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Graded Lie Superalgebras and the Superdimension Formula*

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In this paper, we investigate the structure of graded Lie superalgebras $\mathscr{L} = \bigoplus_{(\alpha, a) \in \Gamma \times \mathscr{A}} \mathscr{L}_{(\alpha, a)}$, where Γ is a countable abelian semigroup and $\mathscr{A}$ is a countable abelian group with a coloring map satisfying a certain finiteness condition. Given a denominator identity for the graded Lie superalgebra $\mathscr{L}$, we derive a superdimension formula for the homogeneous subspaces $\mathscr{L}_{(\alpha, a)}$ $(\alpha \in \Gamma,\ a \in \mathscr{A})$, which enables us to study the structure of graded Lie superalgebras in a unified way. We discuss the applications of our superdimension formula to free Lie superalgebras, generalized Kac–Moody superalgebras, and Monstrous Lie superalgebras. In particular, the product identities for normalized formal power series are interpreted as the denominator identities for free Lie superalgebras. We also give a characterization of replicable functions in terms of product identities and determine the root multiplicities of Monstrous Lie superalgebras.

INTRODUCTION

The *Kac–Moody algebras* were introduced independently by Kac [K1] and Moody [Mo] as generalizations of complex finite dimensional simple Lie algebras. In [K2], Kac discovered a character formula, called the *Weyl–Kac formula* for integrable highest weight modules over symmetrizable Kac–Moody algebras, and showed that the Macdonald identities [M] are equivalent to the denominator identities for affine Kac–Moody algebras. Since then, the theory of Kac–Moody algebras (and more generally infinite dimensional Lie algebras) has attracted extensive research activities due to its rich and significant applications to many areas of mathematics and mathematical physics.

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The Kac–Moody theory has been extended to the theory of *generalized Kac–Moody algebras* by Borcherds in his study of the vertex algebras and *Monstrous Moonshine* [B1–B3, CN]. The structure and the representation theories of generalized Kac–Moody algebras are similar to those of Kac–Moody algebras, and a lot of facts about Kac–Moody algebras can be generalized to generalized Kac–Moody algebras [B2, K5]. For example, in [B2], Borcherds proved a character formula, called the *Weyl–Kac–Borcherds formula*, for the unitarizable highest weight modules over generalized Kac–Moody algebras. The main difference is that the generalized Kac–Moody algebras may have *imaginary simple roots* with norms ≤ 0 whose multiplicity can be > 1. The most interesting example of generalized Kac–Moody algebras may be the *Monster Lie algebra*, which played a crucial role in Borcherds' proof of the Moonshine conjecture [B3].

In [KaK3], we considered general graded Lie algebras $L = \bigoplus_{\alpha \in \Gamma} L_\alpha$ graded by a countable abelian semigroup Γ such that every element in Γ can be expressed as a sum of elements in Γ in only finitely many ways. The Euler–Poincaré principle for the graded Lie algebra $L = \bigoplus_{\alpha \in \Gamma} L_\alpha$ yields the *denominator identity*

$$\prod_{\alpha \in \Gamma} (1 - e^{\alpha})^{\dim L_\alpha} = 1 - \sum_{\alpha \in \Gamma} d(\alpha) e^{\alpha} \qquad \text{with } d(\alpha) \in \mathbf{Z} \ (\alpha \in \Gamma),$$

from which we derived a dimension formula for the homogeneous subspaces L_α $(\alpha \in \Gamma)$. Our dimension formula is a generalization of the root multiplicity formulas for Kac–Moody algebras and generalized Kac–Moody algebras given in [BM, Ka2, Ka3]. The applications of our dimension formula to various Lie algebras such as free Lie algebra, Kac–Moody algebras, and generalized Kac–Moody algebras were discussed in [KaK3] (see also [KaK1]).

On the other hand, since 1970s, the Lie superalgebras and their representation have emerged as fundamental algebraic structure behind several areas in mathematical physics. In [K3], Kac gave a comprehensive presentation of the mathematical theory of Lie superalgebras, and obtained an important classification theorem for finite dimensional simple Lie superalgebras over algebraically closed fields of characteristic zero (see also [S]). In [K4], the notion of *Kac–Moody superalgebras* was introduced, and for the *nondegenerate* Kac–Moody superalgebras, Kac proved the *Weyl–Kac character formula* for the irreducible highest weight modules with dominant integral highest weights, which yields the denominator identity when applied to the 1-dimensional trivial representation. By specializing to various classes of affine Kac–Moody superalgebras and irreducible highest weight modules, he derived a lot of interesting combinatorial identities that are closely related to number theory. Further developments of the theory of Lie superalgebras along this line can be found in [KW1, KW2], where Kac and Wakimoto presented some interesting applications of affine Kac−Moody superalgebras to number theory.

In [Mi, R], the representation theories of nondegenerate Kac−Moody superalgebras and generalized Kac−Moody algebras were combined to give rise to the representation theory of *generalized Kac−Moody superalgebras*. Following the outline of [K4, K5], Miyamoto and Ray independently developed the representation theory of generalized Kac−Moody superalgebras. In particular, they proved the *Weyl−Kac−Borcherds character formula* for the irreducible highest weight modules over generalized Kac−Moody superalgebras with dominant integral highest weights, and hence obtained the denominator identity. In [KaK2], using the Weyl−Kac−Borcherds formula and the denominator identity, we obtain a closed form root multiplicity formula for all generalized Kac−Moody superalgebras, and discussed its applications to several special cases.

In this paper, we consider a very general setting. Let Γ be a countable (usually infinite) abelian semigroup and $\mathscr{A}$ be a countable (usually finite) abelian group satisfying a certain finiteness condition. Consider the class of $(\Gamma \times \mathscr{A})$-graded Lie superalgebras $\mathfrak{L} = \bigoplus_{(\alpha, a) \in \Gamma \times \mathscr{A}} \mathfrak{L}_{(\alpha, a)}$ with $\dim \mathfrak{L}_{(\alpha, a)} < \infty$ for all $(\alpha, a) \in \Gamma \times \mathscr{A}$. There are many interesting and important Lie superalgebras belonging to this class of Lie superalgebras. For example, the free Lie superalgebras generated by graded superspaces with finite dimensional homogeneous subspaces are of this kind, and so are the positive or negative parts of finite dimensional simple Lie superalgebras, Kac−Moody superalgebras, and generalized Kac−Moody superalgebras. In this work, we propose a general method for investigating the structure of graded Lie superalgebras in a unified way, and discuss its applications to various classes of Lie superalgebras such as free Lie superalgebras, generalized Kac−Moody superalgebras, and Monstrous Lie superalgebras.

In Section 1, we recall the basic theory of Lie superalgebras, and show that, if we are given the denominator identity

$$\prod_{(\alpha, a) \in \Gamma \times \mathscr{A}} \left(1 - E^{(\alpha, a)}\right)^{\operatorname{Dim} \mathfrak{L}_{(\alpha, a)}} = 1 - \sum_{(\alpha, a) \in \Gamma \times \mathscr{A}} t(\alpha, a) E^{(\alpha, a)},$$

we can derive a closed form *superdimension formula* for the homogeneous subspaces of graded Lie superalgebras (Theorem 1.2). Our superdimension formula will be expressed in terms of the *Witt partition functions* associated with the partitions of the elements in $\Gamma \times \mathscr{A}$. We believe that the most natural way to derive the denominator identity is to use the Euler−Poincaré principle for the homology of graded Lie superalgebras. However, in this work, we will not use the Euler−Poincaré principle to derive the denominator identity, for not very much has been known about the homology of graded Lie superalgebras. For example, Kostant-type formulas for Kac–Moody superalgebras or generalized Kac–Moody superalgebras are not yet available. Therefore, instead of making use of the Euler–Poincaré principle, we will use the Poincaré–Birkhoff–Witt Theorem for free Lie superalgebras, and the Weyl–Kac–Borcherds formula for generalized Kac–Moody superalgebras, respectively, to derive the denominator identities for the corresponding Lie superalgebras.

