max\left\{\frac{v_\wp(g_i)}{1-q^i},\frac{v_\wp(h_j)-v_\wp(h_0)}{1-q^j}\ \middle|\ g_i\neq 0,\ h_j\neq 0,\ 1\leqslant i\leqslant r,\ 1\leqslant j\leqslant rd\right\}.$$

For a positive integer n, we will write $\rho_{\mathfrak{p}^n}(z)=\mathfrak{p}^n z+g_n(z)$ and $\rho_{T^n}(z)=T^n z+f_n(z)$. Note that $f_n(z)$, $g_n(z)$ are $\mathbb{F}_q$-linear polynomials in z with coefficients in $K_\wp$. Let $z\in K_\wp$ satisfying $v_\wp(z)>\ell$. Then, it follows by induction that

$$v_\wp\big(T^n z\big) < v_\wp\big(f_n(z)\big) \quad \text{and} \tag{5}$$
$$v_\wp\big(\mathfrak{p}^n z\big) < v_\wp\big(g_n(z)\big). \tag{6}$$

Consequently, the strong triangle inequality yields

$$v_\wp\big(\rho_{T^n}(z)\big) = v_\wp\big(T^n z\big) \quad \text{and} \tag{7}$$
$$v_\wp\big(\rho_{\mathfrak{p}^n}(z)\big) = v_\wp\big(\mathfrak{p}^n z\big). \tag{8}$$

Let $N = \{z \in K_\wp \mid v_\wp(z) > \ell\}$. Then, ρ_T gives rise to a self map of N by (7). As N is a $\mathbb{F}_q$-vector space and $\rho_\mathfrak{a}(\mathfrak{a} \in \mathbb{A})$ are $\mathbb{F}_q$-linear maps, N is stable under the $\mathbb{A}$-action via ρ. Let $z \in N$ and $\mathfrak{a} = \sum_{i=0}^n \alpha_i T^i$, $\alpha_i \in \mathbb{F}_q$. Then, $\rho_\mathfrak{a}(z) = \mathfrak{a}z + \sum_{i=0}^d \alpha_i f_i(z)$. If $v_\wp(\mathfrak{a}) = 0$, applying (5) we see that $v_\wp(\rho_\mathfrak{a}(z)) = v_\wp(\mathfrak{a}z) = v_\wp(z)$. For any non-zero $\mathfrak{n} \in \mathbb{A}$, write $\mathfrak{n} = \mathfrak{p}^m\mathfrak{n}'$ where $m = \operatorname{ord}_\mathfrak{p}(\mathfrak{n})$ and $\mathfrak{p} \nmid \mathfrak{n}'$. Since $\rho_\mathfrak{n}(z) = \rho_{\mathfrak{p}^m}(\rho_{\mathfrak{n}'}(z))$, we see that $v_\wp(\rho_\mathfrak{n}(z)) = v_\wp(\mathfrak{p}^m \rho_{\mathfrak{n}'}(z))$ by (8). Therefore, $v_\wp(\rho_\mathfrak{n}(z)) = v_\wp(\rho_{\mathfrak{n}'}(z)) + v_\wp(\mathfrak{p}^m) = v_\wp(z) + v_\wp(\mathfrak{n})$ and (a) is proved.

(b) Assume that $K_\wp$ is of characteristic $\mathfrak{p}$ and set

$$\ell' = \max\left\{\frac{v_\wp(g_i)}{1-q^i}, \frac{v_\wp(h_j) - v_\wp(H_\rho)}{|\mathfrak{p}^f| - q^j} \;\middle|\; g_i \neq 0,\ h_j \neq 0,\ 0 < i \leqslant r,\ fd < j \leqslant rd\right\}.$$

Notice that if $v_\wp(z) > \ell'$ then by the same arguments as in (a), we have

$$v_\wp\big(\rho_{T^n}(z)\big) = v_\wp\big(g_0^n z\big) \quad \text{and} \quad v_\wp\big(\rho_{\mathfrak{p}^n}(z)\big) = v_\wp\big(H_\rho^{u(n)} z^{|\mathfrak{p}^f|^n}\big)$$

where $u(n) = (|\mathfrak{p}^f|^n - 1)/(|\mathfrak{p}^f| - 1)$. As in (a), we let $N = \{z \in K_\wp \mid v_\wp(z) > \ell'\}$. Then N is stable under the $\mathbb{A}$-action and $v_\wp(\rho_\mathfrak{a}(z)) = v_\wp(z)$ for $\mathfrak{a}$ relatively prime to $\mathfrak{p}$. Let $\mathfrak{n} \in \mathbb{A}$ be given and let write $\mathfrak{n} = \mathfrak{p}^m\mathfrak{n}'$ where $m = \operatorname{ord}_\mathfrak{p}(\mathfrak{n})$. Then,

$$\begin{aligned}
v_\wp\big(\rho_\mathfrak{n}(z)\big) &= v_\wp\big(\rho_{\mathfrak{p}^m}\big(\rho_{\mathfrak{n}'}(z)\big)\big) \\
&= v_\wp\big(H_\rho^{u(m)}\big(\rho_{\mathfrak{n}'}(z)\big)^{|\mathfrak{p}^f|^m}\big) \\
&= \big|\mathfrak{p}^f\big|^m v_\wp(z) + \left\{\frac{|\mathfrak{p}^f|^m - 1}{|\mathfrak{p}^f| - 1}\right\} v_\wp(H_\rho).
\end{aligned}$$

This proves (b). □

We let $\ell_{0,\wp}$ to be the smallest integer so that Proposition 3.3 holds. We will need to be able to determine a lower bound of $\ell_{0,\wp}$. As a consequence of Proposition 3.3, if ρ is defined over $\mathcal{O}_\wp$ then we have the following results.

Corollary 3.4. *Assume that ρ is defined over $\mathcal{O}_\wp$.*

(i) *In the case where $K_\wp$ is of generic $\mathbb{A}$-characteristic, if $v_\wp(z) > [v_\wp(\mathfrak{p})/(|\mathfrak{p}| - 1)]$ then*

$$v_\wp\big(\rho_\mathfrak{n}(z)\big) = v_\wp(z) + v_\wp(\mathfrak{n}).$$

(ii) *In the case where* $K_\wp$ *is of finite* $\mathbb{A}$*-characteristic* $\mathfrak{p}$*, if* $v_\wp(H_\rho) = 0$ *and if* $v_\wp(z) > 0$ *then*

$$v_\wp\bigl(\rho_{\mathfrak{n}}(z)\bigr) = \bigl|\mathfrak{p}^f\bigr|^{\mathrm{ord}_{\mathfrak{p}}(\mathfrak{n})} v_\wp(z).$$

Proof. (i) By the commutative relation $\rho_{\mathfrak{p}}\rho_T = \rho_T\rho_{\mathfrak{p}}$ we have that

$$h_l\bigl(g_0^{q^l} - g_0\bigr) = \sum_{i+j=l,\, j<l} g_i h_j^{q^i} - \sum_{i+j=l,\, j<l} h_j g_i^{q^j} \tag{9}$$

where by our convention $g_0 = T$. On the other hand, we note that by assumption we have $v_\wp(g_i) \geqslant 0$. Therefore, $v_\wp(g_i)/(1-q^i) \leqslant 0$. Let ξ be the image of T in $\mathbb{F}_{\mathfrak{p}} = \mathbb{A}/(\mathfrak{p})$. Then ξ generates the finite field $\mathbb{F}_{\mathfrak{p}}$ over $\mathbb{F}_q$. It follows that $\xi, \xi^q, \ldots, \xi^{q^{d-1}}$ are distinct conjugates of ξ over $\mathbb{F}_q$. Therefore, $\mathfrak{p} \nmid (T^{q^i} - T)$ for $i = 1, \ldots, d-1$. As $v_\wp$ is an extension of $\mathfrak{p}$-adic valuation of $k_{\mathfrak{p}} \subset K_\wp$ we see that $v_\wp(T^{q^i} - T) = 0$ for $i = 1, \ldots, d-1$. It follows by induction that

$$v_\wp(h_i) \geqslant v_\wp(h_0) = v_\wp(\mathfrak{p}) \quad \text{for } i = 0, \ldots, d-1.$$

Now, for $i = 1, \ldots, d-1$, we obviously have $v_\wp(\mathfrak{p})/(q^d - 1) \geqslant 0 \geqslant (v_\wp(\mathfrak{p}) - v_\wp(h_i))/(q^i - 1)$. For the case where $i \geqslant d$,

$$\frac{v(\mathfrak{p})}{q^d - 1} \geqslant \frac{v(\mathfrak{p})}{q^i - 1} \geqslant \frac{v(\mathfrak{p}) - v(h_i)}{q^i - 1}.$$

Hence, $v_\wp(\mathfrak{p})/(q^d - 1) \geqslant \ell'$ where ℓ' is as defined in the proof of Proposition 3.3(a). As $v_\wp(z) > [v_\wp(\mathfrak{p})/(q^d - 1)]$ we have

$$v_\wp(z) \geqslant \bigl[v_\wp(\mathfrak{p})/\bigl(q^d - 1\bigr)\bigr] + 1 > v_\wp(\mathfrak{p})/\bigl(q^d - 1\bigr) \geqslant \ell'.$$

Now (i) follows by applying Proposition 3.3(a).

(ii) In the finite characteristic case, we note that under the assumption $v_\wp(H_\rho) = 0$ the constant $\ell' < 0$ in the proof of Proposition 3.3(b). Now (ii) is just a special case of Proposition 3.3(b). □

Remark 2. (a) Let $\mathcal{N}_{0,\wp} = \{x \in K_\wp \mid v_\wp(x) \geqslant \ell_{0,\wp}\}$ and let $\mathcal{N}_{\infty,\wp} = \{x \in K_\wp \mid v_\wp(x) \leqslant \ell_{\infty,\wp}\}$. Then, Propositions 3.1 and 3.3 say that the neighborhood $\mathcal{N}_{0,\wp}$ of 0 and the neighborhood $\mathcal{N}_{\infty,\wp}$ of the point at "infinity" are both invariant under the $\mathbb{A}$-action.

(b) Assume that ρ has good reduction at $\wp$. Then it is not hard to see that $\ell_{\infty,\wp} = -1$. If, furthermore, $\deg(\mathfrak{p})$ is large enough so that $|\mathfrak{p}| > v_\wp(\mathfrak{p}) + 1$, then $\ell_{0,\wp} = 1$.

Corollary 3.5. *Let* $z \in K_\wp$ *be a non-torsion point of* ρ*. Let* $\mathfrak{n}$ *be any non-zero element of* $\mathbb{A}$*.*

(i) *Suppose that* $K_\wp$ *is of generic* $\mathbb{A}$*-characteristic. Then there exists a constant* $c_\wp$ *depending only on* ρ*,* $\wp$ *and* z *such that*

$$v_\wp\bigl(\rho_{\mathfrak{n}}(z)\bigr) \leqslant v_\wp(z) + v_\wp(\mathfrak{n}) + c_\wp.$$

(ii) *Suppose that $K_\wp$ is of finite $\mathbb{A}$-characteristic $\mathfrak{p}$. Then there exits a non-zero $\mathfrak{b} \in \mathbb{A}_+$ such that*

$$v_\wp\big(\rho_\mathfrak{n}(z)\big) \leqslant \big|\mathfrak{p}^f\big|^{\operatorname{ord}_\mathfrak{p}(\mathfrak{n}/\mathfrak{b})} v_\wp\big(\rho_\mathfrak{b}(z)\big) + \left\{\frac{|\mathfrak{p}^f|^{\operatorname{ord}_\mathfrak{p}(\mathfrak{n}/\mathfrak{b})} - 1}{|\mathfrak{p}^f| - 1}\right\} v_\wp(H_\rho)$$

where by convention $\operatorname{ord}_\mathfrak{p}(\mathfrak{n}/\mathfrak{b}) = 0$ if $\mathfrak{n}$ is not divisible by $\mathfrak{b}$.

