Browder's convergence theorem for multivalued mappings without endpoint condition

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We prove Browder's convergence theorem for multivalued nonexpansive mappings in a complete \(\mathbb{R}\)-tree without endpoint condition. This gives an affirmative answer to Jung's question for nonlinear spaces.

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\textbf{1. Introduction}

Let \(E\) be a nonempty subset of a geodesic metric space \((X, d)\). We shall denote the family of nonempty bounded closed subsets of \(E\) by \(BC(E)\), the family of nonempty bounded closed convex subsets of \(E\) by \(BCC(E)\), the family of nonempty compact subsets of \(E\) by \(K(E)\) and the family of nonempty compact convex subsets of \(E\) by \(KC(E)\). Let \(H(\cdot, \cdot)\) be the Hausdorff distance on \(BC(X)\), i.e.,

\[
H(A, B) = \max\left\{\sup_{a \in A} \text{dist}(a, B), \sup_{b \in B} \text{dist}(b, A)\right\}, \quad A, B \in BC(X).
\]

A multivalued mapping \(T : E \rightarrow BC(X)\) is said to be a \textit{contraction} if there exists a constant \(k \in [0, 1)\) such that

\[
H(T(x), T(y)) \leq kd(x, y), \quad x, y \in E.
\]  

If (1.1) is valid when \(k = 1\), then \(T\) is called \textit{nonexpansive}. A point \(x \in E\) is called a \textit{fixed point} of \(T\) if \(x \in T(x)\). A point \(x \in E\) is said to be an \textit{endpoint} of \(T\) if \(x\) is a fixed point of \(T\) and \(T(x) = \{x\}\) (see [29]). We shall denote by \(\text{Fix}(T)\) the set of all fixed points of \(T\) and by \(\text{End}(T)\) the set of all endpoints of \(T\). We see that for each mapping \(T\), \(\text{End}(T) \subseteq \text{Fix}(T)\) and the converse is not true in general. A mapping \(T\) is said to satisfies the \textit{endpoint condition} if \(\text{End}(T) = \text{Fix}(T)\).

\[\text{Fix}(T) = \{x \in E : x \in T(x)\}\]

\[\text{End}(T) = \{x \in E : x \text{ is an endpoint of } T\}\]

\[\text{Fix}(T) \subseteq \text{End}(T)\]

\[\text{End}(T) = \text{Fix}(T)\]

\[\text{Fix}(T) \subseteq \text{End}(T)\]

\[\text{Fix}(T) \cap \text{End}(T) = \emptyset\]

\[\text{Fix}(T) \neq \text{End}(T)\]

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\[\text{End}(T) \neq \emptyset\]

\[\text{Fix}(T) \cup \text{End}(T) = E\]

\[\text{Fix}(T) \cap \text{End}(T) = \emptyset\]

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\[\text{Fix}(T) \cap \text{End}(T) = \emptyset\]

\[\text{Fix}(T) \neq \emptyset\]

\[\text{End}(T) \neq \emptyset\]
Let $E$ be a nonempty closed convex subset of a Banach space $X$ and let $T : E \to E$ be a single-valued nonexpansive mapping. Fix $u \in E$, for each $s \in (0, 1)$, we can define a contraction $t_s : E \to E$ by

$$t_s(x) = su + (1 - s)t(x), \quad x \in E.$$  

Then by Banach’s contraction principle, $t_s$ has a unique fixed point $x_s \in E$, that is,

$$x_s = su + (1 - s)t(x_s). \quad (1.2)$$

In 1967, Browder [5] proved the following theorem.

**Theorem 1.1.** Let $E$ be a nonempty bounded closed convex subset of a Hilbert space $H$ and $t : E \to E$ be nonexpansive. Fix $u \in E$ and let $\{x_s\}$ be defined by (1.2). Then $\{x_s\}$ converges strongly as $s \to 0$ to the point of $\text{Fix}(t)$ nearest to $u$.