In Section 2, we discuss the applications of our superdimension formula to the free Lie superalgebras $\mathfrak{L} = \bigoplus_{(\alpha, a) \in \Gamma \times \mathscr{A}} \mathfrak{L}_{(\alpha, a)}$ generated by graded superspaces $V = \bigoplus_{(\alpha, a) \in \Gamma \times \mathscr{A}} V_{(\alpha, a)}$ with $\dim V_{(\alpha, a)} < \infty$ for all $\alpha \in \Gamma$, $a \in \mathscr{A}$. When applied to free Lie superalgebras, our superdimension formula will be called the *generalized Witt formula* (Theorem 2.1; see also [Ka5]). We also compare the structure of free Lie algebras and free Lie superalgebras generated by the same vector spaces, and discuss the application of the generalized Witt formula to the product identities for normalized formal power series. In particular, any such product identity can be interpreted as the denominator identity for a suitably defined free Lie superalgebra, and we obtain a number of interesting combinatorial identities arising from the identity (2.36). For example, by applying the above idea to the automorphic forms with infinite product expansions given in [B4], we obtain some interesting relations for the Fourier coefficients of the corresponding modular functions.

In Section 3, we apply our superdimension formula to generalized Kac–Moody superalgebra to derive a closed form root multiplicity formula (Theorem 3.4). The *generalized Kac–Moody superalgebras* arise naturally in the context of Monstrous Moonshine [B2, B3], automorphic forms with infinite product expansions [B4, GN1–GN3], and the string theory [HM]. Our root multiplicity formula enables us to study the structure of a generalized Kac–Moody superalgebra $\mathfrak{g}$ as a representation of a Kac–Moody algebra or a Kac–Moody superalgebra $\mathfrak{g}_0$ contained in $\mathfrak{g}$. As an application of this idea, some generalized Kac–Moody algebras will be shown to be the maximal graded Lie algebras with local part $V \oplus \mathfrak{g}_0 \oplus V^*$, where $\mathfrak{g}_0$ is a Kac–Moody superalgebra contained in $\mathfrak{g}$, V is the direct sum of irreducible highest weight modules over $\mathfrak{g}_0$ with highest weight $-\alpha_i$ (α_i runs over all imaginary simple roots counted with multiplicities), and V^* is the contragredient module of V (Proposition 3.6, see also [K3]; for Kac–Moody algebras and generalized Kac–Moody algebras, see [BKM, Ju2, JW, K1]). The choice of $\mathfrak{g}_0$ in our formula gives rise to various expressions of the root multiplicities of $\mathfrak{g}$, which would yield combinatorial identities. We will discuss the applications of this idea elsewhere.

Finally, in Section 4, we define the notion of *Monstrous Lie superalgebras* as generalizations of those given in [B3]. The monstrous Lie superalgebras form a special class of generalized Kac–Moody superalgebras associated with normalized q-series $F(q) = \sum_{n=-1}^{\infty} f(n)q^n$ such that $f(-1) = 1$, $f(0) = 0$, and $f(n) \in \mathbf{Z}$ for all $n \geq 1$. We apply our superdimension formula to obtain a closed form root multiplicity formula for Monstrous Lie superalgebras. We also give an interesting characterization of *replicable functions* in terms of product identities, and determine the root multiplicities of the Monstrous Lie superalgebras associated with the replicable functions.

In [KKK], we will generalize our superdimension formula to the *supertrace formula* for graded Lie superalgebras with group actions, and discuss many interesting applications to various graded Lie superalgebras.

1. THE SUPERDIMENSION FORMULA

1.1. *Lie Superalgebras*

We begin with the basic theory of colored Lie superalgebras (cf. [BMPZ, K3, S]). Let $\mathscr{A}$ be an abelian group and suppose we have a bimultiplicative map $\theta: \mathscr{A} \times \mathscr{A} \to \mathbf{C}^{\times}$ satisfying

$$\begin{aligned} \theta(a+b,c) &= \theta(a,c)\theta(b,c), \\ \theta(a,b+c) &= \theta(a,b)\theta(a,c), \\ \theta(a,b)\theta(b,a) &= 1 \qquad \text{for all } a,b,c \in \mathscr{A}. \end{aligned} \tag{1.1}$$

The map $\theta: \mathscr{A} \times \mathscr{A} \to \mathbf{C}^{\times}$ is called a *coloring map* on $\mathscr{A}$. Note that we have $\theta(a,a) = \pm 1$ for all $a \in \mathscr{A}$. We will write $\psi(a) = \theta(a,a)$ and call it the *sign* of $a \in \mathscr{A}$. Let $\mathscr{A}_0 = \{a \in \mathscr{A} \mid \psi(a) = 1\}$ and $\mathscr{A}_1 = \{a \in \mathscr{A} \mid \psi(a) = -1\}$, which yields a decomposition $\mathscr{A} = \mathscr{A}_0 \cup \mathscr{A}_1$. The elements of $\mathscr{A}$ in $\mathscr{A}_0$ (resp. $\mathscr{A}_1$) are called *even* (resp. *odd*).

A *θ-colored superspace* is a pair (V, θ), where $V = \bigoplus_{a \in \mathscr{A}} V_a$ is an $\mathscr{A}$-graded vector space and $\theta: \mathscr{A} \times \mathscr{A} \to \mathbf{C}^{\times}$ is a coloring map on $\mathscr{A}$. The elements of V_a are called *even* (resp. *odd*) if $\psi(a) = 1$ (resp. $\psi(a) = -1$). For each $a \in \mathscr{A}$, we define the *superdimension* of V_a to be

$$\operatorname{Dim} V_a = \psi(a) \dim V_a. \tag{1.2}$$

Similarly, we define a *θ-colored superalgebra* to be a pair (U, θ), where $U = \bigoplus_{\alpha \in \mathscr{A}} U_a$ is an $\mathscr{A}$-graded associative algebra (i.e., $U_a U_b \subset U_{a+b}$ for all $a, b \in \mathscr{A}$) and $\theta: \mathscr{A} \times \mathscr{A} \to \mathbf{C}^{\times}$ is a coloring map on $\mathscr{A}$. The *direct sum* of $\mathscr{A}$-graded superalgebras is defined in the usual way, but, for θ-colored superalgebras $U = \bigoplus_{a \in \mathscr{A}} U_a$ and $U' = \bigoplus_{a' \in \mathscr{A}} U'_{a'}$, we define the *tensor* *product* of U and U' to be the θ-colored superspace $U \otimes U'$ with the natural $\mathscr{A}$-gradation $u \otimes u' \in (U \otimes U')_{a+a'}$ for $u \in U_a$, $u' \in U'_{a'}$, and the multiplication given by

$$(u \otimes u')(v \otimes v') = \theta(a', b)(uv \otimes u'v')$$

for $u \in U_a$, $v \in U_b$, $u' \in U'_{a'}$, $v' \in U'_{b'}$, $a, a', b, b' \in \mathscr{A}$.

DEFINITION 1.1. *A θ-colored Lie superalgebra* is a θ-colored superspace $\mathfrak{L} = \bigoplus_{a \in \mathscr{A}} \mathfrak{L}_a$ together with a bilinear operation $[\ ,\ ]: \mathfrak{L} \times \mathfrak{L} \to \mathfrak{L}$ satisfying

$$\begin{gathered} [\mathfrak{L}_a, \mathfrak{L}_b] \subset \mathfrak{L}_{a+b}, \\ [x, y] = -\theta(a, b)[y, x], \\ [x, [y, z]] = [[x, y], z] + \theta(a, b)[y, [x, z]] \end{gathered} \tag{1.3}$$

for all $x \in \mathfrak{L}_a$, $y \in \mathfrak{L}_b$, $z \in \mathfrak{L}$, and $a, b \in \mathscr{A}$.

Let $\mathfrak{L}_{(0)} = \bigoplus_{a \in \mathscr{A}_0} \mathfrak{L}_a$ and $\mathfrak{L}_{(1)} = \bigoplus_{b \in \mathscr{A}_1} \mathfrak{L}_b$. Then we have a decomposition $\mathfrak{L} = \mathfrak{L}_{(0)} \oplus \mathfrak{L}_{(1)}$, and the homogeneous elements of $\mathfrak{L}_{(0)}$ (resp. $\mathfrak{L}_{(1)}$) are called *even* (resp. *odd*).