Proof. We first claim that for each $z \in K_\wp$ there exist an $\mathfrak{a} \in \mathbb{A}_+$ such that either $v_\wp(\rho_\mathfrak{a}(z)) \leqslant \ell_{\infty,\wp}$ or $v_\wp(\rho_\mathfrak{a}(z)) \geqslant \ell_{0,\wp}$. Notice that the claim is obviously true if already $v_\wp(z) \leqslant \ell_{\infty,\wp}$ or $v_\wp(z) \geqslant \ell_{0,\wp}$. Let us assume that $v_\wp(z) \geqslant \ell_{\infty,\wp} + 1$ and suppose that there is no $\mathfrak{n} \in \mathbb{A}$ such that $v_\wp(\rho_\mathfrak{n}(z)) \leqslant \ell_{\infty,\wp}$. Let $N_0 := \{x \in K_\wp \mid v_\wp(x) \geqslant \ell_{\infty,\wp} + 1\}$. Then N_0 is a compact neighborhood of 0 with respect to the $\wp$-adic topology. It follows that N_0 is covered by finitely many disks of the form $D(\alpha) = \{x \in K_\wp \mid v_\wp(x - \alpha) \geqslant \ell_{0,\wp}\}$ with $\alpha \in N_0$. By the assumption that $\rho_\mathfrak{n}(z) \in N_0$ for all $\mathfrak{n} \in \mathbb{A}$ and the fact that z is a non-torsion point, we see that its $\mathbb{A}$-orbit $\{\rho_\mathfrak{n}(z) \mid \mathfrak{n} \in \mathbb{A}\}$ is an infinite subset of N_0. Therefore there exist distinct $\mathfrak{n}, \mathfrak{n}' \in \mathbb{A}_+$ such that $v_\wp(\rho_\mathfrak{n}(z) - \rho_{\mathfrak{n}'}(z)) \geqslant \ell_{0,\wp}$. Thus, $v_\wp(\rho_{\mathfrak{n}-\mathfrak{n}'}(z)) = v_\wp(\rho_\mathfrak{n}(z) - \rho_{\mathfrak{n}'}(z)) \geqslant \ell_{0,\wp}$ and our claim is proved by taking $\mathfrak{a}$ to be $\mathfrak{n} - \mathfrak{n}'$ divided by its leading coefficient.

Our claim above implies that for each $z \in K_\wp$, there exists a $\mathfrak{b}_z \in \mathbb{A}_+$ with minimal degree such that either $v_\wp(\rho_{\mathfrak{b}_z}(z)) \leqslant \ell_{\infty,\wp}$ or $v_\wp(\rho_{\mathfrak{b}_z}(z)) \geqslant \ell_{0,\wp}$. Let $z \in K_\wp$ be a given non-torsion point and denote by $\mathfrak{b} = \mathfrak{b}_z$ if no confusion will arise. Let $\mathfrak{n} \in \mathbb{A}$ be a given non-zero element and write $\mathfrak{n} = \mathfrak{c}\mathfrak{b} + \mathfrak{r}$, for unique $\mathfrak{c}, \mathfrak{r} \in \mathbb{A}$ such that $0 \leqslant \deg(\mathfrak{r}) < \deg(\mathfrak{b})$ or $\mathfrak{r} = 0$.

(i) (*$K_\wp$ is of generic characteristic.*) Let

$$c_\wp = \max\Big(\Big\{\big|v_\wp\big(\rho_\mathfrak{r}(z)\big) - v_\wp(z)\big| \;\Big|\; \deg(\mathfrak{r}) < \deg(\mathfrak{b}),\ \mathfrak{r} \neq 0\Big\} \cup \Big\{\big|v_\wp\big(\rho_\mathfrak{b}(z)\big) - v_\wp(z)\big|\Big\}\Big).$$

Let us first treat the case where z has unbounded $\mathbb{A}$-orbit. Then, $v_\wp(\rho_\mathfrak{b}(z)) \leqslant \ell_{\infty,\wp}$. Now $v_\wp(\rho_\mathfrak{n}(z)) = v_\wp(\rho_{\mathfrak{c}\mathfrak{b}}(z) + \rho_\mathfrak{r}(z))$. Applying Proposition 3.1 if $\mathfrak{c} \neq 0$, it is not hard to deduce that $v_\wp(\rho_\mathfrak{n}(z)) - v_\wp(z) \leqslant c_\wp$ in this case.

Next, we assume that z has bounded $\mathbb{A}$-orbit. Then $v_\wp(\rho_\mathfrak{b}(z)) \geqslant \ell_{0,\wp}$. Since $v_\wp(\rho_{\mathfrak{c}\mathfrak{b}}(z)) \geqslant \ell_{0,\wp} > v_\wp(\rho_\mathfrak{r}(z))$, we have

$$v_\wp\big(\rho_\mathfrak{n}(z)\big) = \rho_\mathfrak{r}(z) < \ell_{0,\wp} \leqslant v_\wp\big(\rho_\mathfrak{b}(z)\big) \quad \text{if } \mathfrak{r} \neq 0;$$

otherwise, if $\mathfrak{r} = 0$ then by Proposition 3.3(a), we have

$$\begin{aligned} v_\wp\big(\rho_\mathfrak{n}(z)\big) &= v_\wp\big(\rho_\mathfrak{b}(z)\big) + v_\wp(\mathfrak{c}) \\ &\leqslant v_\wp\big(\rho_\mathfrak{b}(z)\big) + v_\wp(\mathfrak{n}). \end{aligned}$$

The last inequality follows from the fact that $v_\wp(\mathfrak{n}) \geqslant 0$ for any non-zero $\mathfrak{n} \in \mathbb{A}$. In both cases, we see that $v_\wp(\rho_\mathfrak{n}(z)) - v_\wp(z) \leqslant v_\wp(\mathfrak{n}) + c_\wp$ as desired.

(ii) (*$K_\wp$ is of finite characteristic $\mathfrak{p}$.*) Let $\mathfrak{b} = \mathfrak{b}_z$ then either $v_\wp(\rho_\mathfrak{b}(z)) \geqslant \ell_{0,\wp}$ or $v_\wp(\rho_\mathfrak{b}(z)) \leqslant \ell_{\infty,\wp}$. Thus, the proof is reduced to the case where $\mathfrak{b} = 1$ together with the condition that $v_\wp(z) \geqslant \ell_{0,\wp}$ or $v_\wp(z) \leqslant \ell_{\infty,\wp}$.

If $v_\wp(z) \geqslant \ell_{0,\wp}$ then the assertion follows from Proposition 3.3(b). If $v_\wp(z) \leqslant \ell_{\infty,\wp}$ then we observe that $v_\wp(\rho_\mathfrak{n}(z)) = v_\wp(\Delta_\mathfrak{n} z^{|\mathfrak{n}|^r}) \leqslant v_\wp(z)$ by Remark 1(a). Let $m = \operatorname{ord}_\mathfrak{p}(\mathfrak{n})$ and write

$\mathfrak{n} = \mathfrak{p}^m\mathfrak{n}'$. Since $v_\wp(\rho_\mathfrak{n}(z)) \leqslant v_\wp(z)$ it follows that (ii) is true if $m = 0$. Put $y = \rho_{\mathfrak{n}'}(z)$ and note that

$$\rho_{\mathfrak{p}^m}(y) = H_\rho^{u(m)} y^{|\mathfrak{p}^m|^f} + \cdots + \Delta_{\mathfrak{p}^m} y^{|\mathfrak{p}^m|^r}$$

where $u(m) = (|\mathfrak{p}^f|^m - 1)/(|\mathfrak{p}^f| - 1)$. From the proof of Proposition 3.3(b), we see that

$$v_\wp(\rho_{\mathfrak{p}^m} y) = v_\wp\big(\Delta_{\mathfrak{p}^m} y^{|\mathfrak{p}^m|^r}\big) \leqslant v_\wp\big(H_\rho^{u(m)} y^{|\mathfrak{p}^m|^f}\big).$$

The proof is now completed simply by observing that $v_\wp(y) \leqslant v_\wp(z)$ by Remark 1(a). □

Remark 3. (a) Assume that ρ has good reduction at $\wp$. If for $z \in K_\wp$ with $v_\wp(z) = 0$ then $\mathfrak{b}_z \mid L_\wp(1)$ as $L_\wp(1)$ annihilates the finite $\mathbb{A}$-module $\mathbb{F}_\wp$ by Proposition 2.1(b).

(b) Regarding ρ_T as a polynomial map over $K_\wp$ on $\mathbb{P}^1/K_\wp$, one can ask if ρ_T extends to a morphism over $\mathcal{O}_\wp$ on $\mathbb{P}^1/\mathcal{O}_\wp$. This is the case if ρ has good reduction at $\wp$. If ρ does not have good reduction at $\wp$ but if $\wp$ is a finite place of K then a modification of the proof of Propositions 3.1 and 3.3 can show that there exists a *weak Neron model* [11] X for ρ_T.

(c) By definition we have $\mathfrak{b}_z = 1$ if $v_\wp(z) \leqslant \ell_{\infty,\wp}$ or $v_\wp(z) \geqslant \ell_{0,\wp}$. In fact, the set $\{\mathfrak{b}_z \mid z \in K_\wp\}$ is finite. To see this, we note by Proposition 3.3 that $\mathfrak{b}_z$ depends only on the disk $D(z) = z + \mathcal{N}_{0,\wp}$ and is independent of the choice of the center. As there are only finitely many disks of this type in

$$U = \big\{x \in K_\wp \;\big|\; \ell_{\infty,\wp} + 1 \leqslant v_\wp(x) \leqslant \ell_{0,\wp} - 1\big\}$$

and $\mathfrak{b}_z = 1$ for $z \notin U$. It follows that $\{\mathfrak{b}_z \mid z \in K_\wp\}$ is a finite set.

4. $\mathbb{A}$-orbits over K_∞

In this section, we assume that K is of generic $\mathbb{A}$-characteristic and K is viewed as a finite extension of k. Let $D = [K : k]$ be the extension degree of K over k. We fix an infinite place of K. To simplify the notation, we also denote it by ∞ if no confusion arises. Then K_∞ will be the completion of K at ∞ and $\mathcal{O}_\infty$ denotes the ring of integers of K_∞. Similarly, v_∞ will be the normalized valuation of K_∞. Let e_∞ be the ramification index of K_∞ over $k_\infty = \mathbb{F}_q((1/T))$. Note that the degree function deg has a unique extension to K_∞ so that $v_\infty(\cdot) = -e_\infty \deg(\cdot)$. As one usually does for the infinite place of a function field, we will use deg instead of v_∞ as our valuation at the infinite place of K. Let $\mathbf{C}_\infty$ denote the completion of an algebraic closure of K_∞. The extension of the degree function on $\mathbf{C}_\infty$ will still be denoted by deg. For any $\alpha \in \mathbf{C}_\infty$, its absolute value is defined to be $|\alpha| := q^{\deg \alpha}$.