Let $T : E \to BC(E)$ be a multivalued nonexpansive mapping. Fix $u \in E$, for each $s \in (0, 1)$, we define a contraction $G_s : E \to BC(E)$ by

$$G_s(x) = su + (1 - s)T(x), \quad x \in E. \quad (1.3)$$

Then by Nadler’s theorem [20], $G_s$ has a (not necessary unique) fixed point $x_s \in E$, that is,

$$x_s \in su + (1 - s)T(x_s). \quad (1.4)$$

A natural question arises whether Browder’s theorem can be extended to the multivalued case. The first result concerning to this question was proved by Lopez and Xu [17] in 1995. They gave the strong convergence of the net $\{x_s\}$ defined by (1.4) under the endpoint condition. Since then the strong convergence of $\{x_s\}$ has been developed and many of papers have appeared (see e.g., [14,24,13,26,27]). Among other things, Jung [13] obtained the following result.

**Theorem 1.2.** Let $X$ be a uniformly convex Banach space with a uniformly Gâteaux differentiable norm, $E$ a nonempty closed convex subset of $X$, and $T : E \to K(E)$ a nonexpansive mapping. Suppose that $T$ satisfies the endpoint condition. Fix $u \in E$ and let $\{x_s\}$ be defined by (1.4). Then $T$ has a fixed point if and only if $\{x_s\}$ remains bounded as $s \to 0$ and in this case, $\{x_s\}$ converges strongly as $s \to 0$ to a fixed point of $T$.

Jung also posed an open question whether the endpoint condition in Theorem 1.2 can be omitted. In view of Pietramala’s example [22], Shahzad and Zegeye [26] pointed out that it is almost impossible to completely omit this condition for nonexpansive multivalued mappings even in the Euclidean plane $\mathbb{R}^2$. They also improved Jung’s theorem under some mild conditions. On the other hand, the present authors [6] extended Jung’s theorem to a special kind of metric spaces, namely, CAT(0) spaces.

**Theorem 1.3.** Let $E$ be a nonempty closed convex subset of a complete CAT(0) space $X$ and $T : E \to K(E)$ be a nonexpansive mapping. Suppose that $T$ satisfies the endpoint condition. Fix $u \in E$ and let $\{x_s\}$ be defined by (1.4). Then $T$ has a fixed point if and only if $\{x_s\}$ remains bounded as $s \to 0$. In this case, the following statements hold:

(i) $\{x_s\}$ converges to the unique fixed point $z$ of $T$ which is nearest $u$.
(ii) If $\{u_n\}$ is a bounded sequence in $C$ having $\lim_{n \to \infty} \text{dist}(u_n, T(u_n)) = 0$, then

$$d^2(u, z) \leq \mu_n d^2(u, u_n)$$

for all Banach limits $\mu$.

It is well known that the class of Hilbert spaces is a subclass of CAT(0) spaces (see [4]). Thus, we cannot omit the endpoint condition in Theorem 1.3. Summary: there is no any result concerning Browder’s convergence theorem in linear or nonlinear spaces which completely removes the endpoint condition. However, there is a nice subclass of CAT(0) spaces, namely $\mathbb{R}$-trees, such that Browder’s theorem holds without this condition.

2. Preliminaries

For any pair of points $x, y$ in a metric space $(X, d)$, a geodesic path joining these points is an isometry $c$ from a closed interval $[0, 1]$ to $X$ such that $c(0) = x$ and $c(1) = y$. The image of $c$ is called a geodesic segment joining $x$ and $y$. If there exists exactly one geodesic joining $x$ and $y$ we denote by $[x, y]$ the geodesic joining $x$ and $y$. For $x, y \in X$ and $\alpha \in [0, 1]$, we denote the point $z \in [x, y]$ such that $d(x, z) = \alpha d(x, y)$ by $(1 - \alpha)x \oplus \alpha y$. The space $(X, d)$ is said to be a geodesic space if every two points of $X$ are joined by a geodesic, and $X$ is said to be uniquely geodesic if there is exactly one geodesic joining
x and y for each x, y ∈ X. A subset E of X is said to be convex if E includes every geodesic segment joining any two of its points, and E is said to be gated if for any point x /∈ E there is a unique point y_x such that for any z ∈ E,

\[ d(x, z) = d(x, y_x) + d(y_x, z). \]

The point y_x is called the gate of x in E. From the definition of y_x we see that it is also the unique nearest point of x in E.

