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Research Article

Approximating Curve and Strong Convergence of the *CQ* Algorithm for the Split Feasibility Problem

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Using the idea of Tikhonov's regularization, we present properties of the approximating curve for the split feasibility problem (SFP) and obtain the minimum-norm solution of SFP as the strong limit of the approximating curve. It is known that in the infinite-dimensional setting, Byrne's *CQ* algorithm (Byrne, 2002) has only weak convergence. We introduce a modification of Byrne's *CQ* algorithm in such a way that strong convergence is guaranteed and the limit is also the minimum-norm solution of SFP.

1. Introduction

Let C and Q be nonempty closed convex subsets of real Hilbert spaces $\mathscr{H}_1$ and $\mathscr{H}_2$, respectively. The problem under consideration in this article is formulated as finding a point x satisfying the property:

$$x \in C, \qquad Ax \in Q, \tag{1.1}$$

where $A : \mathscr{H}_1 \to \mathscr{H}_2$ is a bounded linear operator. Problem (1.1), referred to by Censor and Elfving [1] as the split feasibility problem (SFP), attracts many authors' attention due to its application in signal processing [1]. Various algorithms have been invented to solve it (see [2–7] and reference therein).

In particular, Byrne [2] introduced the so-called *CQ* algorithm. Take an initial guess $x_0 \in \mathscr{H}_1$ arbitrarily, and define $(x_n)_{n\geq 0}$ recursively as

$$x_{n+1} = P_C\left(I - \gamma A^*(I - P_Q)A\right)x_n, \tag{1.2}$$

where $0 < \gamma < 2/\rho(A^*A)$ and where P_C denotes the projector onto C and $\rho(A^*A)$ is the spectral radius of the self-adjoint operator A^*A. Then the sequence $(x_n)_{n\geq 0}$ generated by (1.2) converges strongly to a solution of SFP whenever $\mathscr{H}_1$ is finite-dimensional and whenever there exists a solution to SFP (1.1).

However, the CQ algorithm need not necessarily converge strongly in the case when $\mathscr{H}_1$ is infinite dimensional. Let us mention that the CQ algorithm can be regarded as a special case of the well-known Krasnosel'skii-Mann algorithm for approximating fixed points of nonexpansive mappings [3]. This iterative method is introduced in [8] and defined as follows. Take an initial guess $x_0 \in C$ arbitrarily, and define $(x_n)_{n\geq 0}$ recursively as

$$x_{n+1} = \alpha_n x_n + (1-\alpha_n)Tx_n, \tag{1.3}$$

where $\alpha_n \in [0,1]$ satisfying $\sum_{n=0}^{\infty} \alpha_n(1-\alpha_n) = \infty$. If T is nonexpansive with a nonempty fixed point set, then the sequence $(x_n)_{n\geq 0}$ generated by (1.3) converges weakly to a fixed point of T. It is known that Krasnosel'skii-Mann algorithm is in general not strongly convergent (see [9, 10] for counterexamples) and neither is the CQ algorithm.

It is therefore the aim of this paper to construct a new algorithm so that strong convergence is guaranteed. The paper is organized as follows. In the next section, some useful lemmas are given. In Section 3, we define the concept of the minimal norm solution of SFP (1.1). Using Tikhonov's regularization, we obtain a continuous curve for approximating such minimal norm solution. Together with some properties of this approximating curve, we introduce, in Section 4, a modification of Byrne's CQ algorithm so that strong convergence is guaranteed and its limit is the minimum-norm solution of SFP (1.1).

2. Preliminaries

Throughout the rest of this paper, I denotes the identity operator on $\mathscr{H}_1$, $\mathrm{Fix}(T)$ the set of the fixed points of an operator T and ∇f the gradient of the functional $f : \mathscr{H}_1 \to \mathbb{R}$. The notation "$\to$" denotes strong convergence and "$\rightharpoonup$" weak convergence.