By the analytic uniformization theorem for Drinfeld modules [7], there exists an $\mathbb{A}$-lattice Λ_ρ over K_∞ and an $\mathbb{F}_q$-linear, ∞-adic entire function $e_\rho(z)$ associated to ρ such that $\rho_\mathfrak{n}(e_\rho(z)) = e_\rho(\mathfrak{n}z)$ for all $\mathfrak{n} \in \mathbb{A}$, where $e_\rho(z)$ is defined by

$$e_\rho(z) = z \prod_{0 \neq \lambda \in \Lambda_\rho} \left(1 - \frac{z}{\lambda}\right).$$

Note that, $\Lambda_\rho = \ker e_\rho$ is the lattice of periods of e_ρ.

4.1. Linear forms in Drinfeld logarithms

We briefly recall some results in [3] on lower bounds of linear forms in logarithms for Drinfeld modules. Following the notations in [3], we let ρ_0 denote *the trivial Drinfeld module* such that $\rho_{0,T}(x) = Tx$. Let $\rho_1, \dots, \rho_n$ denote Drinfeld modules of rank $d_1, \dots, d_n$ over K such that

$$\rho_{i,T} = a_{i,0}\tau^0 + \cdots + a_{i,d_i}\tau^{d_i} \quad \text{for } i = 1, \dots, n.$$

Let $G = (\mathbb{G}_a^{n+1}, \Phi)$ where $\Phi = \rho_0 \times \rho_1 \times \cdots \times \rho_n$ and call G a product of Drinfeld modules $\rho_0, \dots, \rho_n$. Thus, G becomes an $\mathbb{A}$-module via the diagonal $\mathbb{A}$-action. The map

$$\exp_G : \mathbf{C}_\infty^{n+1} \to \mathbf{C}_\infty^{n+1} \quad \text{defined by } \exp_G(z_0, z_1, \dots, z_n) = \bigl(z_0, e_{\rho_1}(z_1), \dots, e_{\rho_1}(z_n)\bigr)$$

will be called the exponential map associated to Φ. Let $\Lambda_1, \dots, \Lambda_n$ be the corresponding lattices of periods of $e_{\rho_1}, \dots, e_{\rho_n}$. Let λ_i be a period of e_{ρ_i} such that $\deg \lambda_i$ is minimal among non-zero periods of e_{ρ_i} for $i = 1, \dots, n$.

Let N be a nonnegative integer. For any point $\mathbf{x} = (x_0, x_1, \dots, x_N) \in \mathbb{P}^N(K)$, let $h(\mathbf{x})$ be its (logarithmic) height which is defined by the following formula

$$h(\mathbf{x}) = h(x_0, \dots, x_N) = \frac{1}{D} \sum_{\wp \in M_K} n_\wp \max\bigl\{-v_\wp(x_i);\ 0 \leqslant i \leqslant N\bigr\}$$

where $n_\wp = \dim_{\mathbb{F}_q} \mathbb{F}_\wp$ (Section 2). Moreover, the height of G is defined to be $h(G) := h(1, a_{1,0}, \dots, a_{n,d_i})$. The following is Theorem 1.1 of [3].

Theorem 4.1. *Let $G = (\mathbb{G}_a^{n+1}, \rho_0 \times \cdots \times \rho_n)$ be a product of $\mathbb{A}$-modules defined over K. Let $\mathcal{H}(\mathbf{x}) = \beta_0 x_0 + \cdots + \beta_n x_n$ be a non-zero linear form defined over K. Let $u_1, \dots, u_n$ be elements of $\mathbf{C}_\infty$ such that, for all $1 \leqslant i \leqslant n$, $e_{\rho_i}(u_i) \in K$. Set $\gamma_i = e_{\rho_i}(u_i)$, and $\delta = [K_\infty(u_1, \dots, u_n) : K_\infty]$ which is finite. Let $B, E, V_1, \dots, V_n$ and h be constants satisfying the following conditions*

$$\log B \geqslant \max\bigl\{e, h(\beta_1), \dots, h(\beta_n)\bigr\},$$
$$\log V_i \geqslant \max\left\{h(\gamma_i), \frac{|u_i|^{d_i}}{D|\lambda_i|^{d_i}}\right\}, \quad \log V_1 \geqslant \cdots \geqslant \log V_n \geqslant e,$$
$$h \geqslant h(G),$$
$$e \leqslant E \leqslant \min\left\{e(D \log V_i)^{1/d_i} \frac{|\lambda_i|}{|u_i|},\ 1 \leqslant i \leqslant n\right\}.$$

Put $d = d_1 + \cdots + d_n$ and $\mathbf{u} = (1, u_1, \dots, u_n) \in \mathbf{C}_\infty^{n+1}$. If $\mathcal{H}(\mathbf{u}) \neq 0$ then there exists a computable positive constant C depending only on q, n and d such that

$$\deg\bigl(\mathcal{H}(\mathbf{u})\bigr) \geqslant -C(Dh)^{d+2}\delta^{d+1}(\log B)\left(\prod_{1 \leqslant i \leqslant n} \log V_i\right)\bigl(\log^+ \delta\bigr)^{d+2}$$
$$\times \bigl(\log(Dh) + \log\log V_1\bigr)$$

$$\times \big(\log\log B + \log(Dh) + \log\log V_1\big)^{d+1-n}$$
$$\times (\log E)^{-(n-1)}$$

where $\log^+(x) = \max\{0, x\}$ *for real* x.

Let $\lambda_1, \dots, \lambda_r$ be a basis for the $\mathbb{A}$-lattice $\Lambda_\rho = \ker e_\rho$. By a successively minimum basis of Λ_ρ we mean a basis $\lambda_1, \dots, \lambda_r$ for the $\mathbb{A}$-lattice Λ_ρ such that $0 < \deg\lambda_1 \leqslant \cdots \leqslant \deg\lambda_r$ and their degrees are minimal with this property. We shall choose $\lambda_1, \dots, \lambda_r$ to be a successive minimum basis of Λ_ρ. Note that, $\lambda_1, \dots, \lambda_r$ may not be unique but their degrees are. The following is a corollary to Theorem 4.1.

Proposition 4.2. *Let* $\mathfrak{n}$ *be a non-constant element of* $\mathbb{A}$ *and let* $\mathfrak{a}_1, \dots, \mathfrak{a}_r$ *be any* r *element of* $\mathbb{A}$. *Let* $u \in \mathbf{C}_\infty$ *be such that* $e_\rho(u) = \mathbf{z} \in K$ *which is not a torsion point of the Drinfeld module* ρ. *Then, for any given* $\epsilon > 0$ *there exists a constant* $C_\epsilon > 0$ *depending only on* ρ *and* u *such that*

$$\deg(\mathfrak{a}_1\lambda_1 + \cdots + \mathfrak{a}_r\lambda_r - \mathfrak{n}u) \geqslant -C_\epsilon(\deg\mathfrak{n})^{1+\epsilon}.$$

Proof. Since $\mathbf{z}$ is not a torsion point of ρ, it follows that $\mathfrak{a}_1\lambda_1 + \cdots + \mathfrak{a}_r\lambda_r - \mathfrak{n}u \neq 0$. It suffices to prove the proposition in the case where

$$\deg(\mathfrak{a}_1\lambda_1 + \cdots + \mathfrak{a}_r\lambda_r - \mathfrak{n}u) < \deg(\mathfrak{n}u).$$

Note that in this case, we must have $\deg(\mathfrak{a}_1\lambda_1 + \cdots + \mathfrak{a}_r\lambda_r) = \deg(\mathfrak{n}u)$. On the other hand, as $\lambda_1, \dots, \lambda_r$ are successive minimum of Λ_ρ, we have [17, Lemma 4.2]

$$\deg(\mathfrak{a}_1\lambda_1 + \cdots + \mathfrak{a}_r\lambda_r) = \max\big\{\deg(\mathfrak{a}_1\lambda_1), \dots, \deg(\mathfrak{a}_r\lambda_r)\big\}.$$

Therefore, $\deg(\mathfrak{a}_i\lambda_i) \leqslant \deg(\mathfrak{n}) + \deg(u) \leqslant c_1\deg(\mathfrak{n})$ for some positive constant c_1 which may be chosen to depend on u only.

Let $G = (\mathbb{G}_a^{r+2}, \Phi)$ be the product of ρ_0 with $(r+1)$-copies of ρ. Let

$$\mathcal{H}(\mathbf{x}) = \mathfrak{a}_1x_1 + \cdots + \mathfrak{a}_rx_r - \mathfrak{n}x_{r+1}.$$

We regard $\mathcal{H}(\mathbf{x})$ as a linear form on $\mathbf{C}_\infty^{r+2}$. Let $\mathbf{u} = (1, \lambda_1, \dots, \lambda_r, u)$. Then, $\mathcal{H}(\mathbf{u})$ is non-zero. Let

$$\log B = \max\big\{e, c_1\deg(\mathfrak{n})\big\}, \tag{10}$$
$$\log V_{r+1} = \max\left\{e, h(\mathbf{z}), \frac{|u|^r}{D|\lambda_1|^r}\right\},$$
$$\log V_i = \max\left\{\frac{|\lambda_i|^r}{D|\lambda_1|^r}, \log V_{i+1}\right\}, \quad 1 \leqslant i \leqslant r,$$
$$h = \max\big\{\deg(g_0), \dots, \deg(g_r)\big\},$$
$$E = \min\left\{e(D\log V_i)^{1/r}\frac{|\lambda_1|}{|\lambda_i|},\ 1 \leqslant i \leqslant r,\ (e\log V_{r+1})^{1/r}\frac{|\lambda_1|}{|u|}\right\}.$$

It is not difficult to check that these constants satisfy the condition required in Theorem 4.1. By Theorem 4.1, $\deg(\mathcal{H}(\mathbf{u}))$ has a lower bound depending on $\deg(\lambda_i), \deg(u)$ and constants $B, E, V_1, \dots, V_{r+1}$. Note that all the above constants but $\log B$ are independent of $\deg(\mathfrak{n})$. Applying Theorem 4.1, we get

$$\deg\big(\mathcal{H}(\mathbf{u})\big) \geqslant -c_2(\log B)(\log\log B + c_3)^{r^2}$$

where in the expression, we use c_2 and c_3 to denote products of all the constants which are independent of $\deg \mathfrak{n}$. We choose the constant c_1 in (10) large enough so that $\log B = c_1 \deg(\mathfrak{n}) \geqslant e$ for all non-constant $\mathfrak{n} \in \mathbb{A}$. Let $\epsilon > 0$ be given. Then the quantity

$$\frac{c_1 c_2(\log(c_1 \deg(\mathfrak{n})) + c_3)^{r^2}}{\deg(\mathfrak{n})^{\epsilon}}$$

is bounded above as $\deg(\mathfrak{n})$ increases. Let c_ϵ be the least upper bound. Then,

$$\deg\big(\mathcal{H}(\mathbf{u})\big) \geqslant -c_\epsilon(\deg \mathfrak{n})^{1+\epsilon}.$$

Note that in the above inequality, c_ϵ is chosen to depend on c_1, c_2, c_3, ϵ and r. Thus, c_ϵ depends on ρ, u and ϵ only. This completes the proof of the proposition. □

4.2. *A lower bound for the degrees of an $\mathbb{A}$-orbit*

Since ρ is a Drinfeld module defined over K_∞, the lattice Λ_ρ is contained in a finite separable extension L of K_∞ [10, Theorem 4.6.9]. Note that $\Lambda_\rho \otimes_{\mathbb{A}} k$ as well as the torsion submodule $\rho_{\text{tor}} = e_\rho(\Lambda_\rho \otimes_{\mathbb{A}} k)$ are also contained in L [10, Remark 4.3.6].