\( \mathbb{R} \)-trees (sometimes called metric trees) were introduced by Tits [28] in 1977. Fixed point theory in \( \mathbb{R} \)-trees was first studied by Kirk [15]. He showed that every continuous mapping defined on a geodesically bounded complete \( \mathbb{R} \)-tree always has a fixed point. Since then fixed point theorems for various types of mappings in \( \mathbb{R} \)-trees has been developed (see e.g., [10,1,19,21,25,3,2]).

**Definition 2.1.** An \( \mathbb{R} \)-tree is a geodesic metric space X such that:

(i) there is a unique geodesic segment \([x, y]\) joining each pair of points x, y ∈ X;

(ii) if \([y, x] \cap [x, z] = [x]\), then \([y, x] \cup [x, z] = [y, z]\).

By (i) and (ii) we have

(iii) if u, v, w ∈ X, then \([u, v] \cap [u, w] = [u, z]\) for some z ∈ X.

An \( \mathbb{R} \)-tree is a special case of a CAT(0) space. For a thorough discussion of these spaces and their applications, see [4]. It is known that in an \( \mathbb{R} \)-tree the gated subsets are precisely the closed convex subsets (see [10]). We now collect some basic properties of \( \mathbb{R} \)-trees.

**Lemma 2.2.** Let X be a complete \( \mathbb{R} \)-tree. Then the following statements hold:

(i) [9, Lemma 2.5] if x, y, z ∈ X and \( \alpha \in [0, 1] \), then

\[ d^2(\alpha x \oplus (1 - \alpha) y, z) \leq \alpha d^2(x, z) + (1 - \alpha) d^2(y, z) - \alpha(1 - \alpha)d^2(x, y); \]

(ii) [9, Lemma 2.3] if x, y, z ∈ X, then \( d(x, z) + d(z, y) = d(x, y) \) if and only if \( z \in [x, y] \);

(iii) [1, Lemma 2.1] if x, y ∈ X and z ∈ [x, y], then \([x, z] \subseteq [x, y]\);

(iv) [19, Lemma 3.1] if A and B are bounded closed convex subsets of X, then, for any u ∈ X, \( d(x, y) \leq H(A, B) \), where the points x, y are respectively the unique nearest points of u in A and B;

(v) [18, Proposition 1] if E is a nonempty closed convex subset of X and \( T : E \to BCC(E) \) is a nonexpansive mapping, then \( \text{Fix}(T) \) is closed and convex.

Let \( \{x_n\} \) be a bounded sequence in X. For x ∈ X, we set

\[ r(x, \{x_n\}) = \limsup_{n \to \infty} d(x, x_n). \]

The asymptotic radius \( r(\{x_n\}) \) of \( \{x_n\} \) is given by

\[ r(\{x_n\}) = \inf\{r(x, \{x_n\}) : x \in X\}, \]

and the asymptotic center \( A(\{x_n\}) \) of \( \{x_n\} \) is the set

\[ A(\{x_n\}) = \{x \in X : r(x, \{x_n\}) = r(\{x_n\})\}. \]

It is known from Proposition 7 of [8] that in an \( \mathbb{R} \)-tree, \( A(\{x_n\}) \) consists of exactly one point. The following lemma can be found in [7].

**Lemma 2.3.** ([7, Proposition 2.1]) If E is a closed convex subset of X and if \( \{x_n\} \) is a bounded sequence in E, then \( A(\{x_n\}) \) is in E.

Recall that a bounded sequence \( \{x_n\} \) in X is said to be regular if \( r(\{x_n\}) = r(\{u_n\}) \) for every subsequence \( \{u_n\} \) of \( \{x_n\} \).

Every bounded sequence in a complete \( \mathbb{R} \)-tree has a regular subsequence (see e.g., [16,11]).

The following lemmas are also needed.