Recall that an operator T from $\mathscr{H}_1$ into itself is called nonexpansive if

$$\|Tx - Ty\| \leq \|x - y\|, \quad x, y \in \mathscr{H}_1; \tag{2.1}$$

contractive if there exists $0 < \alpha < 1$ such that

$$\|Tx - Ty\| \leq \alpha\|x - y\|, \quad x, y \in \mathscr{H}_1; \tag{2.2}$$

monotone if

$$\langle Tx - Ty, x - y\rangle \geq 0, \quad x, y \in \mathscr{H}_1. \tag{2.3}$$

Obviously, contractions are nonexpansive, and if T is nonexpansive, then $I - T$ is monotone (see [11]).

Let P_C denote the projection from $\mathscr{H}_1$ onto a nonempty closed convex subset C of $\mathscr{H}_1$; that is,

$$P_C x = \operatorname*{argmin}_{y \in C} \|x - y\|, \quad x \in \mathscr{H}_1. \tag{2.4}$$

It is well known that $P_C x$ is characterized by the inequality

$$\langle x - P_C x, c - P_C x \rangle \leq 0, \quad c \in C. \tag{2.5}$$

Consequently, P_C is nonexpansive.

The lemma below is referred to as the demiclosedness principle for nonexpansive mappings (see [12]).

Lemma 2.1 (demiclosedness principle). *Let C be a nonempty closed convex subset of $\mathscr{H}_1$ and $T : C \to C$ a nonexpansive mapping with $\operatorname{Fix}(T) \neq \emptyset$. If $(x_n)_{n \geq 1}$ is a sequence in C weakly converging to x and if the sequence $((I - T)x_n)$ converges strongly to y, then $(I - T)x = y$. In particular, if $y = 0$, then $x \in \operatorname{Fix}(T)$.*

Let $f : \mathscr{H}_1 \to \mathbb{R}$ be a a functional. Recall that

(i) f is convex if

$$f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda) f(y), \quad \forall 0 < \lambda < 1, \ \forall x, y \in \mathscr{H}_1; \tag{2.6}$$

(ii) f is strictly convex if the strict less than inequality in (2.6) holds for all distinct $x, y \in \mathscr{H}_1$.

(iii) f is strongly convex if there exists a constant $\alpha > 0$ such that

$$f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda) f(y) - \alpha \|x - y\|^2, \quad \forall 0 < \lambda < 1, \ \forall x, y \in \mathscr{H}_1; \tag{2.7}$$

(iv) f is coercive if $f(x) \to \infty$ whenever $\|x\| \to \infty$. It is easily seen that if f is strongly convex, then it is coercive. See [13] for more details about convex functions.

The following lemma gives the optimality condition for the minimizer of a convex functional over a closed convex subset.

Lemma 2.2 (see [14]). *Let f be a convex and differentiable functional and let C be a closed convex subset of $\mathscr{H}_1$. Then $x \in C$ is a solution of the problem*

$$\operatorname*{minimize}_{x \in C} f(x) \tag{2.8}$$

if and only if $x \in C$ satisfies the following optimality condition:

$$\langle \nabla f(x), v - x \rangle \geq 0, \quad \forall v \in C. \tag{2.9}$$

Moreover, if f is, in addition, strictly convex and coercive, then problem (2.8) *has a unique solution.*

The following is a sufficient condition for a real sequence to converge to zero.

Lemma 2.3 (see [15]). *Let $(a_n)_{n\geq 0}$ be a nonnegative real sequence satisfying*

$$a_{n+1} \leq (1-r_n)a_n + r_n\mu_n, \tag{2.10}$$

where the sequences $(r_n)_{n\geq 0} \subset (0,1)$ and $(\mu_n)_{n\geq 0}$ satisfy the conditions:

(1) $\sum_{n=0}^{\infty} r_n = \infty$;

(2) $\lim_{n\to\infty} r_n = 0$;

(3) *either* $\sum_{n=0}^{\infty} |r_n\mu_n| < \infty$ *or* $\limsup_{n\to\infty}\mu_n \leq 0$.

Then $\lim_{n\to\infty} a_n = 0$.

3. Approximating Curves

The convexly constrained linear problem requires to solve the constrained linear system (cf. [16, 17])

$$\begin{aligned} Ax &= b, \\ x &\in C, \end{aligned} \tag{3.1}$$

where $b \in \mathscr{H}_2$. A classical way to deal with such a possibly ill-posed problem is the well-known Tikhonov regularization, which approximates a solution of problem (3.1) by the unique minimizer of the regularized problem

$$\min_{x\in C} \|Ax-b\|^2 + \alpha\|x\|^2, \tag{3.2}$$

where $\alpha > 0$ is known as the regularization parameter.