Remark 4. (a) Since $|\rho'_{\mathfrak{a}}(z)| = |\mathfrak{a}| > 1$ for every non-constant $\mathfrak{a} \in \mathbb{A}$, it follows that every $\alpha \in \rho_{\text{tor}}$ is a repelling (pre)periodic point for the ∞-adic dynamical systems associated to the maps $\rho_{\mathfrak{a}}(X)$. The closure of ρ_{tor} in $\mathbf{C}_\infty$, denoted by J_ρ, is the Julia set associated to the ∞-adic dynamical systems of $\rho_{\mathfrak{a}}$ for $\mathfrak{a} \in \mathbb{A}$ (cf. [11,12]).

(b) Since ρ_{tor} is bounded and contained in L, it follows that J_ρ is contained in L and is a compact subset of $\mathbf{C}_\infty$ (with respect to the ∞-adic topology). Let $F_\rho = \mathbb{P}^1(\mathbf{C}_\infty) \setminus J_\rho$. Then, J_ρ as well as F_ρ are stable under the $\mathbb{A}$-action.

Let $\mathfrak{n} \in \mathbb{A}$ be given. Recall that

$$\rho_{\mathfrak{n}}(\mathbf{z}) = \rho_{\mathfrak{n}}\big(e_\rho(u)\big) = e_\rho(\mathfrak{n}u) = \mathfrak{n}u \prod_{0 \neq \lambda \in \Lambda_\rho} \left(1 - \frac{\mathfrak{n}u}{\lambda}\right).$$

Therefore, to estimate $\deg(\rho_{\mathfrak{n}}(\mathbf{z}))$ we need to count the elements of the $\mathbb{A}$-lattice Λ_ρ whose degree is bounded by $\deg(\mathfrak{n}u)$.

As Λ_ρ is discrete in $\mathbf{C}_\infty$, there are only finitely many lattice elements $\lambda \in \Lambda_\rho$ whose degrees are bounded above. To simplify our discussion below, we shall assume that $\deg(L^*) = \mathbb{Z}$ as it is not difficult to generalize our argument to the general situation. Put $\ell = \deg(\mathfrak{n}u)$. For any integer s, define the following two sets

$$\mathfrak{L}(s) = \{\lambda \in \Lambda_\rho \mid \deg(\lambda - \mathfrak{n}u) \leqslant \ell - s\},$$
$$\mathcal{L}(s) = \{\lambda \in \Lambda_\rho \mid \deg(\lambda) \leqslant \ell - s\}.$$

Clearly, we have $\mathfrak{L}(s) \supseteq \mathfrak{L}(s+1)$ and $\mathcal{L}(s) \supseteq \mathcal{L}(s+1)$ for all $s \in \mathbb{Z}$. Note that, the set $\mathfrak{L}(s)$ is empty if s is large enough. Moreover, if $\ell \notin \mathbb{Z}$ then $\mathfrak{L}(s)$ is empty for positive integers s. To see this, we note that since $\ell \notin \mathbb{Z}$ it follows $\deg(\lambda) \neq \deg(\mathfrak{n}u)$ for any $\lambda \in \Lambda_\rho$. We thus have either $\deg(\lambda - \mathfrak{n}u) = \deg(\lambda)$ if $\deg(\lambda) > \ell$ or $\deg(\lambda - \mathfrak{n}u) = \ell$ otherwise. Both cases will lead to the conclusion that $\mathfrak{L}(s)$ is empty. On the other hand, the set $\mathcal{L}(s)$ is a finite-dimensional vector space over $\mathbb{F}_q$.

Lemma 4.3. *If $\mathfrak{L}(s)$ is non-empty, then there exists a one-to-one correspondence between sets $\mathfrak{L}(s)$ and $\mathcal{L}(s)$.*

Proof. Let λ_0 be any fixed element of $\mathfrak{L}(s)$ which is non-empty by assumption. Let $f : \mathcal{L}(s) \to \mathfrak{L}(s)$ be defined by $f(\lambda) = \lambda_0 + \lambda$ for $\lambda \in \mathcal{L}(s)$. The map f is well defined since

$$\begin{aligned}\deg\big(f(\lambda) - \mathfrak{n}u\big) &= \deg(\lambda_0 - \mathfrak{n}u + \lambda)\\ &\leqslant \max\{\deg(\lambda_0 - \mathfrak{n}u), \deg(\lambda)\}\\ &\leqslant \ell - s.\end{aligned}$$

Clearly, f is one-to-one. We claim that f is surjective. To see this, let $\lambda' \in \mathfrak{L}(s)$ be given. Then,

$$\begin{aligned}\deg(\lambda' - \lambda_0) &\leqslant \max\{\deg(\lambda' - \mathfrak{n}u), \deg(\lambda_0 - \mathfrak{n}u)\}\\ &\leqslant \ell - s.\end{aligned}$$

It follows that $\lambda' - \lambda_0 \in \mathcal{L}(s)$. Thus $\lambda' = f(\lambda) = \lambda_0 + \lambda$ for some $\lambda \in \mathcal{L}(s)$. This completes the proof of the lemma. □

Let

$$\mathfrak{L}^0(s) = \mathfrak{L}(s) \setminus \mathfrak{L}(s+1) = \{\lambda \in \Lambda_\rho \mid \deg(\lambda - \mathfrak{n}u) = \ell - s\},$$
$$\mathcal{L}^0(s) = \mathcal{L}(s) \setminus \mathcal{L}(s+1) = \{\lambda \in \Lambda_\rho \mid \deg(\lambda) = \ell - s\}.$$

Let $|S|$ denote the cardinality of a finite set S. If $\mathfrak{L}(s+1)$ is non-empty then it follows from Lemma 4.3 that

$$|\mathfrak{L}^0(s)| = |\mathfrak{L}(s)| - |\mathfrak{L}(s+1)| = |\mathcal{L}(s)| - |\mathcal{L}(s+1)| = |\mathcal{L}^0(s)|.$$

Furthermore, if s is the smallest positive integer such that $\mathfrak{L}(s+1)$ is empty, then $|\mathfrak{L}^0(s)| = |\mathfrak{L}(s)| = |\mathcal{L}(s)|$. To ease the notations in our discussion below, we will let $\Lambda'_\rho = \Lambda_\rho \setminus \{0\}$.

Proposition 4.4. *Let $\mathfrak{n} \in \mathbb{A}$ be a non-zero element. Let $u \in \mathbf{C}_\infty$ be such that $e_\rho(u) = \mathbf{z}$ which is not a torsion of ρ. Let $\lambda_0 \in \Lambda'_\rho$ be such that* $\deg(\mathfrak{n}u - \lambda_0) = \min\{\deg(\mathfrak{n}u - \lambda) \mid \lambda \in \Lambda'_\rho\}$. *Then,*

$$\deg \rho_\mathfrak{n}(\mathbf{z}) \geqslant \deg(\mathfrak{n}u) + \deg\left(1 - \frac{\mathfrak{n}u}{\lambda_0}\right).$$

Proof. Note that $\rho_\mathfrak{n}(\mathbf{z}) = \mathfrak{n}u \prod_{\lambda \in \Lambda'_\rho}(1 - \frac{\mathfrak{n}u}{\lambda})$. We split the product according to the two cases: either $\deg(\lambda) > \deg(\mathfrak{n}u)$ or $\deg(\lambda) \leqslant \deg(\mathfrak{n}u)$.

If $\deg(\lambda) > \deg(\mathfrak{n}u)$, then $\deg(1 - \frac{\mathfrak{n}u}{\lambda}) = 0$ which does not contribute anything to the total degree.

If $\deg(\lambda) \leqslant \deg(\mathfrak{n}u)$, then

$$\prod_{\deg(\lambda) \leqslant \deg(\mathfrak{n}u)} \left(1 - \frac{\mathfrak{n}u}{\lambda}\right) = \prod_{\deg(\lambda) < \deg(\mathfrak{n}u)} \left(1 - \frac{\mathfrak{n}u}{\lambda}\right) \prod_{\deg(\lambda) = \deg(\mathfrak{n}u)} \left(1 - \frac{\mathfrak{n}u}{\lambda}\right).$$

If $\deg(\mathfrak{n}u) \notin \mathbb{Z} = \deg(L^*)$, then the case $\deg(\lambda) = \deg(\mathfrak{n}u)$ does not exist. The assertion of the proposition can be verified easily. On the other hand, suppose $\deg(\lambda) = \deg(\mathfrak{n}u)$ but λ does not have cancellation with $\mathfrak{n}u$. Then $\deg(1 - \frac{\mathfrak{n}u}{\lambda}) = 0$. Therefore, we shall assume that $\deg(\mathfrak{n}u) = \ell \in \mathbb{Z}$ and there is some positive integer s such that $\mathfrak{L}(s)$ is non-empty. This is equivalent to saying that there is some λ such that $\deg(1 - \frac{\mathfrak{n}u}{\lambda}) < 0$. Let us fix such a positive integer s.