**Lemma 2.4.** ([6, Lemma 3.2]) Let E be a nonempty closed convex subset of a complete \( \mathbb{R} \)-tree X and \( T : E \to K(E) \) be a nonexpansive mapping. Suppose \( \{x_n\} \) is a sequence in E which is regular and \( \lim_{n \to \infty} d(x_n, T(x_n)) = 0 \). If \( A(\{x_n\}) = \{z\} \), then z is a fixed point of T.
Lemma 2.5. (23, Lemma 2.1) Let $E$ be a nonempty closed convex subset of a complete $\mathbb{R}$-tree $X$ and $t : E \to E$ be a nonexpansive mapping. Let $u \in E$ be fixed. For each $s \in (0, 1)$, the mapping $t_s : E \to E$ defined by

$$t_s(x) = su \oplus (1 - s)t(x) \quad \text{for } x \in E$$

has a unique fixed point.

Recall that a continuous linear functional $\mu$ on $\ell_\infty$, the Banach space of bounded real sequences, is called a Banach limit if $\|\mu\| = \mu(1, 1, \ldots) = 1$ and $\mu_n(a_n) = \mu_n(a_{n+1})$ for all $\{a_n\} \in \ell_\infty$.

Lemma 2.6. (23, Lemma 2.2) Let $E, X, t$ be as in Lemma 2.5. For each $s \in (0, 1)$, let $x_s$ be the fixed point of $t_s$, that is,

$$x_s = t_s(x_s) = su \oplus (1 - s)t(x_s). \tag{2.1}$$

Then $\text{Fix}(t) \neq \emptyset$ if and only if $\{x_s\}$ given by the formula (2.1) remains bounded as $s \to 0$. In this case, the following statements hold:

(i) $\{x_s\}$ converges to the unique fixed point $z$ of $t$ which is nearest $u$;

(ii) $d^2(u, z) \leq \mu(d^2(u, u_n))$ for all Banach limits $\mu$ and all bounded sequences $\{u_n\}$ with $\lim_{n \to \infty} d(u_n, t(u_n)) = 0$.

Lemma 2.7. (23, Theorem 2.3) Let $E$ be a nonempty closed convex subset of a complete $\mathbb{R}$-tree $X$ and $t : E \to E$ be a nonexpansive mapping for which $\text{Fix}(t) \neq \emptyset$. Suppose that $u, z_1 \in E$ are arbitrarily chosen and $\{z_n\}$ is defined by

$$z_{n+1} = \alpha_n u \oplus (1 - \alpha_n) t(z_n), \quad n \geq 1, \tag{2.2}$$

where $\{\alpha_n\}$ is a sequence in $(0, 1)$ satisfying

(C1) $\lim_{n \to \infty} \alpha_n = 0$;

(C2) $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(C3) $\sum_{n=1}^{\infty} |\alpha_n - \alpha_{n+1}| < \infty$ or $\lim_{n \to \infty} (\alpha_n/\alpha_{n+1}) = 1$.

Then $\{z_n\}$ converges to the unique point of $\text{Fix}(t)$ which is nearest to $u$.

3. Main results

Let $E$ be a nonempty closed convex subset of an $\mathbb{R}$-tree $X$. For $x \in X$, we denote by $P_E(x)$ the unique nearest point of $x$ in $E$. The following lemma can be found in [1].

Lemma 3.1. Let $E$ be a nonempty closed convex subset of an $\mathbb{R}$-tree $X$. Then, for any $x, y \in X$, we have either

$$P_E(x) = P_E(y)$$

or

$$d(x, y) = d(x, P_E(x)) + d(P_E(x), P_E(y)) + d(P_E(y), y).$$

As a consequence of Lemma 3.1, we obtain

Lemma 3.2. Let $E$ be a nonempty subset of a complete $\mathbb{R}$-tree $X$ and $T : E \to \text{BCC}(E)$ be a nonexpansive mapping such that $\text{Fix}(T) \neq \emptyset$. Let $u \in E$ and $z \in \text{Fix}(T)$. Then for each $x \in E$, we have either

$$d(z, P_{T(x)}(u)) \leq d(z, x) \quad \text{or} \quad P_{T(x)}(u) \in [u, z].$$

Proof. By Lemma 3.1, we have either

$$P_{T(x)}(u) = P_{T(x)}(z)$$

or

$$d(u, z) = d(u, P_{T(x)}(u)) + d(P_{T(x)}(u), P_{T(x)}(z)) + d(P_{T(x)}(z), z).$$