The *universal enveloping algebra* of a θ-colored Lie superalgebra $\mathfrak{L} = \bigoplus_{a \in \mathscr{A}} \mathfrak{L}_a$ is the pair $(U(\mathfrak{L}), \iota)$, where $U(\mathfrak{L})$ is a θ-colored superalgebra and $\iota: \mathfrak{L} \to U(\mathfrak{L})$ is a linear mapping satisfying

$$\iota([x, y]) = \iota(x)\iota(y) - \theta(a, b)\iota(y)\iota(x) \qquad \text{for } x \in \mathfrak{L}_a,\ y \in \mathfrak{L}_b$$

such that for any θ-colored superalgebra $U = \bigoplus_{a \in \mathscr{A}} U_a$ and a linear mapping $j: \mathfrak{L} \to U$ satisfying

$$j([x, y]) = j(x)j(y) - \theta(a, b)j(y)j(x) \qquad \text{for } x \in \mathfrak{L}_a,\ y \in \mathfrak{L}_b,$$

there exists a unique homomorphism $\psi: U(\mathfrak{L}) \to U$ of θ-colored superalgebras satisfying $\psi \circ \iota = j$.

Let $X = \{x_\alpha \mid \alpha \in \Lambda\}$ (resp. $Y = \{y_\beta \mid \beta \in \Omega\}$) be a homogeneous basis of the subspace $\mathfrak{L}_{(0)} = \bigoplus_{a \in \mathscr{A}_0} \mathfrak{L}_a$ (resp. $\mathfrak{L}_{(1)} = \bigoplus_{b \in \mathscr{A}_1} \mathfrak{L}_b$). Then, by the Poincaré–Birkhoff–Witt Theorem, the elements of the form

$$x_{\alpha_1} x_{\alpha_2} \cdots x_{\alpha_k} y_{\beta_1} y_{\beta_2} \cdots y_{\beta_l} \qquad \text{with } \alpha_1 \le \cdots \le \alpha_k,\ \beta_1 < \cdots < \beta_l \tag{1.4}$$

together with 1 form a basis of the universal enveloping algebra $U(\mathfrak{L})$ of $\mathfrak{L}$ (cf. [BMPZ, K3, S]).

1.2. *Supercharacters and Denominator Identity*

Let $\mathscr{A}$ be a countable (usually finite) abelian group with a coloring map $\theta: \mathscr{A} \times \mathscr{A} \to \mathbf{C}^{\times}$, and let Γ be a countable (usually infinite) abelian semigroup satisfying the following condition:

$$\text{every element } (\alpha, a) \in \Gamma \times \mathscr{A} \text{ can be written as a sum of elements in } \Gamma \times \mathscr{A} \text{ in only finitely many ways.} \tag{1.5}$$

For a $(\Gamma \times \mathscr{A})$-graded θ-colored superspace $V = \bigoplus_{(\alpha, a) \in \Gamma \times \mathscr{A}} V_{(\alpha, a)}$, we define the $(\Gamma \times \mathscr{A})$-*graded character* of V to be

$$\mathrm{ch}_{\Gamma \times \mathscr{A}} V = \sum_{(\alpha, a) \in \Gamma \times \mathscr{A}} \dim V_{(\alpha, a)} e^{(\alpha, a)},$$

where the $e^{(\alpha, a)}$ are the basis elements of the semigroup algebra $\mathbf{C}[\Gamma \times \mathscr{A}]$ with the multiplication $e^{(\alpha, a)} e^{(\beta, b)} = e^{(\alpha+\beta, a+b)}$ for all $\alpha, \beta \in \Gamma$, $a, b \in \mathscr{A}$. (Actually, $\mathrm{ch}_{\Gamma \times \mathscr{A}} V$ itself is usually an element of $\mathbf{C}[[\Gamma \times \mathscr{A}]]$, the completion of $\mathbf{C}[\Gamma \times \mathscr{A}]$.)

We define the *superdimension* of the homogeneous subspace $V_{(\alpha, a)}$ by

$$\mathrm{Dim}\, V_{(\alpha, a)} = \psi(a) \dim V_{(\alpha, a)} \qquad \text{for } (\alpha, a) \in \Gamma \times \mathscr{A}. \tag{1.6}$$

Also, we introduce another basis element of the semigroup algebra $\mathbf{C}[\Gamma \times \mathscr{A}]$ by setting

$$E^{(\alpha, a)} = \psi(a) e^{(\alpha, a)} \qquad \text{for } (\alpha, a) \in \Gamma \times \mathscr{A}.$$

Then it is easy to verify that $E^{(\alpha, a)} E^{(\beta, b)} = E^{(\alpha+\beta, a+b)}$ for all $\alpha, \beta \in \Gamma$, $a, b \in \mathscr{A}$.

With these notations, we define the $(\Gamma \times \mathscr{A})$-*graded supercharacter* of V by

$$\mathrm{Ch}_{\Gamma \times \mathscr{A}} V = \sum_{(\alpha, a) \in \Gamma \times \mathscr{A}} \left(\mathrm{Dim}\, V_{(\alpha, a)}\right) E^{(\alpha, a)}. \tag{1.7}$$

Note that the supercharacter $\mathrm{Ch}_{\Gamma \times \mathscr{A}} V$ is obtained from the character $\mathrm{ch}_{\Gamma \times \mathscr{A}} V$ by replacing $\dim V_{(\alpha, a)} = \psi(a) \mathrm{Dim}\, V_{(\alpha, a)}$ and $e^{(\alpha, a)} = \psi(a) E^{(\alpha, a)}$ for $\alpha \in \Gamma$, $a \in \mathscr{A}$. Since $\psi(a)^2 = 1$, we have $\mathrm{Ch}_{\Gamma \times \mathscr{A}} V = \mathrm{ch}_{\Gamma \times \mathscr{A}} V$. The only (but important) difference is that, in the supercharacter $\mathrm{Ch}_{\Gamma \times \mathscr{A}} V$, we allow the negative coefficients. This implies that if we are given a formal power series

$$T_{\Gamma\times\mathscr{A}} = \sum_{(\alpha,a)\in\Gamma\times\mathscr{A}} t(\alpha,a)E^{(\alpha,a)}$$
$$\text{with } t(\alpha,a)\in\mathbf{Z} \text{ for all } \alpha\in\Gamma,\ a\in\mathscr{A}, \tag{1.8}$$

we can interpret the series $T_{\Gamma\times\mathscr{A}}$ as the $(\Gamma\times\mathscr{A})$-graded supercharacter of a $(\Gamma\times\mathscr{A})$-graded superspace $V=\bigoplus_{(\alpha,a)\in\Gamma\times\mathscr{A}} V_{(\alpha,a)}$ such that $\operatorname{Dim} V_{(\alpha,a)} = t_{(\alpha,a)}$ for all $\alpha\in\Gamma$, $a\in\mathscr{A}$.

For a $(\Gamma\times\mathscr{A})$-graded superspace $V=\bigoplus_{(\alpha,a)\in\Gamma\times\mathscr{A}} V_{(\alpha,a)}$, let $V_\alpha = \bigoplus_{a\in\mathscr{A}} V_{(\alpha,a)}$, and define

$$\operatorname{Dim} V_\alpha = \sum_{a\in\mathscr{A}} \operatorname{Dim} V_{(\alpha,a)} = \sum_{a\in\mathscr{A}_0} \dim V_{(\alpha,a)} - \sum_{a\in\mathscr{A}_1} \dim V_{(\alpha,a)}. \tag{1.9}$$

(We assume that $\operatorname{Dim} V_\alpha$ is well-defined for all $\alpha\in\Gamma$. That is, $|\operatorname{Dim} V_\alpha| < \infty$ for all $\alpha\in\Gamma$.) By specializing $E^{(\alpha,a)} = E^\alpha$ for all $a\in\mathscr{A}$, we obtain the Γ-*graded supercharacter* $\operatorname{Ch}_\Gamma V$ of the Γ-graded superspace $V=\bigoplus_{\alpha\in\Gamma} V_\alpha$:

$$\operatorname{Ch}_\Gamma V = \sum_{\alpha\in\Gamma} (\operatorname{Dim} V_\alpha) E^\alpha. \tag{1.10}$$

Again, this implies that if we are given a formal power series

$$T_\Gamma = \sum_{\alpha\in\Gamma} t(\alpha)E^\alpha \qquad \text{with } t(\alpha)\in\mathbf{Z} \text{ for all } \alpha\in\Gamma, \tag{1.11}$$

we can interpret the series T_Γ as the Γ-graded supercharacter of a Γ-graded superspace $\bigoplus_{\alpha\in\Gamma} V_\alpha$ such that $\operatorname{Dim} V_\alpha = t(\alpha)$ for all $\alpha\in\Gamma$.