As we remark above, we have either $|\mathfrak{L}^0(s)| = |\mathcal{L}^0(s)|$ when $\mathfrak{L}(s+1)$ is non-empty or $|\mathfrak{L}^0(s)| = |\mathcal{L}(s)|$ if $\mathfrak{L}(s+1)$ is empty. If $|\mathfrak{L}^0(s)| = |\mathcal{L}^0(s)|$ then $\deg(1 - \frac{\mathfrak{n}u}{\lambda}) = -s$ for $\lambda \in \mathfrak{L}^0(s)$ and $\deg(1 - \frac{\mathfrak{n}u}{\lambda}) = s$ for $\lambda \in \mathcal{L}^0(s)$. Hence,

$$\deg\left(\prod_{\lambda \in \mathcal{L}^0(s)} \left(1 - \frac{\mathfrak{n}u}{\lambda}\right) \prod_{\lambda \in \mathfrak{L}^0(s)} \left(1 - \frac{\mathfrak{n}u}{\lambda}\right)\right) = 0.$$

Suppose $|\mathfrak{L}^0(s)| = |\mathcal{L}(s)|$, hence $\mathfrak{L}(s+1)$ is empty and any $\lambda_0 \in \mathfrak{L}(s)$ satisfies $\deg(\mathfrak{n}u - \lambda_0) = \min\{\deg(\mathfrak{n}u - \lambda) \mid \lambda \in \Lambda'_\rho\}$. We need to estimate

$$\deg\left(\prod_{0 \neq \lambda \in \mathcal{L}(s)} \left(1 - \frac{\mathfrak{n}u}{\lambda}\right) \prod_{\lambda \in \mathfrak{L}^0(s)} \left(1 - \frac{\mathfrak{n}u}{\lambda}\right)\right).$$

We fix an element $\lambda_0 \in \mathfrak{L}^0(s)$ and write the above product as

$$\left(\prod_{0 \neq \lambda \in \mathcal{L}(s)} \left(1 - \frac{\mathfrak{n}u}{\lambda}\right) \prod_{\lambda_0 \neq \lambda \in \mathfrak{L}^0(s)} \left(1 - \frac{\mathfrak{n}u}{\lambda}\right)\right)\left(1 - \frac{\mathfrak{n}u}{\lambda_0}\right).$$

Note that $\deg(1 - \frac{\mathfrak{n}u}{\lambda}) \geqslant s$ for $\lambda \in \mathcal{L}(s) \setminus \{0\}$ provided that $\mathcal{L}(s)$ has more than one element. In any case, we always have

$$\deg\left(\prod_{0 \neq \lambda \in L(s)} \left(1 - \frac{\mathfrak{n}u}{\lambda}\right) \prod_{\lambda \in \mathfrak{L}^0(s)} \left(1 - \frac{\mathfrak{n}u}{\lambda}\right)\right) \geqslant \deg\left(1 - \frac{\mathfrak{n}u}{\lambda_0}\right).$$

Combining all the computations above, we conclude that

$$\begin{aligned}\deg \rho_{\mathfrak{n}}(\mathbf{z}) &= \deg\left(\mathfrak{n}u \prod_{0\neq\lambda\in\Lambda_\rho}\left(1-\frac{\mathfrak{n}u}{\lambda}\right)\right)\\ &\geqslant \deg(\mathfrak{n}u) + \deg\left(1-\frac{\mathfrak{n}u}{\lambda_0}\right).\end{aligned}$$

This complete the proof of the proposition. □

As J_ρ and F_ρ are stable under the $\mathbb{A}$-action given by ρ (see Remark 4), if $z \notin J_\rho$, then its $\mathbb{A}$-orbit will stay away from J_ρ. In fact, as a result of Proposition 4.4, its $\mathbb{A}$-orbit is ∞-adically unbounded.

Corollary 4.5. *Let $z \in \mathbf{C}_\infty \setminus J_\rho$. Then there exist a positive constant $\delta_{\infty,z}$ depending only on ρ and z such that*

$$\deg\bigl(\rho_{\mathfrak{n}}(z)\bigr) = \delta_{\infty,z}|\mathfrak{n}|^r + O(1)$$

for non-zero $\mathfrak{n} \in \mathbb{A}$, where the constant in $O(1)$ also depends on ρ and z only.

Proof. Note that J_ρ is compact and hence is bounded with respect to the ∞-adic topology. Therefore

$$s = \sup\bigl\{\deg(\zeta) \bigm| \zeta \in J_\rho\bigr\}$$

exists. By examining the Newton polygon of $\rho_T(X)$, it follows that $s \geqslant -\deg(\Delta_T)/(q^r-1)$. Let $z \in \mathbf{C}_\infty \setminus J_\rho$ be given. Suppose that $\deg(z) > s$, then

$$\begin{aligned}\deg\bigl(\rho_{\mathfrak{n}}(z)\bigr) &= \deg\left(\Delta_{\mathfrak{n}} \prod_{\zeta\in\rho[\mathfrak{n}]}(z-\zeta)\right)\\ &= |\mathfrak{n}|^r \deg(z) + \deg(\Delta_{\mathfrak{n}})\\ &= |\mathfrak{n}|^r\left\{\deg(z) + \frac{\deg(\Delta_T)}{q^r-1}\right\} - \frac{\deg(\Delta_T)}{q^r-1}.\end{aligned}$$

Hence, the corollary is true in this case. Suppose on the other hand that $\deg(z) \leqslant s$ and let $u \in \mathbf{C}_\infty$ be as in the proof of Proposition 4.4 such that $e_\rho(u) = z$. Then,

$$\deg\bigl(\rho_{\mathfrak{n}}(z)\bigr) \geqslant \deg(\mathfrak{n}u) + \deg(1-\mathfrak{n}u/\lambda)$$

for some $\lambda \in \Lambda_\rho$. On the other hand, $\deg(1-\mathfrak{n}u/\lambda)$ is bounded below since $z \notin J_\rho$. Therefore, there exits a $\mathfrak{b} \in \mathbb{A}$ such that $\deg(\rho_{\mathfrak{b}}(z)) > s$ by Proposition 4.4.

The remaining arguments is the same as in the proof of Corollary 3.2. We therefore omit the rest of the proof. □

Given $z \in \mathbf{C}_\infty$ it is a question in Diophantine approximation on how close to 0 $\rho_\mathfrak{n}(z)$ can be as $\mathfrak{n}$ varies in $\mathbb{A}$. In the case where z is algebraic over k, we have a control on the lower bound of $\deg(\rho_\mathfrak{n}(\mathbf{z}))$ in terms of $\deg(\mathfrak{n})$.

Theorem 4.6. *Let* $\mathbf{z} \in K \setminus \rho_{\mathrm{tor}}(K)$ *and let* ϵ *be a given positive real number. Then, there exists a positive constant* c_ϵ *such that* $\deg(\rho_\mathfrak{n}(\mathbf{z})) \geqslant -c_\epsilon(\deg \mathfrak{n})^{1+\epsilon}$ *for all* $\mathfrak{n} \in \mathbb{A} \setminus \mathbb{F}_q$*. Furthermore,* c_ϵ *is independent of* $\mathfrak{n}$.

Proof. Let $\lambda_0 \in \Lambda_\rho$ be an element satisfying the condition of Proposition 4.4. Increasing the degree of $\mathfrak{n}$ if necessary, we may assume that there exists a non-zero $\lambda \in \Lambda_\rho$ such that $\deg(\lambda) \leqslant \deg(\mathfrak{n}u)$. Then, we have that $\deg(\lambda_0) \leqslant \deg(\mathfrak{n}u)$. It follows from Proposition 4.4 that

$$\begin{aligned} \deg\bigl(\rho_\mathfrak{n}(w)\bigr) &\geqslant \deg(\mathfrak{n}u) + \deg\left(1 - \frac{\mathfrak{n}u}{\lambda_0}\right) \\ &= \deg\left(\frac{\mathfrak{n}u}{\lambda_0}\right) + \deg(\mathfrak{n}u - \lambda_0) \\ &\geqslant \deg(\mathfrak{n}u - \lambda_0) \\ &\geqslant -c_\epsilon(\deg \mathfrak{n})^{1+\epsilon}. \end{aligned}$$

Notice that the last inequality and the existence of the constant c_ϵ follows from Proposition 4.2. The assertion of the theorem is proved. □

5. Proof of the main result

We resume our notations in Section 2. Let $\mathbf{z} \in {}^\rho K$ be a given non-torsion point for the Drinfeld module ρ. Let $\wp \in M_K$. Following [4,6], one can associate to the commuting family $\{\rho_\mathfrak{n} \mid \mathfrak{n} \in \mathbb{A}\}$ of morphisms *the canonical local height* $\hat{h}_\wp(x)$ of $x \in K_\wp$ defined by

$$\hat{h}_\wp(x) = \lim_{\deg(\mathfrak{n}) \to \infty} \frac{\max\{0, -v_\wp(\rho_\mathfrak{n}(x))\}}{|\mathfrak{n}|^r}.$$

The existence of the limit is proved in [6] for the case of Drinfeld modules and in [4] for general cases. In our situation, one can deduce from Corollary 3.5 for finite places $\wp$ and Corollary 4.5 for infinite place $\wp$ that the limit exists. For instance, it is clear from the definition that the orbit $\{\rho_\mathfrak{n}(x) \mid \mathfrak{n} \in \mathbb{A}\}$ of x is $\wp$-adically bounded if and only if $\hat{h}_\wp(x) = 0$. Suppose that x has $\wp$-adically unbounded $\mathbb{A}$-orbit, then Corollaries 3.5(i) and 4.5 say that there exists positive constant $\delta_{\wp,x}$ which depends on the place $\wp$ and x such that $v(\rho_\mathfrak{n}(x)) = -\delta_{\wp,x}|\mathfrak{n}|^r + O(1)$. It follows that $\hat{h}_\wp(x) = \delta_{\wp,x}$. Put

$$\hat{h}_\rho(\mathbf{z}) = \sum_{\wp \in M_K} n_\wp \hat{h}_\wp(\mathbf{z}).$$

We notice that $\hat{h}_\wp(\mathbf{z}) = 0$ for almost all places $\wp \in M_K$. Therefore, the sum of the right-hand side is actually a finite sum. Furthermore, we have $\hat{h}_\rho(\mathbf{z}) = \sum_{\wp \in \mathcal{T}_\mathbf{z}} \delta_{\wp,\mathbf{z}} > 0$ if and only if $\mathbf{z}$ is a non-torsion for ρ. The function $\hat{h}_\rho$ called the canonical height associated to the Drinfeld module ρ enjoys the property that $\hat{h}_\rho(\rho_{\mathfrak{n}}(\mathbf{z})) = |\mathfrak{n}|^r \hat{h}_\rho(\mathbf{z})$ (see [6], also [4] for general situation).

Let $\mathcal{T}_{\mathbf{z}}$ be the set of places of K where $\mathbf{z}$ has unbounded orbit. Note that $\mathcal{T}_{\mathbf{z}}$ is a finite subset of M_K. Moreover, it is non-empty if and only if $\mathbf{z}$ is a non-torsion point for ρ.

5.1. Generic $\mathbb{A}$-characteristic

Let $\mathfrak{n} \in \mathbb{A}_+$ be a non-constant monic polynomial. Let $\mu : \mathbb{A}_+ \to \{-1, 0, 1\}$ be the Möbius function on $\mathbb{A}_+$. The following elementary fact about the Möbius function is well known.

Lemma 5.1. *Let* $\mathfrak{n} \in \mathbb{A}_+$ *then*

$$\sum_{\mathfrak{m}|\mathfrak{n}} \mu\left(\frac{\mathfrak{n}}{\mathfrak{m}}\right) = \Delta(\mathfrak{n}) = \begin{cases} 1 & \text{if } \mathfrak{n} = 1; \\ 0 & \text{otherwise}, \end{cases}$$

where the above sum is over all the monic divisors of $\mathfrak{n}$.

Define the *primitive* $\mathfrak{n}$*th division polynomial* $\Phi_{\rho,\mathfrak{n}}(X)$ of the Drinfeld module ρ as follows

$$\Phi_{\rho,\mathfrak{n}}(X) = \prod_{\mathfrak{m}|\mathfrak{n}} \rho_{\mathfrak{m}}(X)^{\mu(\frac{\mathfrak{n}}{\mathfrak{m}})}.$$

Lemma 5.2. *Let* $\mathfrak{n} \in \mathbb{A}_+$ *be given and let* $\Phi_{\rho,\mathfrak{n}}(X)$ *be the primitive* $\mathfrak{n}$*th division polynomial of* ρ.