If $P_{T(x)}(u) = P_{T(x)}(z)$, then

$$d(z, P_{T(x)}(u)) = \text{dist}(z, T(x)) \leq H(T(z), T(x)) \leq d(z, x).$$
If \( P_{T(x)}(u) \neq P_{T(x)}(z) \), then
\[
d(u, z) = d(u, P_{T(x)}(u)) + d(P_{T(x)}(u), P_{T(x)}(z)) + d(P_{T(x)}(z), z)
= d(u, P_{T(x)}(u)) + d(P_{T(x)}(u), z).
\]
By Lemma 2.2(ii), we have \( P_{T(x)}(u) \in [u, z] \). \( \Box \)

**Lemma 3.3.** Let \( E \) be a nonempty closed convex subset of a complete \( \mathbb{R} \)-tree \( X \) and \( T : E \to BCC(E) \) be a multivalued nonexpansive mapping. Fix \( u \in E \) and define \( f : E \to E \) by \( f(x) = P_{T(x)}(u) \) for \( x \in E \). For each \( s \in (0, 1) \), we define \( t_s : E \to E \) by
\[
t_s(x) = su \oplus (1 - s)f(x), \quad x \in E.
\]
Then \( t_s \) has a unique fixed point.

**Proof.** By Lemma 2.2(iv), \( f \) is nonexpansive. The conclusion follows from Lemma 2.5. \( \Box \)

**Lemma 3.4.** Let \( E, X, T, u, f \) be as in Lemma 3.3. For each \( s \in (0, 1) \), let \( x_s \) be the fixed point of \( t_s \), that is,
\[
x_s = su \oplus (1 - s)f(x_s).
\]
If \( z \in \text{Fix}(T) \), then \( x_s \in [u, z] \), equivalently,
\[
d(u, x_s) + d(x_s, z) = d(u, z).
\]

**Proof.** If \( u \in \text{Fix}(T) \), then \( u = P_{T(u)}(u) = f(u) \). Thus
\[
d(f(x_s), u) = d(f(x_s), f(u)) \leq H(T(x_s), T(u)) \leq d(x_s, u) = sd(f(x_s), u).
\]
This implies that \( u = f(x_s) = x_s \). Then the conclusion follows. Now, if \( u \notin \text{Fix}(T) \), then \( x_s \neq f(x_s) \). Otherwise, \( d(u, f(x_s)) = d(u, x_s) = (1 - s)d(u, f(x_s)) \) which implies \( u = f(x_s) = x_s \), contradicting the fact that \( u \notin \text{Fix}(T) \). By Lemma 3.2, we have either
\[
d(z, f(x_s)) \leq d(z, x_s) \quad \text{or} \quad d(x_s) \in [u, z].
\]
If \( d(z, f(x_s)) \leq d(z, x_s) \), then the gate of \( z \) in \([u, f(x_s)]\) lies in \((x_s, f(x_s))\). Hence, in any case, \( x_s \in [u, z] \). \( \Box \)

**Lemma 3.5.** Let \( E \) be a nonempty closed convex subset of a complete \( \mathbb{R} \)-tree \( X \) and \( T : E \to KC(E) \) be a nonexpansive mapping. Fix \( u \in E \) and define \( f : E \to E \) by \( f(x) = P_{T(x)}(u) \). Then the following statements hold:

(i) \( \text{Fix}(f) \neq \emptyset \) if and only if \( \text{Fix}(T) \neq \emptyset \);
(ii) if \( \text{Fix}(f) \neq \emptyset \) and \( x \) and \( y \) are respectively the unique nearest points of \( u \) in \( \text{Fix}(f) \) and \( \text{Fix}(T) \), then \( x = y \).