We now consider the $(\Gamma\times\mathscr{A})$-graded θ-colored Lie superalgebras

$$\mathfrak{L} = \bigoplus_{(\alpha,a)\in\Gamma\times\mathscr{A}} \mathfrak{L}_{(\alpha,a)} \qquad \text{with } \dim \mathfrak{L}_{(\alpha,a)} < \infty \text{ for all } \alpha\in\Gamma,\ a\in\mathscr{A}.$$

Let $\mathfrak{L}_{(0)} = \bigoplus_{(\alpha,a)\in\Gamma\times\mathscr{A}_0} \mathfrak{L}_{(\alpha,a)}$ and $\mathfrak{L}_{(1)} = \bigoplus_{(\beta,b)\in\Gamma\times\mathscr{A}_1} \mathfrak{L}_{(\beta,b)}$. Then, by the Poincaré–Birkhoff–Witt Theorem, we have

$$\operatorname{ch}_{\Gamma\times\mathscr{A}} U(\mathfrak{L}) = \frac{\prod_{(\beta,b)\in\Gamma\times\mathscr{A}_1} \left(1+e^{(\beta,b)}\right)^{\dim \mathfrak{L}_{(\beta,b)}}}{\prod_{(\alpha,a)\in\Gamma\times\mathscr{A}_0} \left(1-e^{(\alpha,a)}\right)^{\dim \mathfrak{L}_{(\alpha,a)}}}.$$

Hence the $(\Gamma \times \mathscr{A})$-graded supercharacter of the universal enveloping algebra $U(\mathfrak{L})$ of $\mathfrak{L}$ is equal to

$$\mathrm{Ch}_{\Gamma\times\mathscr{A}} U(\mathfrak{L}) = \prod_{(\alpha, a)\in\Gamma\times\mathscr{A}} \left(1 - E^{(\alpha, a)}\right)^{-\mathrm{Dim}\,\mathfrak{L}_{(\alpha, a)}}. \tag{1.12}$$

Let

$$\mathscr{D}(\mathfrak{L}) = \prod_{(\alpha, a)\in\Gamma\times\mathscr{A}} \left(1 - E^{(\alpha, a)}\right)^{\mathrm{Dim}\,\mathfrak{L}_{(\alpha, a)}}. \tag{1.13}$$

We will call $\mathscr{D}(\mathfrak{L})$ the *denominator function* of the $(\Gamma \times \mathscr{A})$-graded θ-colored Lie superalgebra $\mathfrak{L} = \bigoplus_{(\alpha, a)\in\Gamma\times\mathscr{A}} \mathfrak{L}_{(\alpha, a)}$.

Suppose we have a product identity for the denominator function $\mathscr{D}(\mathfrak{L})$ of the form

$$\prod_{(\alpha, a)\in\Gamma\times\mathscr{A}} \left(1 - E^{(\alpha, a)}\right)^{\mathrm{Dim}\,\mathfrak{L}_{(\alpha, a)}} = 1 - T_{\Gamma\times\mathscr{A}} \tag{1.14}$$

for some formal power series $T_{\Gamma\times\mathscr{A}} = \sum_{(\alpha, a)\in\Gamma\times\mathscr{A}} t(\alpha, a) E^{(\alpha, a)}$ with $t(\alpha, a) \in \mathbf{Z}$ for all $\alpha \in \Gamma$, $a \in \mathscr{A}$. Then the identity (1.14) will be called the *denominator identity* for the $(\Gamma \times \mathscr{A})$-graded Lie superalgebra $\mathfrak{L} = \bigoplus_{(\alpha, a)\in\Gamma\times\mathscr{A}} \mathfrak{L}_{(\alpha, a)}$.

1.3. *The Superdimension Formula and Its Specialization*

Let $P(\Gamma \times \mathscr{A}) = \{(\alpha, a) \in \Gamma \times \mathscr{A} \mid t(\alpha, a) \neq 0\}$, and let $\{(\tau_i, b_j) \in \Gamma \times \mathscr{A} \mid i, j = 1, 2, 3, \ldots\}$ be an enumeration of the set $P(\Gamma \times \mathscr{A})$. For $(\tau, b) \in \Gamma \times \mathscr{A}$, we define

$$T(\tau, b) = \left\{ s = (s_{ij})_{i, j=1}^{\infty} \,\middle|\, s_{ij} \in \mathbf{Z}_{\geq 0},\ \sum s_{ij}(\tau_i, b_j) = (\tau, b) \right\}, \tag{1.15}$$

which is the set of all partitions of (τ, b) into a sum of (τ_i, b_j)'s. By our finiteness condition (1.5) on $\Gamma \times \mathscr{A}$, the set $T(\tau, b)$ must be finite. For a partition $s = (s_{ij}) \in T(\tau, b)$, we will use the notation $|s| = \sum s_{ij}$ and $s! = \prod (s_{ij}!)$ (cf. [Bo]). For $(\tau, b) \in \Gamma \times \mathscr{A}$, we define a function

$$W(\tau, b) = \sum_{s\in T(\tau, b)} \frac{(|s| - 1)!}{s!} \prod t(\tau_i, b_j)^{s_{ij}}. \tag{1.16}$$

We will call $W(\tau, b)$ the *Witt partition function*. In the next theorem, using the denominator identity (1.14), we derive a closed form formula for the superdimensions $\mathrm{Dim}\,\mathfrak{L}_{(\alpha, a)}$ $(\alpha \in \Gamma,\ a \in \mathscr{A})$ in terms of Witt partition functions.

THEOREM 1.2. *Let $\mathscr{A}$ be a countable abelian group with a coloring map θ: $\mathscr{A} \times \mathscr{A} \to \mathbf{C}^{\times}$ and Γ be a countable abelian semigroup satisfying the finiteness condition* (1.5). *Let $\mathfrak{L} = \bigoplus_{(\alpha, a) \in \Gamma \times \mathscr{A}} \mathfrak{L}_{(\alpha, a)}$ be a $(\Gamma \times \mathscr{A})$-graded θ-colored Lie superalgebra with $\dim \mathfrak{L}_{(\alpha, a)} < \infty$ for all $\alpha \in \Gamma$, $a \in \mathscr{A}$. Suppose we have a denominator identity*

$$\prod_{(\alpha, a) \in \Gamma \times \mathscr{A}} \left(1 - E^{(\alpha, a)}\right)^{\operatorname{Dim} \mathfrak{L}_{(\alpha, a)}} = 1 - T_{\Gamma \times \mathscr{A}}$$

for some formal power series $T_{\Gamma \times \mathscr{A}} = \sum_{(\alpha, a) \in \Gamma \times \mathscr{A}} t(\alpha, a) E^{(\alpha, a)}$ with $t(\alpha, a) \in \mathbf{Z}$ for all $(\alpha, a) \in \Gamma \times \mathscr{A}$.

Then for any $(\alpha, a) \in \Gamma \times \mathscr{A}$, we have

$$\operatorname{Dim} \mathfrak{L}_{(\alpha, a)} = \sum_{\substack{d > 0 \\ (\alpha, a) = d(\tau, b)}} \frac{1}{d} \mu(d) W(\tau, b), \tag{1.17}$$

where μ is the classical Möbius function.