(a) $\Phi_{\rho,\mathfrak{n}}(X)$ *is a polynomial in* X *with coefficients in* K.
(b) $\rho_{\mathfrak{n}}(X) = \prod_{\mathfrak{m}|\mathfrak{n}} \Phi_{\rho,\mathfrak{m}}(X)$.
(c) *Let* $\alpha \in \bar{K}$ *then* $\Phi_{\rho,\mathfrak{n}}(\alpha) = 0$ *if and only if* $\rho_{\mathfrak{n}}(\alpha) = 0$ *and* $\rho_{\mathfrak{m}}(\alpha) \neq 0$ *for any proper divisor* $\mathfrak{m}$ *of* $\mathfrak{n}$.
(d) *If* ρ *has good reduction at finite place* $\wp$ *of* K, *then all the coefficients of* $\Phi_{\rho,\mathfrak{n}}(X)$ *are* $\wp$*-integral with the leading coefficient a* $\wp$*-unit.*

Proof. (a)–(c) follows from the Möbius inversion formula and the fact that $\rho_{\mathfrak{n}}(X)$ are separable polynomials for non-zero $\mathfrak{n}$. (d) follows from the fact that $\rho_{\mathfrak{n}}(X)$ has all its coefficients in $\mathcal{O}_\wp$ and $\Delta_{\mathfrak{n}}$ is a $\wp$-unit if ρ has good reduction at $\wp$. □

For a finite place $\wp$ of K, recall from Section 2.2 the notations $F_i = \mathcal{O}_\wp/\mathcal{M}^i$ and that $\chi_1 = L_\wp(1)$ annihilates the finite Drinfeld module ${}^\rho\mathbb{F}_{\mathfrak{p}}$ where $L_{\mathfrak{p}}(X)$ is the characteristic polynomial associated to $\mathrm{Frob}_{\mathfrak{p}}$. The following proposition is a modification of [15, Lemma 4].

Proposition 5.3. *Let* $\wp$ *be a finite place of* K *such that* ρ *has good reduction at* $\wp$. *Let* $z \in \mathcal{O}_\wp$ *be a non-torsion point of* ρ. *Let* $\mathfrak{n} \in \mathbb{A}_+$ *be a monic polynomial with* $\deg(\mathfrak{n}) > 2D\log_q(D+1)$. *If* $v_\wp(\Phi_{\rho,\mathfrak{n}}(z)) > 0$ *but* $\mathfrak{n}$ *is not the order of* $\bar{z}_\wp \equiv z \pmod{\pi_\wp}$ *in* ${}^\rho\mathbb{F}_\wp$, *then*

$$v_\wp\big(\Phi_{\rho,\mathfrak{n}}(z)\big) \leqslant v_\wp(\mathfrak{n}).$$

Proof. (Cf. [15, Lemma 4].) For each positive integer i, let $(\mathfrak{a}_i)$ be the ideal of annihilators of $z \pmod{\pi_\wp^i}$ which is viewed as an element of ${}^\rho F_i$. We clearly have the following descending chain of ideals

$$(\mathfrak{a}_1) \supseteq (\mathfrak{a}_2) \supseteq \cdots (\mathfrak{a}_n) \supseteq \cdots$$

and that $\chi_i = \mathfrak{p}^{i-1}\chi_1 \in (\mathfrak{a}_i)$ by Lemma 2.2. Set $l = [v_\wp(\mathfrak{p})/(|\mathfrak{p}| - 1)]$. Then, $l \leqslant v_\wp(\mathfrak{p}) \leqslant D$. Let $\mathfrak{n}$ be given such that $\deg(\mathfrak{n}) > 2D\log_q(D+1)$.

By Corollary 3.4(i), there exists a nonnegative integer $l' \leqslant l$ such that for all $z \in \mathcal{O}_\wp$ with $v_\wp(z) > l'$

$$v_\wp\big(\rho_\mathfrak{n}(z)\big) = v_\wp(z) + v_\wp(\mathfrak{n}).$$

Let l_0 be the smallest nonnegative integer such that the above equality holds. We claim that $\mathfrak{a}_{l_0+1}$ is a proper divisor of $\mathfrak{n}$.

Note that $v_\wp(\rho_\mathfrak{n}(z)) \geqslant v_\wp(\Phi_{\rho,\mathfrak{n}}(z)) > 0$, this means that $\mathfrak{n}$ annihilates $\bar{z}_\wp$. Hence we have $\mathfrak{a}_1 \mid \mathfrak{n}$. On the other hand, $\mathfrak{n}$ is not the order of $\bar{z}_\wp$ in ${}^\rho\mathbb{F}_\wp$ by assumption, there must exist a proper divisor $\mathfrak{m}$ of $\mathfrak{n}$ such that $v_\wp(\rho_\mathfrak{m}(z)) > 0$. It follows that $\mathfrak{a}_1 \mid \mathfrak{m}$.

If $l_0 = 0$ then the claim is true since $\mathfrak{a}_1 \mid \mathfrak{m} \mid \mathfrak{n}$ and $\mathfrak{m}$ is a proper divisor of $\mathfrak{n}$. Now we assume that $l_0 \geqslant 1$ and let t be the positive integer such that $\mathfrak{a}_t \mid \mathfrak{n}$ but $\mathfrak{a}_{t+1} \nmid \mathfrak{n}$. We want to show that $t > l_0$. Suppose to the contrary that $t \leqslant l_0$. By definition, if for some positive integer $i \leqslant l_0$, $\mathfrak{a}_i \mid \mathfrak{m}$ but $\mathfrak{a}_{i+1} \nmid \mathfrak{m}$, then $v_\wp(\rho_\mathfrak{m}(z)) = i$. Therefore,

$$\begin{aligned} v_\wp\big(\Phi_{\rho,\mathfrak{n}}(z)\big) &= \sum_{\mathfrak{m}\mid\mathfrak{n}} \mu\left(\frac{\mathfrak{n}}{\mathfrak{m}}\right) v_\wp\big(\rho_\mathfrak{m}(z)\big) \\ &= \sum_{i=1}^{t} \sum_{\mathfrak{a}_i\mid\mathfrak{m}\mid\mathfrak{n}} \mu\left(\frac{\mathfrak{n}}{\mathfrak{m}}\right) \\ &= \sum_{i=1}^{t} \Delta\left(\frac{\mathfrak{n}}{\mathfrak{a}_i}\right). \end{aligned} \tag{11}$$

On the other hand, we have that

$$\deg(\mathfrak{a}_t) \leqslant \deg(\mathfrak{a}_{l+1}) \leqslant \deg\big(\mathfrak{p}^l\chi_1\big).$$

As $1 \leqslant l_0 \leqslant [v_\wp(\mathfrak{p})/(|\mathfrak{p}| - 1)]$, we see that $\deg(\mathfrak{p}) \leqslant \log_q(v_\wp(\mathfrak{p}) + 1) \leqslant \log_q(D+1)$. Moreover, by Proposition 2.1, we have $\deg(\chi_1) = \deg(\mathfrak{a}_r) = n_\wp \leqslant D\deg(\mathfrak{p})$. Thus,

$$\begin{aligned} \deg(\mathfrak{a}_{l+1}) &\leqslant l\deg(\mathfrak{p}) + D\deg(\mathfrak{p}) \\ &\leqslant 2D\log_q(D+1) < \deg(\mathfrak{n}). \end{aligned}$$

It follows that $\mathfrak{a}_t$ is a proper divisor of $\mathfrak{n}$ if $t \leqslant l_0$. However, if $\mathfrak{a}_t$ is a proper divisor of $\mathfrak{n}$ then the sum in (11) equals 0 which cannot happen since $v_\wp(\Phi_{\rho,\mathfrak{n}}(z)) > 0$ by assumption. Therefore, we must have $t > l_0$. Then, $\mathfrak{a}_{l_0+1} \mid \mathfrak{n}$. But, $\deg(\mathfrak{a}_{l_0+1}) \leqslant \deg(\mathfrak{a}_{l+1}) < \deg(\mathfrak{n})$, it follows that $\mathfrak{a}_{l_0+1}$ is a proper divisor of $\mathfrak{n}$ as claimed.

As in [15, Lemma 4], we have

$$\begin{aligned} v_{\wp}\big(\Phi_{\rho,\mathfrak{n}}(z)\big) &= \sum_{i=1}^{l_0} \sum_{\mathfrak{a}_i|\mathfrak{m}|\mathfrak{n}} \mu\left(\frac{\mathfrak{n}}{\mathfrak{m}}\right) + \sum_{\mathfrak{a}_{l_0+1}|\mathfrak{m}|\mathfrak{n}} \mu\left(\frac{\mathfrak{n}}{\mathfrak{m}}\right) v_{\wp}\big(\rho_{\mathfrak{m}}(z)\big) \\ &= \sum_{i=1}^{l_0} \sum_{\mathfrak{a}_i|\mathfrak{m}|\mathfrak{n}} \mu\left(\frac{\mathfrak{n}}{\mathfrak{m}}\right) + \sum_{\mathfrak{a}_{l_0+1}|\mathfrak{m}|\mathfrak{n}} \mu\left(\frac{\mathfrak{n}}{\mathfrak{m}}\right) \big(v_{\wp}\big(\rho_{\mathfrak{a}_{l_0+1}}(z)\big) - l_0\big) \\ &\quad + \sum_{\mathfrak{a}_{l_0+1}|\mathfrak{m}|\mathfrak{n}} \mu\left(\frac{\mathfrak{n}}{\mathfrak{m}}\right) v_{\wp}\left(\frac{\mathfrak{m}}{\mathfrak{a}_{l_0+1}}\right). \end{aligned}$$

Now $\mathfrak{a}_{l_0+1}$ is a proper divisor of $\mathfrak{n}$,

$$\sum_{\mathfrak{a}_i|\mathfrak{m}|\mathfrak{n}} \mu\left(\frac{\mathfrak{n}}{\mathfrak{m}}\right) = 0$$

for $i = 1, 2, \ldots, l_0 + 1$. Hence,

$$\begin{aligned} v_{\wp}\big(\Phi_{\rho,\mathfrak{n}}(z)\big) &= \sum_{\mathfrak{a}_{l_0+1}|\mathfrak{m}|\mathfrak{n}} \mu\left(\frac{\mathfrak{n}}{\mathfrak{m}}\right) v_{\wp}\left(\frac{\mathfrak{m}}{\mathfrak{a}_{l_0+1}}\right) \\ &= \begin{cases} v_{\wp}(\mathfrak{p}) & \text{if } \frac{\mathfrak{n}}{\mathfrak{a}_{l_0+1}} \text{ is a power of } \mathfrak{p}, \\ 0 & \text{otherwise}, \end{cases} \\ &\leqslant v_{\wp}(\mathfrak{n}). \end{aligned}$$

This proves the assertion of the proposition. □

Proof of Theorem 2.3. (*The case of generic $\mathbb{A}$-characteristic.*) Let $\mathbf{z} \in {}^{\rho}K$ be a given non-torsion point of ρ. Recall the set $\mathcal{S}_{\rho,\mathbf{z}}$ consists of places $\wp$ of K where ρ has bad reduction at $\wp$ or $\mathbf{z}$ is not $\wp$-integral. Since adding a finite set of places to $\mathcal{S}_{\rho,\mathbf{z}}$ will not affect the truth of the theorem, in the following we will enlarge $\mathcal{S}_{\rho,\mathbf{z}}$ to include the non-empty finite subset $\mathcal{T}_{\mathbf{z}}$ of M_K. Thus, we have $\mathcal{T}_{\mathbf{z}} \subset \mathcal{S}_{\rho,\mathbf{z}}$ and if $\wp \notin S_{\rho,\mathbf{z}}$ then ρ has good reduction at $\wp$ and $\mathbf{z}$ is $\wp$-integral with $\wp$-adically bounded $\mathbb{A}$-orbits.