We now try to transfer this idea of Tikhonov's regularization method for solving the constrained linear inverse problem (3.1) to the case of SFP (1.1).

It is not hard to find that SFP (1.1) is equivalent to the minimization problem

$$\min_{x\in C} \left\| (I-P_Q)Ax \right\|. \tag{3.3}$$

Motivated by Tikhonov's regularization, we consider the minimization problem

$$\min_{x\in C} \left\| (I-P_Q)Ax \right\|^2 + \alpha\|x\|^2, \tag{3.4}$$

where $\alpha > 0$ is the regularization parameter. Denote by x_α the unique solution of (3.4); namely,

$$x_\alpha := \operatorname*{argmin}_{x\in C} \left\{ \left\| (I-P_Q)Ax \right\|^2 + \alpha\|x\|^2 \right\}. \tag{3.5}$$

Proposition 3.1. *For any $\alpha > 0$, the minimizer x_α given by* (3.5) *is uniquely defined. Moreover, x_α is characterized by the inequality*

$$\langle A^*(I - P_Q)Ax_\alpha + \alpha x_\alpha, c - x_\alpha \rangle \geq 0, \quad c \in C. \tag{3.6}$$

Proof. Let

$$f(x) = \|(I - P_Q)Ax\|^2, \qquad f_\alpha(x) = f(x) + \alpha\|x\|^2. \tag{3.7}$$

Since f is convex and differentiable with gradient (see [13])

$$\nabla f(x) = 2A^*(I - P_Q)Ax, \tag{3.8}$$

f_α is strictly convex, coercive, and differentiable with gradient

$$\nabla f_\alpha(x) = 2A^*(I - P_Q)Ax + 2\alpha x. \tag{3.9}$$

Thus, applying Lemma 2.2 gets the assertion (3.6), as desired. □

The next result collects some useful properties of $(x_\alpha)_{\alpha>0}$.

Proposition 3.2. *The following assertions hold.*

(a) $\|x_\alpha\|$ *is decreasing for* $\alpha \in (0, \infty)$.

(b) $\|(I - P_Q)Ax_\alpha\|$ *is increasing for* $\alpha \in (0, \infty)$.

(c) $\alpha \mapsto x_\alpha$ *defines a continuous curve from* $(0, \infty)$ *to* $\mathscr{H}_1$.

Proof. Let $\alpha > \beta > 0$ be fixed. Since x_α and x_β are the (unique) minimizers of f_α and f_β, respectively, we get

$$\|(I - P_Q)Ax_\alpha\|^2 + \alpha\|x_\alpha\|^2 \leq \|(I - P_Q)Ax_\beta\|^2 + \alpha\|x_\beta\|^2, \tag{3.10}$$

$$\|(I - P_Q)Ax_\beta\|^2 + \beta\|x_\beta\|^2 \leq \|(I - P_Q)Ax_\alpha\|^2 + \beta\|x_\alpha\|^2. \tag{3.11}$$

Adding up (3.10) and (3.11) yields

$$\alpha\|x_\alpha\|^2 + \beta\|x_\beta\|^2 \leq \alpha\|x_\beta\|^2 + \beta\|x_\alpha\|^2, \tag{3.12}$$

which implies that $\|x_\alpha\| \leq \|x_\beta\|$. Hence (a) holds.

It follows from (3.11) that

$$\|(I - P_Q)Ax_\beta\|^2 \leq \|(I - P_Q)Ax_\alpha\|^2 + \beta\left(\|x_\alpha\|^2 - \|x_\beta\|^2\right), \tag{3.13}$$

which together with (a) implies

$$\left\|(I-P_Q)Ax_\beta\right\| \leq \left\|(I-P_Q)Ax_\alpha\right\|, \tag{3.14}$$

and therefore (b) holds.