Proof. By the denominator identity (1.14), we have

$$\begin{aligned}
\prod_{(\alpha, a) \in \Gamma \times \mathscr{A}} \left(1 - E^{(\alpha, a)}\right)^{-\operatorname{Dim} \mathfrak{L}_{(\alpha, a)}} &= \frac{1}{1 - T_{\Gamma \times \mathscr{A}}} \\
&= \frac{1}{1 - \sum_{(\alpha, a) \in \Gamma \times \mathscr{A}} t(\alpha, a) E^{(\alpha, a)}} \\
&= \frac{1}{1 - \sum_{i, j = 1}^{\infty} t(\tau_i, b_j) E^{(\tau_i, b_j)}}.
\end{aligned}$$

Using the formal power series $\log(1 - t) = -\sum_{k=1}^{\infty} \frac{1}{k} t^k$, we obtain from the left-hand side

$$\begin{aligned}
&\log\left(\prod_{(\alpha, a) \in \Gamma \times \mathscr{A}} \left(1 - E^{(\alpha, a)}\right)^{-\operatorname{Dim} \mathfrak{L}_{(\alpha, a)}} \right) \\
&\quad = - \sum_{(\alpha, a) \in \Gamma \times \mathscr{A}} \operatorname{Dim} \mathfrak{L}_{(\alpha, a)} \log\left(1 - E^{(\alpha, a)}\right) \\
&\quad = \sum_{(\alpha, a) \in \Gamma \times \mathscr{A}} \operatorname{Dim} \mathfrak{L}_{(\alpha, a)} \sum_{k=1}^{\infty} \frac{1}{k} E^{k(\alpha, a)} \\
&\quad = \sum_{(\alpha, a) \in \Gamma \times \mathscr{A}} \sum_{k=1}^{\infty} \frac{1}{k} \operatorname{Dim} \mathfrak{L}_{(\alpha, a)} E^{k(\alpha, a)}.
\end{aligned}$$

On the other hand, the right-hand side yields

$$\begin{aligned}
\log\left(\frac{1}{1-T_{\Gamma\times\mathscr{A}}}\right) &= \log\left(\frac{1}{1-\sum_{i,j=1}^{\infty} t(\tau_i,b_j)E^{(\tau_i,b_j)}}\right)\\
&= -\log\left(1-\sum_{i,j=1}^{\infty} t(\tau_i,b_j)E^{(\tau_i,b_j)}\right)\\
&= \sum_{k=1}^{\infty}\frac{1}{k}\left(\sum_{i,j=1}^{\infty} t(\tau_i,b_j)E^{(\tau_i,b_j)}\right)^k\\
&= \sum_{k=1}^{\infty}\frac{1}{k}\sum_{\substack{s=(s_{ij})\\ s_{ij}\in\mathbf{Z}_{\geq 0}\\ \sum s_{ij}=k}}\frac{(\sum s_{ij})!}{\prod(s_{ij}!)}\left(\prod t(\tau_i,b_j)^{s_{ij}}\right)E^{\sum s_{ij}(\tau_i,b_j)}\\
&= \sum_{(\tau,b)\in\Gamma\times\mathscr{A}}\left(\sum_{s\in T(\tau,b)}\frac{(|s|-1)!}{s!}\prod t(\tau_i,b_j)^{s_{ij}}\right)E^{(\tau,b)}\\
&= \sum_{(\tau,b)\in\Gamma\times\mathscr{A}} W(\tau,b)E^{(\tau,b)}.
\end{aligned}$$

Therefore, we have

$$W(\tau,b) = \sum_{\substack{k>0\\ (\tau,b)=k(\alpha,a)}}\frac{1}{k}\operatorname{Dim}\mathfrak{L}_{(\alpha,a)}.$$

Hence, by Möbius inversion, we obtain

$$\operatorname{Dim}\mathfrak{L}_{(\alpha,a)} = \sum_{\substack{d>0\\ (\alpha,a)=d(\tau,b)}}\frac{1}{d}\mu(d)W(\tau,b).$$

∎

Remark. For a fixed $d\in\mathbf{Z}_{>0}$, there may exist more than one $(\tau,b)\in\Gamma\times\mathscr{A}$ satisfying $(\alpha,a)=d(\tau,b)$. In this case, we need to take the sum over all of those (τ,b)'s.

As a special case of Theorem 1.2, we obtain a superdimension formula for ordinary $(\Gamma\times\mathbf{Z}_2)$-graded Lie superalgebras.

COROLLARY 1.3. *Let* $\mathfrak{L} = \bigoplus_{(\alpha, i) \in \Gamma \times \mathbf{Z}_2} \mathfrak{L}_{(\alpha, i)}$ *be a* $(\Gamma \times \mathbf{Z}_2)$*-graded Lie superalgebra with* $\dim \mathfrak{L}_{(\alpha, i)} < \infty$ *for all* $\alpha \in \Gamma$, $i \in \mathbf{Z}_2$. *Suppose we have a denominator identity*

$$\prod_{(\alpha, i) \in \Gamma \times \mathbf{Z}_2} (1 - E^{(\alpha, i)})^{\operatorname{Dim} \mathfrak{L}_{(\alpha, i)}} = 1 - T_{\Gamma \times \mathbf{Z}_2}$$

for some formal power series $T_{\Gamma \times \mathbf{Z}_2} = \sum_{(\alpha, i) \in \Gamma \times \mathbf{Z}_2} t(\alpha, i) E^{(\alpha, i)}$ *with* $t(\alpha, i) \in \mathbf{Z}$ *for all* $(\alpha, i) \in \Gamma \times \mathbf{Z}_2$.

Then we have

$$\operatorname{Dim} \mathfrak{L}_{(\alpha, 0)} = \sum_{\substack{d > 0 \\ \alpha = d\tau}} \frac{1}{d} \mu(d) W(\tau, 0) + \sum_{\substack{d > 0, \text{ even} \\ \alpha = d\tau}} \frac{1}{d} \mu(d) W(\tau, 1), \tag{1.18}$$

$$\operatorname{Dim} \mathfrak{L}_{(\alpha, 1)} = \sum_{\substack{d > 0, \text{ odd} \\ \alpha = d\tau}} \frac{1}{d} \mu(d) W(\tau, 1). \tag{1.19}$$

Proof. By Theorem 1.2, we have

$$\operatorname{Dim} \mathfrak{L}_{(\alpha, i)} = \sum_{\substack{d > 0 \\ (\alpha, i) = d(\tau, k)}} \frac{1}{d} \mu(d) W(\tau, k),$$

for $\alpha, \tau \in \Gamma$ and $i, k \in \mathbf{Z}_2$. If $(\alpha, 0) = d(\tau, k)$ for some $d \in \mathbf{Z}_{>0}$, then $\alpha = d\tau$ and $dk \equiv 0 \pmod 2$. Hence, if $k = 0$, then d can be any positive integer dividing α, and if $k = 1$, then d must be even, dividing α, which yields (1.18). Similarly, if $(\alpha, 1) = d(\tau, k)$, then $k = 1$ and d must be odd, dividing α. Hence we obtain (1.19). ∎

Suppose we have the denominator identity (1.14) for a $(\Gamma \times \mathscr{A})$-graded θ-colored Lie superalgebra $\mathfrak{L} = \bigoplus_{(\alpha, a) \in \Gamma \times \mathscr{A}} \mathfrak{L}_{(\alpha, a)}$. For each $\alpha \in \Gamma$, let $\mathfrak{L}_\alpha = \sum_{a \in \mathscr{A}} \mathfrak{L}_{(\alpha, a)}$, and assume that the superdimension $\operatorname{Dim} \mathfrak{L}_\alpha = \sum_{a \in \mathscr{A}} \operatorname{Dim} \mathfrak{L}_{(\alpha, a)}$ is well-defined. Then we get a Γ-graded Lie superalgebra $\mathfrak{L} = \bigoplus_{\alpha \in \Gamma} \mathfrak{L}_\alpha$ with $\dim \mathfrak{L}_\alpha < \infty$ for all $\alpha \in \Gamma$. Furthermore, assume that $t(\alpha) \stackrel{\text{def}}{=} \sum_{a \in \mathscr{A}} t(\alpha, a)$ is also well-defined. Then, by specializing $E^{(\alpha, a)} = E^\alpha$ for all $a \in \mathscr{A}$, we obtain a formal power series $T_\Gamma = \sum_{\alpha \in \Gamma} t(\alpha) E^\alpha$ with $t(\alpha) \in \mathbf{Z}$ for all $\alpha \in \Gamma$, and the denominator identity (1.14) yields a product identity

$$\prod_{\alpha \in \Gamma} (1 - E^\alpha)^{\operatorname{DIm} \mathfrak{L}_\alpha} = 1 - T_\Gamma = 1 - \sum_{\alpha \in \Gamma} t(\alpha) E^\alpha. \tag{1.20}$$

The identity (1.20) will also be called the *denominator identity* for the Γ-graded Lie superalgebra $\mathfrak{L} = \bigoplus_{\alpha \in \Gamma} \mathfrak{L}_{\alpha}$.