Let $\mathfrak{n} \in \mathbb{A}$ be given with $\deg(\mathfrak{n}) > 2D \log_q(D+1)$. Suppose that there is no $\wp \in M_K \setminus \mathcal{S}_{\rho,\mathbf{z}}$ such that $\mathfrak{n} = \mathfrak{d}_{\wp}$. The idea of the proof is to compute $\sum_{\wp \in M_K} n_{\wp} v_{\wp}(\Phi_{\rho,\mathfrak{n}}(\mathbf{z}))$ in two ways.

On the one hand, by the product formula for K we have

$$\sum_{\wp \in M_k} n_{\wp} v_{\wp}\big(\Phi_{\rho,\mathfrak{n}}(\mathbf{z})\big) = 0.$$

On the other hand, we split the sum into three sub-sums: the sums over $\wp \notin S_{\rho,\mathbf{z}}$, $\wp \in \mathcal{S}_{\rho,\mathbf{z}} \setminus \mathcal{T}_{\mathbf{z}}$ and $\wp \in \mathcal{T}_{\mathbf{z}}$. Compare these two results we will get a contradiction if $\deg(\mathfrak{n})$ is large enough.

First, for $\wp \notin \mathcal{S}_{\rho,\mathbf{z}}$, by Proposition 5.3 we have the following

$$\sum_{\wp \notin \mathcal{S}_{\rho,\mathbf{z}}} n_\wp v_\wp\big(\Phi_{\rho,\mathfrak{n}}(\mathbf{z})\big) \leqslant \sum_{\wp \notin \mathcal{S}_{\rho,\mathbf{z}}} n_\wp v_\wp(\mathfrak{n}) \leqslant D \deg(\mathfrak{n}).$$

If $\wp \in \mathcal{S}_{\rho,\mathbf{z}} \setminus \mathcal{T}_{\mathbf{z}}$, then we apply Corollary 3.5(i) and Theorem 4.6 that there exist constant $c_\wp$ or $c_\epsilon > 0$ which depend only on $\wp$, $\mathbf{z}$ such that

$$v_\wp\big(\rho_\mathfrak{n}(\mathbf{z})\big) \leqslant \begin{cases} v_\wp(z) + v_\wp(\mathfrak{n}) + c_\wp & \text{if } \wp \in \mathcal{S}_{\rho,\wp} \cap M_K^0, \\ c_\epsilon(\deg(\mathfrak{n}))^{1+\epsilon} & \text{if } \wp \in \mathcal{S}_{\rho,\mathbf{z}} \cap M_K^\infty, \end{cases}$$

where we use the relation $v_\wp(\cdot) = -e_\wp \deg(\cdot)$ for $\wp \in M_K^\infty$. We may choose the largest c_ϵ working for all infinite $\wp$. In the following, we will simply take $\epsilon = 1/2$ and by choosing $C = c_{1/2} > 0$ large enough, we might as well assume that $v_\wp(\rho_\mathfrak{n}(\mathbf{z})) \leqslant C(\deg(\mathfrak{n}))^{3/2}$ for all $\wp \in \mathcal{S}_{\rho,\mathbf{z}} \setminus \mathcal{T}_{\mathbf{z}}$. Thus for any $\wp \in \mathcal{S}_{\rho,\mathbf{z}} \setminus \mathcal{T}_{\mathbf{z}}$ we have

$$v_\wp\big(\Phi_{\rho,\mathfrak{n}}(\mathbf{z})\big) = \sum_{\mathfrak{m}|\mathfrak{n}} \mu\left(\frac{\mathfrak{n}}{\mathfrak{m}}\right) v_\wp\big(\rho_\mathfrak{m}(\mathbf{z})\big) \leqslant C 2^{\nu(\mathfrak{n})}\big(\deg(\mathfrak{n})\big)^{3/2}$$

where $\nu(\mathfrak{n})$ denotes the number of monic irreducible factors of $\mathfrak{n}$. It follows that

$$\sum_{\wp \in \mathcal{S}_{\rho,\mathbf{z}} \setminus \mathcal{T}_{\mathbf{z}}} n_\wp v_\wp\big(\Phi_{\rho,\mathfrak{n}}(\mathbf{z})\big) \leqslant CD|\mathcal{S}_{\rho,\mathbf{z}}| 2^{\nu(\mathfrak{n})}\big(\deg(\mathfrak{n})\big)^{3/2}.$$

Finally, for $\wp \in \mathcal{T}_{\mathbf{z}}$

$$\begin{aligned} v_\wp\big(\Phi_{\rho,\mathfrak{n}}(\mathbf{z})\big) &= \sum_{\mathfrak{m}|\mathfrak{n}} \mu\left(\frac{\mathfrak{n}}{\mathfrak{m}}\right) v_\wp\big(\rho_\mathfrak{m}(\mathbf{z})\big) \\ &= \sum_{\mathfrak{m}|\mathfrak{n}} \mu\left(\frac{\mathfrak{n}}{\mathfrak{m}}\right)\big\{-\delta_\wp(\mathbf{z})|\mathfrak{m}|^r + O(1)\big\} \\ &= -\varphi_r(\mathfrak{n})\delta_\wp(\mathbf{z}) + O\big(2^{\nu(\mathfrak{n})}\big) \end{aligned}$$

where

$$\varphi_r(\mathfrak{n}) = |\mathfrak{n}|^r \prod_{\mathfrak{p}|\mathfrak{n}} \left(1 - \frac{1}{|\mathfrak{p}|^r}\right).$$

Consequently,

$$\sum_{\wp \in \mathcal{T}_{\mathbf{z}}} n_\wp v_\wp\big(\Phi_{\rho,\mathfrak{n}}(\mathbf{z})\big) = -\varphi_r(\mathfrak{n})\hat{h}_\rho(\mathbf{z}) + O\big(2^{\nu(\mathfrak{n})}\big).$$

Combining our computation above, we conclude that

$$\sum_{\wp\in M_k} n_\wp v_\wp\big(\Phi_{\rho,\mathfrak{n}}(\mathbf{z})\big) \leqslant -\varphi_r(\mathfrak{n})\hat{h}_\rho(\mathbf{z}) + 2^{\nu(\mathfrak{n})}\big(CD|\mathcal{S}_{\rho,\mathbf{z}}|\big(\deg(\mathfrak{n})\big)^{3/2} + O(1)\big) + D\deg(\mathfrak{n}).$$

However, given any real number $0<\gamma<r$, there exists a positive constant d_γ such that

$$\frac{2^{\nu(\mathfrak{n})}}{\varphi_r(\mathfrak{n})} \leqslant \frac{d_\gamma}{|\mathfrak{n}|^\gamma}.$$

Thus,

$$0=\sum_{\wp\in M_k} n_\wp v_\wp\big(\Phi_{\rho,\mathfrak{n}}(\mathbf{z})\big) \leqslant \varphi_r(\mathfrak{n})\bigg(-\hat{h}_\rho(\mathbf{z}) + \frac{d_\gamma}{|\mathfrak{n}|^\gamma}\big[E\deg(\mathfrak{n})^{3/2}+O(1)\big]\bigg)$$

for some constant E. The inequality will give a contradiction if $\deg(\mathfrak{n})$ becomes large. Therefore, there exists a constant N such that if $\deg(\mathfrak{n})>N$ then $\mathfrak{n}=\mathfrak{n}_\wp$ for some $\wp\in M_K\setminus\mathcal{S}_{\rho,\mathbf{z}}$. $\square$

5.2. Finite $\mathbb{A}$-characteristic

We assume that K is of $\mathbb{A}$-characteristic $\mathfrak{p}$. Then

$$\rho_\mathfrak{p} = h_{fd}\tau^{fd}+\cdots+h_{rd}\tau^{rd}, \quad h_{fd}\neq 0,$$

for some integer $0<f\leqslant r$. For the case where $f=r$ (the supersingular case), the order $\mathfrak{d}_\wp$ of $\mathbf{z}$ (mod $\wp$) is prime to $\mathfrak{p}$. As it is not difficult to modify our argument below to this case (see Remark 5), we omit the proof for the supersingular case. We will assume that $f<r$ from now on. Following Section 3 we put $H_\rho = h_{fd}$. Note that every root of $\rho_\mathfrak{n}(X)$ is of multiplicity $|\mathfrak{p}^f|^{\mathrm{ord}_\mathfrak{p}(\mathfrak{n})}$. Thus, $\rho_\mathfrak{n}(X)$ is a separable polynomial if and only if $\mathfrak{n}$ is prime to $\mathfrak{p}$. The *primitive* $\mathfrak{n}$*th division polynomial* $\Phi_{\rho,\mathfrak{n}}(X)$ of the Drinfeld module ρ is defined as follows:

$$\Phi_{\rho,\mathfrak{n}}(X) = \prod_{\mathfrak{m}|\mathfrak{n}} \rho_\mathfrak{m}(X)^{\mu(\frac{\mathfrak{n}}{\mathfrak{m}})|\mathfrak{p}^f|^{\mathrm{ord}_\mathfrak{p}(\mathfrak{n}/\mathfrak{m})}}.$$

Note that if $\mathfrak{n}$ is prime to $\mathfrak{p}$ then the definition of $\Phi_{\rho,\mathfrak{n}}(X)$ is the same as in the case of generic $\mathbb{A}$-characteristic.