By Proposition 3.1, we have that

$$\left\langle A^*(I-P_Q)Ax_\alpha + \alpha x_\alpha, x_\beta - x_\alpha\right\rangle \geq 0, \tag{3.15}$$

and also that

$$\left\langle A^*(I-P_Q)Ax_\beta + \beta x_\beta, x_\alpha - x_\beta\right\rangle \geq 0. \tag{3.16}$$

Adding up (3.15) and (3.16), we get

$$\left\langle \alpha x_\alpha - \beta x_\beta, x_\alpha - x_\beta\right\rangle \leq \left\langle (I-P_Q)Ax_\alpha - (I-P_Q)Ax_\beta, Ax_\beta - Ax_\alpha\right\rangle. \tag{3.17}$$

Since $I-P_Q$ is monotone, we obtain from the last relation

$$\alpha\left\|x_\alpha - x_\beta\right\|^2 \leq (\beta-\alpha)\left\langle x_\beta, x_\alpha - x_\beta\right\rangle. \tag{3.18}$$

It turns out that

$$\left\|x_\alpha - x_\beta\right\| \leq \frac{|\alpha-\beta|}{\alpha}\left\|x_\beta\right\|. \tag{3.19}$$

Thus (c) holds. □

Let $\mathcal{F} = C \cap A^{-1}(Q)$, where $A^{-1}(Q) = \{x \in \mathcal{H}_1 : Ax \in Q\}$. In what follows, we assume that $\mathcal{F} \neq \emptyset$; that is, the solution set of SFP (1.1) is nonempty. The fact that $\mathcal{F}$ is nonempty closed convex set thus allows us to introduce the concept of minimum-norm solution of SFP (1.1).

Definition 3.3. An element $\tilde{x} \in \mathcal{F}$ is said to be the minimal norm solution of SFP (1.1) if $\|\tilde{x}\| = \inf_{x\in\mathcal{F}}\|x\|$. In other words, $\tilde{x}$ is the projection of the origin onto the solution set $\mathcal{F}$ of SFP (1.1). Thus the minimum-norm solution $\tilde{x}$ for SFP (1.1) exists and is unique.

Theorem 3.4. *Let x_α be given as* (3.5). *Then x_α converges strongly as $\alpha \to 0$ to the minimum-norm solution $\tilde{x}$ of SFP* (1.1).

Proof. We first show that the inequality

$$\|x_\alpha\| \leq \|\tilde{x}\| \tag{3.20}$$

holds for any $0 < \alpha < \infty$. To this end, observe that

$$\left\|(I-P_Q)Ax_\alpha\right\|^2 + \alpha\|x_\alpha\|^2 \leq \left\|(I-P_Q)A\tilde{x}\right\|^2 + \alpha\|\tilde{x}\|^2. \tag{3.21}$$

Since $\tilde{x} \in C \cap A^{-1}(Q)$, $(I - P_Q)A\tilde{x} = 0$. It follows from (3.21) that

$$\|x_\alpha\|^2 \le \|\tilde{x}\|^2 - \frac{\left\|(I - P_Q)Ax_\alpha\right\|^2}{\alpha} \le \|\tilde{x}\|^2, \tag{3.22}$$

and (3.20) is proven.

Let now $(\alpha_n)_{n\ge0}$ be a sequence such that $\alpha_n \to 0$ as $n \to \infty$ and let x_{α_n} be abbreviated as x_n. All we need to prove is that $(x_n)_{n\ge0}$ contains a subsequence converging strongly to $\tilde{x}$. Since $(x_n)_{n\ge0}$ is bounded and since C is bounded convex, by passing to a subsequence if necessary, we may assume that $(x_n)_{n\ge0}$ converges weakly to a point $w \in C$. By Proposition 3.1, we deduce that

$$\left\langle A^*(I - P_Q)Ax_n + \alpha_n x_n, \tilde{x} - x_n \right\rangle \ge 0. \tag{3.23}$$

It turns out that

$$\left\langle (I - P_Q)Ax_n, A(\tilde{x} - x_n)\right\rangle \ge \alpha_n \langle x_n, x_n - \tilde{x}\rangle. \tag{3.24}$$

Since $A\tilde{x} \in Q$, the characterizing inequality (2.5) gives

$$\left\langle (I - P_Q)Ax_n, A\tilde{x} - P_Q Ax_n \right\rangle \le 0, \tag{3.25}$$

and this implies that

$$\left\|(I - P_Q)Ax_n\right\|^2 \le \langle (I - P_Q)Ax_n, A(x_n - \tilde{x})\rangle. \tag{3.26}$$