Let $P(\Gamma) = \{\alpha \in \Gamma \mid t(\alpha) \neq 0\}$, and let $\{\tau_i \mid i = 1, 2, 3, \dots\}$ be an enumeration of the set $P(\Gamma)$. For $\tau \in \Gamma$, define

$$T(\tau) = \left\{ s = (s_i)_{i=1}^{\infty} \mid s_i \in \mathbf{Z}_{\geq 0}, \sum s_i \tau_i = \tau \right\}, \tag{1.21}$$

which is the set of all partitions of τ into a sum of τ_i's. Again, by our finiteness condition (1.5), the set $T(\tau)$ must be finite. For a partition $s \in T(\tau)$, we will also use the notation $|s| = \sum s_i$ and $s! = \prod (s_i!)$, and for $\tau \in \Gamma$, we define the *Witt partition function* $W(\tau)$ by

$$W(\tau) = \sum_{s \in T(\tau)} \frac{(|s| - 1)!}{s!} \prod t(\tau_i)^{s_i}. \tag{1.22}$$

Then, by Theorem 1.2, we obtain a closed form formula for the superdimensions $\operatorname{Dim} \mathfrak{L}_{\alpha}$ for all $\alpha \in \Gamma$.

PROPOSITION 1.4. *Let $\mathfrak{L} = \bigoplus_{\alpha \in \Gamma} \mathfrak{L}_{\alpha}$ be a Γ-graded Lie superalgebra with* $\dim \mathfrak{L}_{\alpha} < \infty$ *for all $\alpha \in \Gamma$, and suppose we have a denominator identity*

$$\prod_{\alpha \in \Gamma} (1 - E^{\alpha})^{\operatorname{Dim} \mathfrak{L}_{\alpha}} = 1 - T_{\Gamma}$$

for some formal power series $T_{\Gamma} = \sum_{\alpha \in \Gamma} t(\alpha) E^{\alpha}$ with $t(\alpha) \in \mathbf{Z}$ for all $\alpha \in \Gamma$.

Then, for any $\alpha \in \Gamma$, we have

$$\operatorname{Dim} \mathfrak{L}_{\alpha} = \sum_{\substack{d > 0 \\ \alpha = d\tau}} \frac{1}{d} \mu(d) W(\tau), \tag{1.23}$$

where μ is the classical Möbius function.

Remark. As in Theorem 1.2, for a fixed $d \in \mathbf{Z}_{>0}$, there may exist more than one $\tau \in \Gamma$ satisfying $\alpha = d\tau$. In this case, we need to take the sum over all of those τ's.

2. FREE LIE SUPERALGEBRAS

2.1. *Generalized Witt Formula*

Let $\mathscr{A}$ be a countable abelian group with a coloring map $\theta: \mathscr{A} \times \mathscr{A} \to \mathbf{C}^{\times}$ and Γ be a countable abelian semigroup satisfying the finiteness condition (1.5). Let $V = \bigoplus_{(\alpha, a) \in \Gamma \times \mathscr{A}} V_{(\alpha, a)}$ be a $(\Gamma \times \mathscr{A})$-graded θ-colored superspace with finite dimensional homogeneous subspaes. We denote the superdimension of $V_{(\alpha,a)}$ by $t(\alpha,a) = \operatorname{Dim} V_{(\alpha,a)} = \psi(a)\dim V_{(\alpha,a)}$. Then the $(\Gamma\times\mathscr{A})$-graded supercharacter of V can be written as

$$\operatorname{Ch}_{\Gamma\times\mathscr{A}} V = \sum_{(\alpha,a)\in\Gamma\times\mathscr{A}} t(\alpha,a)E^{(\alpha,a)}. \tag{2.1}$$

Let $\mathfrak{L}$ be the free Lie superalgebra generated by V. Then $\mathfrak{L}$ has a $(\Gamma\times\mathscr{A})$-gradation $\mathfrak{L} = \bigoplus_{(\alpha,a)\in\Gamma\times\mathscr{A}} \mathfrak{L}_{(\alpha,a)}$ induced by V, and as we have seen in Section 1, the Poincaré−Birkhoff−Witt Theorem yields

$$\operatorname{Ch}_{\Gamma\times\mathscr{A}} U(\mathfrak{L}) = \prod_{(\alpha,a)\in\Gamma\times\mathscr{A}} (1-E^{(\alpha,)})^{-\operatorname{Dim} L_{(\alpha,a)}}. \tag{2.2}$$

On the other hand, since $\mathfrak{L}$ is free on V, the universal enveloping algebra $U(\mathfrak{L})$ of $\mathfrak{L}$ is the tensor algebra $\mathscr{T}(V) = \mathbf{C}\oplus V\oplus V^{\otimes 2}\oplus\cdots$ of V. It follows that

$$\begin{aligned}\operatorname{Ch}_{\Gamma\times\mathscr{A}} U(\mathfrak{L}) &= \operatorname{Ch}_{\Gamma\times\mathscr{A}}\mathscr{T}(V) = 1 + \operatorname{Ch}_{\Gamma\times\mathscr{A}} V + (\operatorname{Ch}_{\Gamma\times\mathscr{A}} V)^2 + \cdots \\ &= \frac{1}{1-\operatorname{Ch}_{\Gamma\times\mathscr{A}} V} = \frac{1}{1-\sum_{(\alpha,a)\in\Gamma\times\mathscr{A}} t(\alpha,a)E^{(\alpha,a)}}.\end{aligned} \tag{2.3}$$

Hence we obtain a *denominator identity* for the $(\Gamma\times\mathscr{A})$-graded free Lie superalgebra $\mathfrak{L} = \bigoplus_{(\alpha,a)\in\Gamma\times\mathscr{A}} \mathfrak{L}_{(\alpha,a)}$:

$$\begin{aligned}\prod_{(\alpha,a)\in\Gamma\times\mathscr{A}} (1-E^{(\alpha,a)})^{\operatorname{Dim}\mathfrak{L}_{(\alpha,a)}} &= 1-\operatorname{Ch}_{\Gamma\times\mathscr{A}} V \\ &= 1-\sum_{(\alpha,a)\in\Gamma\times\mathscr{A}} t(\alpha,a)E^{(\alpha,a)}.\end{aligned} \tag{2.4}$$

Let $P(V,\Gamma\times\mathscr{A}) = \{(\alpha,a)\in\Gamma\times\mathscr{A} \mid \dim V_{(\alpha,a)}\neq 0\}$ and let $\{(\tau_i,b_j) \mid i,j=1,2,3,\dots\}$ be an enumeration of the set $P(V,\Gamma\times\mathscr{A})$. For each $(\tau,b)\in\Gamma\times\mathscr{A}$, we denote by $T(\tau,b)$ the set of all partitions of (τ,b) into a sum of (τ_i,b_j)'s as defined in (1.15), and let $W(\tau,b)$ be the Witt partition function as defined in (1.16). Then our superdimension formula (1.17) yields the following generalized version of the Witt formula for the free Lie superalgebras generated by $(\Gamma\times\mathscr{A})$-graded superspaces:

THEOREM 2.1 (cf. [Ka5]). *Let $\mathscr{A}$ be a countable abelian group with a coloring map θ: $\mathscr{A}\times\mathscr{A}\to\mathbf{C}^\times$ and Γ be a countable abelian semigroup satisfying the finiteness condition* (1.5). *Let $V = \bigoplus_{(\alpha,a)\in\Gamma\times\mathscr{A}} V_{(\alpha,a)}$ be a $(\Gamma\times\mathscr{A})$-graded θ-colored superspace with finite dimensional homogeneous subspaces, and let $\mathfrak{L} = \bigoplus_{(\alpha,a)\in\Gamma\times\mathscr{A}} \mathfrak{L}_{(\alpha,a)}$ be the free Lie superalgebra generated by V.*

Then for any $(\alpha, a) \in \Gamma \times \mathscr{A}$, *we have*

$$\operatorname{Dim} \mathfrak{L}_{(\alpha, a)} = \sum_{\substack{d>0 \\ (\alpha, a) = d(\tau, b)}} \frac{1}{d} \mu(d) W(\tau, b), \tag{2.5}$$

where μ *is the classical Möbius function.*

COROLLARY 2.2. *Let* $V = \bigoplus_{(\alpha, i) \in \Gamma \times \mathbf{Z}_2} V_{(\alpha, i)}$ *be a* $(\Gamma \times \mathbf{Z}_2)$*-graded superspace with finite dimensional homogeneous subspaces and let* $\mathfrak{L} = \bigoplus_{(\alpha, i) \in \Gamma \times \mathbf{Z}_2} \mathfrak{L}_{(\alpha, i)}$ *be the free Lie superalgebra generated by* V.