**Proof.** (i) Clearly \( \text{Fix}(f) \subseteq \text{Fix}(T) \). Thus one direction is obvious. Conversely, let \( p \in \text{Fix}(T) \). Then by (3.2), \( d(x_s, p) \leq d(u, p) \) for all \( s \in (0, 1) \). This implies that \( \{x_s\} \) is bounded. Hence by Lemma 2.6, \( \text{Fix}(f) \neq \emptyset \).

(ii) Let a sequence \( \{x_n\} \) in \((0, 1)\) converges to \( 0 \) and \( x_0 := x_{x_0} \). Then obtain a regular subsequence \( \{x_{n_k}\} \) of \( \{x_n\} \) and denote \( A(\{x_{n_k}\}) = \{v\} \). For \( k \in \mathbb{N} \), \( x_{n_k} = s_{n_k}u \oplus (1 - s_{n_k})f(x_{n_k}) \). Thus
\[
d(x_{n_k}, f(x_{n_k})) = s_{n_k}d(u, f(x_{n_k})) \to 0 \quad \text{as} \quad k \to \infty.
\]
By Lemma 2.4, \( v \in \text{Fix}(f) \subseteq \text{Fix}(T) \). Next, we show that \( \{x_{n_k}\} \) converges to \( v \). Suppose that \( \varepsilon = \limsup_{k \to \infty} d^2(x_{n_k}, v) > 0 \) and denote \( r = \limsup_{k \to \infty} d^2(x_{n_k}, u) \). Then \( 0 < \varepsilon < r \). Let \( \alpha \in (0, 1) \) be such that \( 0 < \alpha r < \varepsilon \). This implies that
\[
\alpha^2 r < \alpha \varepsilon.
\]
Let \( w = \alpha u \oplus (1 - \alpha)v \). Then by Lemma 2.2(i) and (3.2), we have
\[
d^2(x_{n_k}, w) \leq \alpha d^2(x_{n_k}, u) + (1 - \alpha) d^2(x_{n_k}, v) - \alpha(1 - \alpha) d^2(u, v)
< \alpha^2 d^2(x_{n_k}, u) + (1 - \alpha)^2 d^2(x_{n_k}, v).
\]
This together with (3.3) imply that
\[
\limsup_{n \to \infty} d^2(x_{n_k}, w) \leq \alpha^2 r + (1 - \alpha) \varepsilon < \varepsilon,
\]
contradicting the fact that \( A(\{x_{n_k}\}) = \{v\} \). Therefore \( \lim_{k \to \infty} x_{n_k} = v \). Since \( x, y \in \text{Fix}(T) \), then by (3.2), we have
\[ d(u, x_n) + d(x_n, x) = d(u, x) \]

and

\[ d(u, x_n) + d(x_n, y) = d(u, y). \]

These imply, by letting \( k \to \infty \), that

\[ d(u, v) + d(v, x) = d(u, x) \tag{3.4} \]

and

\[ d(u, v) + d(v, y) = d(u, y). \tag{3.5} \]

Since \( x \) and \( y \) are respectively the unique nearest points of \( u \) in \( \text{Fix}(f) \) and \( \text{Fix}(T) \), by (3.4) and (3.5) we have \( x = v = y \) and the proof is complete. \( \square \)

The following theorem is our main result.

**Theorem 3.6.** Let \( E \) be a nonempty closed convex subset of a complete \( \mathbb{R} \)-tree \( X, T : E \to K\text{C}(E) \) be a multivalued nonexpansive mapping and \( u \in E \). Then \( T \) has a fixed point if and only if \( \{x_s\} \) given by (3.1) remains bounded as \( s \to 0 \). In this case, the following statements hold:

(i) \( \{x_s\} \) converges to the unique fixed point \( z \) of \( T \) which is nearest \( u \).

(ii) If \( \{u_n\} \) is a bounded sequence in \( E \) having \( \lim_{n \to \infty} \text{dist}(u_n, T(u_n)) = 0 \), then

\[ d^2(u, z) \leq \mu_n d^2(u, u_n) \]

for all Banach limits \( \mu \).