Now by combining (3.26) and (3.24), we get

$$\left\|(I - P_Q)Ax_n\right\|^2 \le \alpha_n \langle x_n, \tilde{x} - x_n\rangle \le 2\alpha_n \|\tilde{x}\|^2, \tag{3.27}$$

where the last inequality follows from (3.20). Consequently, we get

$$\lim_{n\to\infty} \left\|(I - P_Q)Ax_n\right\| = 0. \tag{3.28}$$

Note that A is also weakly continuous and hence $Ax_n \rightharpoonup Aw$. Now due to (3.28), we can use the demiclosedness principle (Lemma 2.1) to conclude that $(I - P_Q)Aw = 0$. That is, $Aw \in Q$ or $w \in A^{-1}(Q)$; therefore, $w \in \mathcal{F}$. We next prove that $w = \tilde{x}$ and this finishes the proof. To see this, we have that the weak convergence to w of $\{x_n\}$ together with (3.20) implies that

$$\|w\| \le \liminf_{n\to\infty} \|x_n\| \le \|\tilde{x}\| = \min\{\|x\| : x \in \mathcal{F}\}. \tag{3.29}$$

This shows that w is also a point in $\mathcal{F}$ which assumes minimum norm. Due to uniqueness of minimum-norm element, we must have $w = \tilde{x}$. □

Remark 3.5. The above argument shows that if the solution set $\mathcal{F}$ of SFP (1.1) is empty, then the net of norms, $(\|x_\alpha\|)$, diverges to ∞ as $\alpha \to 0$.

4. A Modified CQ Algorithm

It is a standard way to use contractions to approximate nonexpansive mappings. We follow this idea and use contractions to approximate the nonexpansive mapping $I - \gamma A^*(I - P_Q)A$ in order to modify Byrne's CQ algorithm. More precisely, we introduce the following algorithm which is viewed as a modification of Byrne's CQ algorithm. The purpose for such a modification lies in the hope of strong convergence.

Algorithm 4.1. For an arbitrary guess x_0, the sequence $(x_n)_{n\geq 0}$ is generated by the iterative algorithm

$$x_{n+1} = P_C\left[(1-\alpha_n)\left(I - \gamma A^*\left(I - P_Q\right)A\right)\right]x_n, \tag{4.1}$$

where $(\alpha_n)_{n\geq 0}$ is a sequence in $(0,1)$ such that

(1) $\lim_{n\to\infty}\alpha_n = 0$;

(2) $\sum_{n=0}^{\infty}\alpha_n = \infty$;

(3) either $\sum_{n=0}^{\infty}|\alpha_{n+1} - \alpha_n| < \infty$ or $\lim_{n\to\infty}|\alpha_{n+1} - \alpha_n|/\alpha_n = 0$.

Note that a prototype of (α_n) is $\alpha_n = (1+n)^{-1}$ for all $n \geq 0$.

To prove the convergence of algorithm (4.1) (see Theorem 4.3 below), we need a lemma below.

Lemma 4.2. *Set $U = I - \gamma A^*(I - P_Q)A$, where $0 < \gamma < 2/\rho(A^*A)$ with $\rho(A^*A)$ being the spectral radius of the self-adjoint operator A^*A.*

(i) *U is an averaged mapping; namely, $U = (1-\beta)I + \beta V$, where $\beta \in (0,1)$ is a constant and V is a nonexpansive mapping from $\mathcal{H}_1$ into itself.*

(ii) $\mathrm{Fix}(U) = A^{-1}(Q)$*; consequently,* $\mathrm{Fix}(P_C U) = \mathrm{Fix}(P_C) \cap \mathrm{Fix}(U) = \mathcal{F} = C \cap A^{-1}(Q)$.

Proof. (i) That U which is averaged is actually proved in [3].