Lemma 5.4. *Let $\mathfrak{n}\in\mathbb{A}_+$ be given and let $\Phi_{\rho,\mathfrak{n}}(X)$ be the primitive $\mathfrak{n}$th division polynomial of ρ.*

(a) *$\Phi_{\rho,\mathfrak{n}}(X)$ is a polynomial in X with coefficients in K whose roots have multiplicities $|\mathfrak{p}^f|^{\mathrm{ord}_\mathfrak{p}(\mathfrak{n})}$.*
(b) *$\rho_\mathfrak{n}(X)=\prod_{\mathfrak{m}|\mathfrak{n}}\Phi_{\rho,\mathfrak{m}}(X)^{|\mathfrak{p}^f|^{\mathrm{ord}_\mathfrak{p}(\mathfrak{n}/\mathfrak{m})}}$.*
(c) *Let $\alpha\in\bar{K}$. Then $\Phi_{\rho,\mathfrak{n}}(\alpha)=0$ if and only if $\rho_\mathfrak{n}(\alpha)=0$ and $\rho_\mathfrak{m}(\alpha)\neq 0$ for any proper divisor $\mathfrak{m}$ of $\mathfrak{n}$.*
(d) *If ρ has good reduction at finite place $\wp$ of K, then all the coefficients of $\Phi_{\rho,\mathfrak{n}}(X)$ are $\wp$-integral with the leading coefficient a $\wp$-unit.*

Proof. Let $f(X) \in K(X)$ be a rational function with coefficients in K. As usual, the vanishing order of $f(X)$ at $a \in \bar{K}$ is denoted by $\mathrm{ord}_a(f(X))$. Let $\alpha \in \rho[\mathfrak{n}]$ be a given $\mathfrak{n}$-torsion, then

$$\mathrm{ord}_\alpha\big(\Phi_{\rho,\mathfrak{n}}(X)\big) = \sum_{\mathfrak{m}|\mathfrak{n}} \mu\left(\frac{\mathfrak{n}}{\mathfrak{m}}\right)\left|\mathfrak{p}^f\right|^{\mathrm{ord}_\mathfrak{p}(\mathfrak{n}/\mathfrak{m})} \mathrm{ord}_\alpha\big(\rho_\mathfrak{m}(X)\big).$$

Assume that, as a torsion of the Drinfeld module ρ, α is of order $\mathfrak{a}$ then

$$\begin{aligned}\mathrm{ord}_\alpha\big(\Phi_{\rho,\mathfrak{n}}(X)\big) &= \sum_{\mathfrak{a}|\mathfrak{m}|\mathfrak{n}} \mu\left(\frac{\mathfrak{n}}{\mathfrak{m}}\right)\left|\mathfrak{p}^f\right|^{\mathrm{ord}_\mathfrak{p}(\mathfrak{n}/\mathfrak{m})}\left|\mathfrak{p}^f\right|^{\mathrm{ord}_\mathfrak{p}(\mathfrak{m})} \\ &= \sum_{\mathfrak{a}|\mathfrak{m}|\mathfrak{n}} \mu\left(\frac{\mathfrak{n}}{\mathfrak{m}}\right)\left|\mathfrak{p}^f\right|^{\mathrm{ord}_\mathfrak{p}(\mathfrak{n})} \\ &= \begin{cases} \left|\mathfrak{p}^f\right|^{\mathrm{ord}_\mathfrak{p}(\mathfrak{n})} & \text{if } \mathfrak{n} = \mathfrak{a}, \\ 0 & \text{otherwise.}\end{cases}\end{aligned}$$

Now (a) and (c) follows from the above computation of the vanishing order of $\Phi_{\rho,\mathfrak{n}}$ at $\alpha \in \rho[\mathfrak{n}]$. The proof of (b) is a straightforward application of the Möbius inversion formula, we omit it here. The proof of (d) is the same as that of Lemma 5.2(d). □

Proposition 5.5. *Let $\wp$ be a place of K where ρ has good reduction and $v_\wp(H_\rho) = 0$. Let $z \in \mathcal{O}_\wp$ be a non-torsion point of ρ. Let $\mathfrak{n} \in \mathbb{A}_+$ be a monic polynomial. Then $v_\wp(\Phi_{\rho,\mathfrak{n}}(z)) > 0$ if and only if $\mathfrak{n}$ is the order of $\bar{z}_\wp \equiv z \pmod{\pi_\wp}$ in ${}^\rho\mathbb{F}_\wp$.*

Proof. Recall that the reduction map $\rho \mapsto \bar{\rho}_\wp$ induces an injection of the torsion submodule $\rho[\mathfrak{n}] \hookrightarrow {}^\rho\bar{\mathbb{F}}_\wp$ for $\mathfrak{n}$ relatively prime to $\mathfrak{p}$. Moreover, it also follows from the assumption $v_\wp(H_\rho) = 0$ that (ignoring the multiplicities) the set of $\mathfrak{n}$-torsion points $\rho[\mathfrak{n}]$ is mapped bijectively to the set of $\mathfrak{n}$-torsion points $\bar{\rho}[\mathfrak{n}]$ for arbitrary $\mathfrak{n}$ under the reduction map. Consequently, two distinct roots of $\Phi_{\rho,\mathfrak{n}}(X)$ remain distinct under the reduction map. The proposition now follows. □

We now proceed to prove the main theorem for the finite $\mathbb{A}$-characteristic case. Note that in this case the results of Section 4 are not needed. The idea of the proof is the same as that of the case of generic $\mathbb{A}$-characteristic.

Proof of Theorem 2.3. (*The case of finite $\mathbb{A}$-characteristic and ρ is ordinary.*) Let $\mathbf{z} \in {}^\rho K$ be a given non-torsion point of ρ. We enlarge $S_{\rho,\mathbf{z}}$ to include $\mathcal{T}_\mathbf{z}$ and a finite set the places $\wp$ of K such that $v_\wp(H_\rho) \neq 0$. Let us still denote it by $S_{\rho,\mathbf{z}}$.

Let $\mathfrak{n} \in \mathbb{A}_+$ be given and suppose that there is no $\wp \in M_K \setminus \mathcal{S}_{\rho,\mathbf{z}}$ such that $\mathfrak{n} = \mathfrak{d}_\wp$. Then, by Proposition 5.5,

$$\sum_{\wp \notin \mathcal{S}_{\rho,\mathbf{z}}} n_\wp v_\wp\big(\Phi_{\rho,\mathfrak{n}}(\mathbf{z})\big) = 0.$$

Let us now look at places $\wp \in \mathcal{S}_{\rho,\mathbf{z}}$. If $\wp \in \mathcal{S}_{\rho,\mathbf{z}} \setminus \mathcal{T}_\mathbf{z}$ then by Corollary 3.5(ii), there is a $\mathfrak{b}_\wp$ depending only on $\mathbf{z}$ and $\wp$ such that

$$v_\wp\big(\rho_{\mathfrak{n}}(\mathbf{z})\big) \leqslant \big|\mathfrak{p}^f\big|^{\mathrm{ord}_{\mathfrak{p}}(\mathfrak{n}/\mathfrak{b}_\wp)} v_\wp\big(\rho_{\mathfrak{b}_\wp}(\mathbf{z})\big) + \left\{\frac{|\mathfrak{p}^f|^{\mathrm{ord}_{\mathfrak{p}}(\mathfrak{n}/\mathfrak{b}_\wp)} - 1}{|\mathfrak{p}^f| - 1}\right\} v_\wp(H_\rho). \tag{12}$$

Consequently,

$$\big|v_\wp\big(\rho_{\mathfrak{n}}(\mathbf{z})\big)\big| \leqslant A_\wp \big|\mathfrak{p}^f\big|^{\mathrm{ord}_{\mathfrak{p}}(\mathfrak{n}/\mathfrak{b}_\wp)}$$

for some constant $A_\wp$ depending only on ρ, $\wp$ and $\mathbf{z}$. From this, we have

$$\begin{aligned}\Big|\sum_{\wp\in\mathcal{S}_{\rho,\mathbf{z}}\setminus\mathcal{T}_{\mathbf{z}}} n_\wp v_\wp\big(\Phi_{\rho,\mathfrak{n}}(\mathbf{z})\big)\Big| &\leqslant \sum_{\wp\in\mathcal{S}_{\rho,\mathbf{z}}\setminus\mathcal{T}_{\mathbf{z}}} n_\wp \Big|\sum_{\mathfrak{m}|\mathfrak{n}} \mu\Big(\frac{\mathfrak{n}}{\mathfrak{m}}\Big)\big|\mathfrak{p}^f\big|^{\mathrm{ord}_{\mathfrak{p}}(\mathfrak{n}/\mathfrak{m})} v_\wp\big(\rho_{\mathfrak{m}}(\mathbf{z})\big)\Big| \\ &\leqslant D|\mathcal{S}_{\rho,\mathbf{z}}|E_{\rho,\mathbf{z}} 2^{\nu(\mathfrak{n})}\big|\mathfrak{p}^f\big|^{\mathrm{ord}_{\mathfrak{p}}(\mathfrak{n})}\end{aligned}$$

for some positive constant $E_{\rho,\mathbf{z}}$ depending on ρ and $\mathbf{z}$. On the other hand, for $\wp \in \mathcal{T}_{\mathbf{z}}$

$$\begin{aligned}v_\wp\big(\Phi_{\rho,\mathfrak{n}}(\mathbf{z})\big) &= \sum_{\mathfrak{m}|\mathfrak{n}} \mu\Big(\frac{\mathfrak{n}}{\mathfrak{m}}\Big) v_\wp\big(\rho_{\mathfrak{m}}(\mathbf{z})\big) \\ &= \sum_{\mathfrak{m}|\mathfrak{n}} \mu\Big(\frac{\mathfrak{n}}{\mathfrak{m}}\Big)\big\{-\delta_\wp(\mathbf{z})|\mathfrak{m}|^r + O(1)\big\} \\ &= -\varphi_r(\mathfrak{n})\delta_\wp(\mathbf{z}) + O\big(2^{\nu(\mathfrak{n})}\big),\end{aligned}$$

and

$$\sum_{\wp\in\mathcal{T}_\wp} n_\wp v_\wp\big(\Phi_{\rho,\mathfrak{n}}(\mathbf{z})\big) = -\varphi_r(\mathfrak{n})\hat{h}_\rho(\mathbf{z}) + O\big(2^{\nu(\mathfrak{n})}\big).$$

By our assumption $f < r$, it follows that there is a positive constant d_γ such that

$$\frac{2^{\nu(\mathfrak{n})}|\mathfrak{p}^f|^{\mathrm{ord}_{\mathfrak{p}}(\mathfrak{n})}}{\varphi_r(\mathfrak{n})} \leqslant \frac{d_\gamma}{|\mathfrak{n}|^\gamma}$$

for any given positive $\gamma < r - f$. Then,

$$0 = \sum_{\wp\in M_K} n_\wp v_\wp\big(\Phi_{\rho,\mathfrak{n}}(\mathbf{z})\big) \leqslant \varphi_r(\mathfrak{n})\bigg(-\hat{h}_\rho(\mathbf{z}) + \frac{d_\gamma}{|\mathfrak{n}|^\gamma}\big[D|\mathcal{S}_{\rho,\mathbf{z}}|E_{\rho,\mathbf{z}} + o(1)\big]\bigg)$$

which again gives contradiction as the degree of $\mathfrak{n}$ gets large. This concludes the proof of the main theorem. □

Remark 5. (a) For the supersingular case, we only consider those $\mathfrak{n} \in \mathbb{A}_+$ which are prime to $\mathfrak{p}$. Then the inequality (12) reduces to

$$\left|v_{\wp}\left(\rho_{\mathfrak{n}}(\mathbf{z})\right)\right| \leqslant v_{\wp}\left(\rho_{\mathfrak{b}_{\wp}}(\mathbf{z})\right).$$

Now it is not difficult to modify the remaining arguments in the proof to deduce the result for the supersingular case.

(b) For the number field case, let a be a non-zero element which is not a root of unity in an algebraic number field F. Schinzel [15] proved that there is an effective computable bound $n_0(d)$ depending on the degree d of a so that for every $n > n_0(d)$ there is a prime ideal $\mathfrak{P}$ of F such that n is the multiplicative order of a modulo $\wp$.

By a careful analysis, one can actually show that the constants appearing in Section 3 as well as the constants involved at an infinite place can be effectively determined. Thus, the exceptional orders $\mathfrak{n}$ (that is, $\mathfrak{n} \neq \mathfrak{d}_{\wp}$ for any $\wp \in M_K$) for a given $\mathbf{z}$ can also be effectively determined. It is an interesting question whether or not the exceptional order has a uniform bound depending on K only.

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