**Proof.** We note that \( x_s = su \oplus (1 - s)f(x_s) \) and \( f \) is nonexpansive. Thus by Lemma 2.6 and Lemma 3.5(i), \( T \) has a fixed point if and only if \( \{x_s\} \) is bounded.

(i) Follows from Lemma 2.6(i) and Lemma 3.5(ii).

(ii) The proof is similar to the one given in [6]. For convenience of the reader we include the details. Let \( \{u_n\} \) be a bounded sequence in \( E \) such that \( \lim_n \text{dist}(u_n, T(u_n)) = 0 \). Let \( \mu \) be a Banach limit and suppose \( \mu_n d^2(u, u_n) < \rho < \gamma < d^2(u, z) \). Thus there exists a subsequence \( \{u_{n_k}\} \) with

\[ d^2(u, u_{n_k}) < \gamma \quad \text{for all } k. \tag{3.6} \]

Otherwise \( d^2(u, u_{n_k}) \geq \gamma \) for all large \( n \) which implies \( \mu_n d^2(u, u_n) \geq \gamma > \rho \), a contradiction, and therefore (3.6) holds. We can assume that \( \{u_{n_k}\} \) is a regular subsequence. Since \( \lim_{k \to \infty} \text{dist}(u_{n_k}, T(u_{n_k})) = 0 \), if \( A(\{u_{n_k}\}) = \{w\} \), then \( w \in \text{Fix}(T) \) by Lemma 2.4. Then by (3.6) and Lemma 2.3, \( w \in \overline{B}(u, \sqrt{\gamma}) \) which is contradicting the fact that \( z \) is the nearest point in \( \text{Fix}(T) \) to \( u \). This concludes that \( d^2(u, z) \leq \mu_n d^2(u, u_n) \). \( \square \)

Let \( T : E \to K\text{C}(E) \) be a nonexpansive mapping and \( \{\alpha_n\} \) be a sequence in \( (0, 1) \). Fix \( u, z_1 \in C \). Let \( y_1 \in T(z_1) \) be the gate of \( u \) in \( T(z_1) \). Define

\[ z_2 = \alpha_1 u \oplus (1 - \alpha_1) y_1. \]

Let \( y_2 \in T(z_2) \) be the gate of \( u \) in \( T(z_2) \). Inductively, we have

\[ z_{n+1} = \alpha_n u \oplus (1 - \alpha_n) y_n, \tag{3.7} \]

where \( y_n \in T(z_n) \) is the gate of \( u \) in \( T(z_n) \).

Now, we obtain a strong convergence theorem of Halpern's iteration [12] for multivalued nonexpansive mappings.

**Theorem 3.7.** Let \( E \) be a nonempty closed convex subset of a complete \( \mathbb{R} \)-tree \( X, T : E \to K\text{C}(E) \) be multivalued nonexpansive mapping such that \( \text{Fix}(T) \neq \emptyset \). Suppose that \( u, z_1 \in E \) are arbitrarily chosen and \( \{z_n\} \) is defined by (3.7), where \( \{\alpha_n\} \) is a sequence in \( (0, 1) \) satisfying

\[ \begin{align*}
(C1) & \ \lim_{n \to \infty} \alpha_n = 0; \\
(C2) & \ \sum_{n=1}^{\infty} \alpha_n = \infty; \\
(C3) & \ \sum_{n=1}^{\infty} |\alpha_n - \alpha_{n+1}| < \infty \quad \text{or} \quad \lim_{n \to \infty} (\alpha_n/\alpha_{n+1}) = 1.
\end{align*} \]

Then \( \{z_n\} \) converges to the unique point of \( \text{Fix}(T) \) which is nearest to \( u \).
Proof. We observe that $z_{n+1} = \alpha_n u \oplus (1 - \alpha_n)f(z_n)$. The conclusion follows from Lemmas 2.7 and 3.5.

Finally, we finish the paper with the following question:

**Question 3.8.** In the original Jung’s theorem, the mapping $T$ is assumed to take compact values while in Theorem 3.6 $T$ takes compact and convex values. Does Theorem 3.6 hold if $T$ is only assumed to take compact values?

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**References**