To see (ii), we first observe that the inclusion $A^{-1}(Q) \subset \mathrm{Fix}(U)$ holds trivially. It remains to prove the implication: $x = Ux \Rightarrow Ax \in Q$. To see this, we notice that the relation $x = Ux$ is equivalent to the relation $x = x - \gamma A^*(I - P_Q)Ax$. It turns out that

$$A^*\left(I - P_Q\right)Ax = 0. \tag{4.2}$$

Since the solution set $\mathcal{F} = C \cap A^{-1}(Q) \neq \emptyset$, we can take $z \in \mathcal{F}$. Now since $Az \in Q$, we have by (2.5),

$$\left\langle \left(I - P_Q\right)Ax, Az - P_Q Ax\right\rangle \leq 0. \tag{4.3}$$

It follows from (4.2) and (4.3) that

$$\begin{aligned}\left\|(I-P_Q)Ax\right\|^2 &= \langle (I-P_Q)Ax, Ax-Az\rangle + \langle (I-P_Q)Ax, Az-P_QAx\rangle \\ &\le \langle (I-P_Q)Ax, Ax-Az\rangle \\ &= \langle A^*(I-P_Q)Ax, x-z\rangle \\ &= 0.\end{aligned} \tag{4.4}$$

This shows that $Ax = P_Q(Ax) \in Q$; that is, $x \in A^{-1}(Q)$.

Finally, since $\operatorname{Fix}(P_C) \cap \operatorname{Fix}(U) = C \cap A^{-1}(Q) = \mathcal{F} \neq \emptyset$, and both P_C and U are averaged, we have $\operatorname{Fix}(P_C U) = \operatorname{Fix}(P_C) \cap \operatorname{Fix}(U) = \mathcal{F}$. □

Theorem 4.3. *The sequence* $(x_n)_{n\ge 0}$ *generated by algorithm* (4.1) *converges strongly to the minimum-norm solution* $\tilde{x}$ *of SFP* (1.1)*.*

Proof. Define operators T_n and T on $\mathcal{H}_1$ by

$$\begin{gathered}T_n x := P_C\left[(1-\alpha_n)(I-\gamma A^*(I-P_Q)A)\right]x = P_C[(1-\alpha_n)U]x, \quad x \in \mathcal{H}_1, \\ Tx := P_C\left[(I-\gamma A^*(I-P_Q)A)\right]x = P_C U x, \quad x \in \mathcal{H}_1,\end{gathered} \tag{4.5}$$

where $U = I - \gamma A^*(I-P_Q)A$ is averaged by Lemma 4.2.

It is readily seen that T_n is a contraction with contractive constant $1-\alpha_n$. Namely,

$$\left\|T_n x - T_n y\right\| \le (1-\alpha_n)\left\|x-y\right\|, \quad x, y \in \mathcal{H}_1. \tag{4.6}$$

Also we may rewrite algorithm (4.1) as

$$x_{n+1} = T_n x_n = P_C[(1-\alpha_n)U]x_n. \tag{4.7}$$

We first prove that (x_n) is a bounded sequence. Indeed, since $\mathcal{F} \neq \emptyset$, we can take any $\hat{x} \in \mathcal{F}$ (thus $\hat{x} = U\hat{x}$ by Lemma 4.2) to deduce that

$$\|x_{n+1} - \hat{x}\| = \|T_n x_n - \hat{x}\| \le \|T_n x_n - T_n \hat{x}\| + \|T_n \hat{x} - \hat{x}\|. \tag{4.8}$$

Note that

$$\begin{aligned}\|T_n\hat{x} - \hat{x}\| &= \|P_C[(1-\alpha_n)U]\hat{x} - P_C U\hat{x}\| \\ &\le \|(1-\alpha_n)U\hat{x} - U\hat{x}\| \\ &= \alpha_n\|U\hat{x}\| = \alpha_n\|\hat{x}\|.\end{aligned} \tag{4.9}$$

Substituting (4.9) into (4.8), we get

$$\begin{aligned}\|x_{n+1}-\widehat{x}\| &\le (1-\alpha_n)\|x_n-\widehat{x}\|+\alpha_n\|\widehat{x}\| \\ &\le \max\{\|x_n-\widehat{x}\|,\|\widehat{x}\|\}.\end{aligned} \tag{4.10}$$

By induction, we can easily show that, for all $n \ge 0$,

$$\|x_n-\widehat{x}\| \le \max\{\|x_0-\widehat{x}\|,\|\widehat{x}\|\}. \tag{4.11}$$

In particular, (x_n) is bounded.