Then for any $\alpha \in \Gamma$, *we have*

$$\operatorname{Dim} \mathfrak{L}_{(\alpha, 0)} = \sum_{\substack{d>0 \\ \alpha = d\tau}} \frac{1}{d} \mu(d) W(\tau, 0) + \sum_{\substack{d>0, \text{ even} \\ \alpha = d\tau}} \frac{1}{d} \mu(d) W(\tau, 1), \tag{2.6}$$

$$\operatorname{Dim} \mathfrak{L}_{(\alpha, 1)} = \sum_{\substack{d>0, \text{ odd} \\ \alpha = d\tau}} \frac{1}{d} \mu(d) W(\tau, 1), \tag{2.7}$$

where μ *is the classical Möbius function.*

Furthermore, let $V_\alpha = \bigoplus_{a \in \mathscr{A}} V_{(\alpha, a)}$ and $\mathfrak{L}_\alpha = \bigoplus_{a \in \mathscr{A}} \mathfrak{L}_{(\alpha, a)}$. Then V becomes a Γ-graded superspace, and the free Lie superalgebra $\mathfrak{L}$ on V has a Γ-gradation $\mathfrak{L} = \bigoplus_{\alpha \in \Gamma} \mathfrak{L}_\alpha$ induced by V. If $t(\alpha) \overset{\text{def}}{=} \sum_{a \in \mathscr{A}} t(\alpha, a)$ is well-defined, then $\operatorname{Dim} V_\alpha$ and $\operatorname{Dim} \mathfrak{L}_\alpha$ are also well-defined, and by specializing $E^{(\alpha, a)} = E^\alpha$ for all $a \in \mathscr{A}$ in (2.4), we obtain a *denominator identity* for the free Lie superalgebra $\mathfrak{L} = \bigoplus_{\alpha \in \Gamma} \mathfrak{L}_\alpha$:

$$\prod_{\alpha \in \Gamma} (1 - E^\alpha)^{\operatorname{Dim} \mathfrak{L}_\alpha} = 1 - \operatorname{Ch}_\Gamma V = 1 - \sum_{\alpha \in \Gamma} t(\alpha) E^\alpha. \tag{2.8}$$

As in Section 1, let $P(V, \Gamma) = \{\alpha \in \Gamma \mid t(\alpha) \neq 0\}$ and let $\{\tau_i \mid i = 1, 2, 3, \cdots\}$ be an enumeration of $P(V, \Gamma)$. For $\tau \in \Gamma$, define the set $T(\tau)$ of all partitions of τ into a sum of τ_i's as in (1.21) and define the Witt partition function $W(\tau)$ by (1.22). Then, by Theorem 2.1, we obtain:

PROPOSITION 2.3. *Let* $V = \bigoplus_{\alpha \in \Gamma} V_\alpha$ *be a* Γ*-graded superspace with finite dimensional homogeneous subspaces and let* $\mathfrak{L} = \bigoplus_{\alpha \in \Gamma} \mathfrak{L}_\alpha$ *be the free Lie superalgebra generated by* V.

Then for any $\alpha \in \Gamma$, *we have*

$$\operatorname{Dim} \mathfrak{L}_\alpha = \sum_{\substack{d>0 \\ \alpha = d\tau}} \frac{1}{d} \mu(d) W(\tau), \tag{2.9}$$

where μ *is the classical Möbius function.*

Remark. The formulas (2.5)–(2.7) and (2.9) will be called the *generalized Witt formulas* for free Lie superalgebras.

EXAMPLE 2.4. In this example, we will consider the simplest generalization of the classical Witt formula for free Lie algebras to free Lie superalgebras generated by finite dimensional vector spaces. Consider the superspace $V = V_{(0)} \oplus V_{(1)}$ with $\dim V_{(0)} = r$ and $\dim V_{(1)} = s$ for some $r, s \in \mathbf{Z}_{>0}$, and let $\mathfrak{L}$ be the free Lie superalgebra generated by V. Then, by setting $\deg v = 1$ for all $v \in V$, the free Lie superalgebra $\mathfrak{L}$ has a $\mathbf{Z}_{>0}$-gradation $\mathfrak{L} = \bigoplus_{n=1}^{\infty} \mathfrak{L}_n$ induced by V. Since $P(V, \mathbf{Z}_{>0}) = \{1\}$ and $\operatorname{Dim} V = r - s$, the denominator identity for $\mathfrak{L}$ is equal to

$$\prod_{n=1}^{\infty} (1 - q^n)^{\operatorname{Dim} \mathfrak{L}_n} = 1 - (r - s)q,$$

and the Witt partition function $W(n)$ is given by

$$W(n) = \frac{(n-1)!}{n!}(r-s)^n = \frac{1}{n}(r-s)^n.$$

Hence, by the generalized Witt formula (2.9), we have

$$\operatorname{Dim} \mathfrak{L}_n = \frac{1}{n} \sum_{d|n} \mu(d)(r-s)^{n/d}. \tag{2.10}$$

Therefore the denominator identity for $\mathfrak{L}$ yields the following product identity:

$$\prod_{n=1}^{\infty} (1 - q^n)^{\frac{1}{n}\sum_{d|n} \mu(d) r^{n/d}} = 1 - rq \qquad \text{for all } r \in \mathbf{Z}. \tag{2.11}$$

On the other hand, the superspace $V = V_{(0)} \oplus V_{(1)}$ can be regarded as the $(\mathbf{Z}_{>0} \times \mathbf{Z}_2)$-graded superspace $V = V_{(1,0)} \oplus V_{(1,1)}$ with $\operatorname{Dim} V_{(1,0)} = r$, $\operatorname{Dim} V_{(1,1)} = -s$. Hence the free Lie superalgebra $\mathfrak{L}$ generated by V has a $(\mathbf{Z}_{>0} \times \mathbf{Z}_2)$-gradation $\mathfrak{L} = \bigoplus_{(n,i) \in \mathbf{Z}_{>0} \times \mathbf{Z}_2} \mathfrak{L}_{(n,i)}$ induced by V, and by letting $Q = E^{(1,0)}$, $z = E^{(0,1)}$, we obtain the denominator identity for the $(\mathbf{Z}_{>0} \times \mathbf{Z}_2)$-graded free Lie superalgebra $\mathfrak{L} = \bigoplus_{(n,i) \in \mathbf{Z}_{>0} \times \mathbf{Z}_2} \mathfrak{L}_{(n,i)}$:

$$\prod_{n=1}^{\infty} (1 - q^n)^{\operatorname{Dim} \mathfrak{L}_{(n,0)}} \prod_{n=1}^{\infty} (1 - q^n z)^{\operatorname{Dim} \mathfrak{L}_{(n,1)}} = 1 - q(r - sz),$$

where $z^2 = 1$.

Therefore, by (2.6) and (2.7), we obtain

$$\operatorname{Dim} \mathfrak{L}_{(n,0)} = \frac{1}{n} \sum_{d|n} \mu(d) \sum_{j=0}^{\left[\frac{n}{2d}\right]} \binom{\frac{n}{d}}{2j} r^{\frac{n}{d}-2j} s^{2j} - \frac{1}{n} \sum_{\substack{d|n \\ d:\text{ even}}} \mu(d) \sum_{j=0}^{\left[\frac{n}{2d}\right]} \binom{\frac{n}{d}}{2j+1} r^{\frac{n}{d}-2j-1} s^{2j+1}$$

$$= \frac{1}{n} \sum_{d|n} \mu(d) (r-s)^{\frac{n}{d}} + \frac{1}{n} \sum_{\substack{d|n \\ d:\text{ odd}}} \mu(d) \sum_{j=0}^{\left[\frac{n}{2d}\right]} \binom{\frac{n}{d}}{2j+1} r^{\frac{n}{d}-2j-1} s^{2j+1},$$

$$\operatorname{Dim} \mathfrak{L}_{(n,1)} = -\frac{1}{n} \sum_{\substack{d|n \\ d:\text{ odd}}} \mu(d) \sum_{j=0}^{\left[\frac{n}{2d}\right]} \binom{\frac{n}{d}}{2j+1} r^{\frac{n}{d}-2j-1} s^{2j+1}.$$

Moreover, combining the denominator identity with (2.11), we obtain a product identity

$$\prod_{n=1}^{\infty} \left(\frac{1-q^n}{1-q^n z} \right)^{\frac{1}{n} \sum_{d|n,\, d:\text{ odd}} \mu(d) \sum_{j=0}^{[n/2d]} \binom{n/d}{2j+1} r^{(n/d)-2j-1} s^{2j+1}} = \frac{1-q(r-sz)}{1-(r-s)q}, \tag{2.12}$$

where $z^2 = 1$, $r, s \in \mathbf{Z}_{>0}$.