We now claim that

$$\lim_{n\to\infty}\|x_{n+1}-x_n\|=0. \tag{4.12}$$

To see this, we compute

$$\begin{aligned}\|x_{n+1}-x_n\| &= \|T_nx_n-T_{n-1}x_{n-1}\| \\ &\le \|T_nx_n-T_nx_{n-1}\|+\|T_nx_{n-1}-T_{n-1}x_{n-1}\| \\ &\le (1-\alpha_n)\|x_n-x_{n-1}\|+\|T_nx_{n-1}-T_{n-1}x_{n-1}\|.\end{aligned} \tag{4.13}$$

Letting $M>0$ be a constant such that $M>\|Ux_n\|$ for all $n\ge 0$, we find

$$\begin{aligned}\|T_nx_{n-1}-T_{n-1}x_{n-1}\| &= \|P_C[(1-\alpha_n)Ux_{n-1}]-P_C[(1-\alpha_{n-1})Ux_{n-1}]\| \\ &\le \|(1-\alpha_n)Ux_{n-1}-(1-\alpha_{n-1})Ux_{n-1}\| \\ &= \|(\alpha_n-\alpha_{n-1})Ux_{n-1}\| \\ &\le M|\alpha_n-\alpha_{n-1}|.\end{aligned} \tag{4.14}$$

Substituting (4.14) into (4.13), we arrive at

$$\|x_{n+1}-x_n\| \le (1-\alpha_n)\|x_n-x_{n-1}\|+M|\alpha_n-\alpha_{n-1}|. \tag{4.15}$$

By virtue of the assumptions (a)–(c), we can apply Lemma 2.3 to (4.15) to obtain (4.12). Consequently we also have

$$\lim_{n\to\infty}\|x_n-Tx_n\|=0. \tag{4.16}$$

This follows from the following computations:

$$\begin{aligned}\|x_n - Tx_n\| &\le \|x_n - x_{n+1}\| + \|T_n x_n - Tx_n\| \\ &\le \|x_n - x_{n+1}\| + \|(1-\alpha_n)Ux_n - Ux_n\| \\ &\le \|x_n - x_{n+1}\| + M\alpha_n \longrightarrow 0.\end{aligned} \tag{4.17}$$

Therefore, the demiclosedness principle (Lemma 2.1) ensures that each weak limit point of (x_n) is a fixed point of the nonexpansive mapping $T = P_C U$, that is, a point of the solution set $\mathcal{F}$ of SFP (1.1).

One of the key ingredients of the proof is the following conclusion:

$$\limsup_{n\to\infty}\langle Ux_n - \tilde{x}, -\tilde{x}\rangle \le 0, \tag{4.18}$$

where $\tilde{x}$ is the minimum-norm element of $\mathcal{F}$ (i.e., the projection $P_{\mathcal{F}}(0)$). Since

$$\langle Ux_n - \tilde{x}, -\tilde{x}\rangle = \langle Ux_n - x_n, -\tilde{x}\rangle + \langle x_n - \tilde{x}, -\tilde{x}\rangle, \tag{4.19}$$

to prove (4.18), it suffices to prove that

$$\lim_{n\to\infty}\|Ux_n - x_n\| = 0, \tag{4.20}$$

$$\limsup_{n\to\infty}\langle x_n - \tilde{x}, -\tilde{x}\rangle \le 0. \tag{4.21}$$

To prove (4.20), we use Lemma 4.2 to get $\tilde{x} \in \mathrm{Fix}(U)$ and U is averaged. Write $U = (1-\beta)I + \beta V$ for some $\beta \in (0,1)$ and nonexpansive mapping V. Then we derive, by taking a point $z \in \mathcal{F}$, that

$$\begin{aligned}\|x_{n+1} - z\|^2 &= \|P_C[(1-\alpha_n)U]x_n - z\|^2 \\ &\le \|(1-\alpha_n)Ux_n - z\|^2 \\ &= \|(1-\alpha_n)(Ux_n - z) + \alpha_n(-z)\|^2 \\ &\le (1-\alpha_n)\|Ux_n - z\|^2 + \alpha_n\|z\|^2 \\ &\le \left\|(1-\beta)(x_n - z) + \beta(Vx_n - z)\right\|^2 + \alpha_n\|z\|^2 \\ &= (1-\beta)\|x_n - z\|^2 + \beta\|Vx_n - z\|^2 \\ &\quad - \beta(1-\beta)\|x_n - Vx_n\|^2 + \alpha_n\|z\|^2 \\ &\le \|x_n - z\|^2 - \beta(1-\beta)\|x_n - Vx_n\|^2 + \alpha_n\|z\|^2 \quad (\text{as } z = Vz).\end{aligned} \tag{4.22}$$

It turns out that (for some constant $M > \|x_n - z\|$ for all n)

$$\begin{aligned}\beta(1-\beta)\|x_n - Vx_n\|^2 &\leq \|x_n - z\|^2 - \|x_{n+1} - z\|^2 + \alpha_n\|z\|^2 \\ &\leq 2M\big|\|x_n - z\| - \|x_{n+1} - z\|\big| + \alpha_n\|z\|^2 \\ &\leq 2M\|x_n - x_{n+1}\| + \alpha_n\|z\|^2 \\ &\longrightarrow 0 \quad \text{by (4.12)}.\end{aligned} \tag{4.23}$$

Now since $I - U = \beta(I - V)$, (4.23) implies (4.20).

To prove (4.21), we take a subsequence $(x_{n'})$ of (x_n) so that

$$\limsup_{n\to\infty}\langle x_n - \widetilde{x}, -\widetilde{x}\rangle = \lim_{n'\to\infty}\langle x_{n'} - \widetilde{x}, -\widetilde{x}\rangle. \tag{4.24}$$

Since (x_n) is bounded, we may further assume with no loss of generality that $(x_{n'})$ converges weakly to a point $\widehat{x}$. Noticing that $\widehat{x} \in \mathrm{Fix}(T) = \mathcal{F}$ and that $\widetilde{x}$ is the projection of the origin onto $\mathcal{F}$, and applying (2.5), we arrive at

$$\limsup_{n\to\infty}\langle x_n - \widetilde{x}, -\widetilde{x}\rangle = \lim_{n'\to\infty}\langle x_{n'} - \widetilde{x}, -\widetilde{x}\rangle = \langle -\widetilde{x}, \widehat{x} - \widetilde{x}\rangle \leq 0. \tag{4.25}$$

This is (4.21).

Finally we prove $x_n \to \widetilde{x}$ in norm. To see this, we compute

$$\begin{aligned}\|x_{n+1} - \widetilde{x}\|^2 &= \|P_C[(1-\alpha_n)Ux_n] - P_C[U\widetilde{x}]\|^2 \\ &\leq \|(1-\alpha_n)Ux_n - U\widetilde{x}\|^2 \\ &\leq \|(1-\alpha_n)Ux_n - \widetilde{x}\|^2 \quad (\text{as } \widetilde{x} = U\widetilde{x}) \\ &= \|(1-\alpha_n)(Ux_n - \widetilde{x}) + \alpha_n(-\widetilde{x})\|^2 \\ &= (1-\alpha_n)^2\|Ux_n - \widetilde{x}\|^2 \\ &\quad + 2\alpha_n(1-\alpha_n)\langle Ux_n - \widetilde{x}, -\widetilde{x}\rangle + \alpha_n^2\|\widetilde{x}\|^2 \\ &\leq (1-\alpha_n)\|x_n - \widetilde{x}\|^2 \\ &\quad + 2\alpha_n(1-\alpha_n)\langle Ux_n - \widetilde{x}, -\widetilde{x}\rangle + \alpha_n^2\|\widetilde{x}\|^2 \\ &\leq (1-\alpha_n)\|x_n - \widetilde{x}\|^2 + \alpha_n\delta_n,\end{aligned} \tag{4.26}$$

where

$$\delta_n := 2(1-\alpha_n)\langle Ux_n - \widetilde{x}, -\widetilde{x}\rangle + \alpha_n\|\widetilde{x}\|^2 \tag{4.27}$$

satisfies the property (due to (4.18))

$$\limsup_{n\to\infty}\delta_n \leq 0. \tag{4.28}$$

We therefore can apply Lemma 2.3 to (4.26) to conclude that $\|x_n - \tilde{x}\|^2 \to 0$. This completes the proof. □

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