Remark. Since the identity (2.11) holds for all $r \in \mathbf{Z}$, we have

$$\prod_{n=1}^{\infty} (1 - a^n q^n)^{\frac{1}{n} \sum_{d|n} \mu(d) r^{n/d}} = 1 - raq = \prod_{n=1}^{\infty} (1-q^n)^{\frac{1}{n} \sum_{d|n} \mu(d) a^{n/d} r^{n/d}} \tag{2.13}$$

for all $a, r \in \mathbf{Z}$, which was called the *exchange principle* in [KaK1].

2.2. *Free Lie Algebras and Free Lie Superalgebras*

In this subsection, we will discuss the relation of the free Lie algebra and free Lie superalgebra generated by the same vector space. Let $\mathscr{A}$ be a countable abelian group with a coloring map $\theta: \mathscr{A} \times \mathscr{A} \to \mathbf{C}^{\times}$ and Γ be a countable abelian semigroup satisfying the finiteness condition (1.5). Let $V = \bigoplus_{(\alpha, a) \in \Gamma \times \mathscr{A}} V_{(\alpha, a)}$ be a $(\Gamma \times \mathscr{A})$-graded θ-colored superspace with finite dimensional homogeneous subspaces. First, by neglecting the coloring map θ on $\mathscr{A}$, consider V as a $(\Gamma \times \mathscr{A})$-graded vector space (not a superspace), and let $L = \bigoplus_{(\alpha, a) \in \Gamma \times \mathscr{A}} L_{(\alpha, a)}$ be the free Lie algebra generated by V. Hence the denominator identity for L is equal to

$$\prod_{(\alpha, a) \in \Gamma \times \mathscr{A}} \left(1 - e^{(\alpha, a)}\right)^{\dim L_{(\alpha, a)}} = 1 - \sum_{(\alpha, a) \in \Gamma \times \mathscr{A}} \dim V_{(\alpha, a)} e^{(\alpha, a)}. \tag{2.14}$$

On the other hand, by taking the coloring map θ into account, consider V as a $(\Gamma \times \mathscr{A})$-graded θ-colored superspace, and let $\mathfrak{L} = \bigoplus_{(\alpha, a) \in \Gamma \times \mathscr{A}} \mathfrak{L}_{(\alpha, a)}$ be the free Lie superalgebra generated by V. Then the denominator identity for the free Lie superalgebra $\mathfrak{L}$ is the same as

$$\prod_{(\alpha, a) \in \Gamma \times \mathscr{A}} \left(1 - E^{(\alpha, a)}\right)^{\dim \mathfrak{L}_{(\alpha, a)}} = 1 - \sum_{(\alpha, a) \in \Gamma \times \mathscr{A}} t(\alpha, a) E^{(\alpha, a)}, \tag{2.15}$$

where $t(\alpha, a) = \operatorname{Dim} V_{(\alpha, a)} = \psi(a) \dim V_{(\alpha, a)}$ and $E^{(\alpha, a)} = \psi(a) e^{(\alpha, a)}$. Note that the right-hand sides of the identities (2.14) and (2.15) are the same.

Let $C(\alpha, a) = \psi(a) \dim L_{(\alpha, a)}$. Then the left-hand side of (2.14) is equal to

$$\begin{aligned}
&\prod_{(\alpha, a) \in \Gamma \times \mathscr{A}_0} \left(1 - E^{(\alpha, a)}\right)^{C(\alpha, a)} \prod_{(\beta, b) \in \Gamma \times \mathscr{A}_1} \left(1 + E^{(\beta, b)}\right)^{-C(\beta, b)} \\
&\quad = \prod_{(\alpha, a) \in \Gamma \times \mathscr{A}} \left(1 - E^{(\alpha, a)}\right)^{C(\alpha, a)} \prod_{(\beta, b) \in \Gamma \times \mathscr{A}_1} \left(1 - E^{2(\beta, b)}\right)^{-C(\beta, b)} \\
&\quad = \prod_{(\alpha, a) \in \Gamma \times \mathscr{A}} \left(1 - E^{(\alpha, a)}\right)^{\psi(a) \dim L_{(\alpha, a)}} \prod_{\substack{(\alpha, a) = 2(\beta, b) \\ (\beta, b) \in \Gamma \times \mathscr{A}_1}} \left(1 - E^{(\alpha, a)}\right)^{\dim L_{(\beta, b)}}.
\end{aligned}$$

Hence the denominator identity for the free Lie superalgebra $\mathfrak{L}$ is equal to

$$\begin{aligned}
&\prod_{(\alpha, a) \in \Gamma \times \mathscr{A}} \left(1 - E^{(\alpha, a)}\right)^{\psi(a) \dim L_{(\alpha, a)}} \prod_{\substack{(\alpha, a) = 2(\beta, b) \\ (\beta, b) \in \Gamma \times \mathscr{A}_1}} \left(1 - E^{(\alpha, a)}\right)^{\dim L_{(\beta, b)}} \\
&\quad = 1 - \sum_{(\alpha, a) \in \Gamma \times \mathscr{A}} t(\alpha, a) E^{(\alpha, a)}.
\end{aligned} \tag{2.16}$$

Therefore, we obtain:

PROPOSITION 2.5. *Let $\mathscr{A}$ be a countable abelian group with a coloring map $\theta\colon \mathscr{A}\times\mathscr{A}\to \mathbf{C}^{\times}$ and Γ be a countable abelian semigroup satisfying the finiteness condition* (1.5). *For a $(\Gamma\times\mathscr{A})$-graded vector space $V=\bigoplus_{(\alpha,a)\in\Gamma\times\mathscr{A}} V_{(\alpha,a)}$ with finite dimensional homogeneous subspaces, let $L=\bigoplus_{(\alpha,a)\in\Gamma\times\mathscr{A}} L_{(\alpha,a)}$ be the free Lie algebra generated by V, and let $\mathfrak{L}=\bigoplus_{(\alpha,a)\in\Gamma\times\mathscr{A}} \mathfrak{L}_{(\alpha,a)}$ be the θ-colored free Lie superalgebra generated by V.*

Then, for any $(\alpha,a)\in\Gamma\times\mathscr{A}$, we have

$$\operatorname{Dim}\mathfrak{L}_{(\alpha,a)}=\psi(a)\dim L_{(\alpha,a)}+\sum_{\substack{(\alpha,a)=2(\beta,b)\\ (\beta,b)\in\Gamma\times\mathscr{A}_1}}\dim L_{(\beta,b)}. \quad (2.17)$$

EXAMPLE 2.6. Let $\mathscr{A}$ be a countable abelian group with a coloring map θ and $V=\bigoplus_{a\in\mathscr{A}}V_a$ be a θ-colored superspace. Suppose $\dim V_{(0)}=r$ and $\dim V_{(1)}=s$, where $V_{(0)}=\bigoplus_{a\in\mathscr{A}_0}V_a$, $V_{(1)}=\bigoplus_{b\in\mathscr{A}_1}V_b$. Let $\{x_1,\dots,x_r\}$ be a basis of $V_{(0)}$ and $\{x_{r+1},\dots,x_{r+s}\}$ be a basis of $V_{(1)}$.

Consider V as a vector space and let L be the free Lie algebra generated by V. Let $\Gamma=\mathbf{Z}^{r+s}_{\geq 0}\setminus\{0\}$. By defining $\deg x_i=(0,\dots,1,\dots,0)$ with 1 in the ith place, L becomes a Γ-graded Lie algebra. For each $\alpha=(\alpha_1,\dots,\alpha_{r+s})\in\Gamma$, set $|\alpha|=\sum\alpha_i$ and $\alpha!=\prod\alpha_i!$. Since $P(V,\Gamma)=\{(1,0,\dots,0),\dots,(0,\dots,0,1)\}$, we have $T(\alpha)=\{(\alpha_1,\dots,\alpha_{r+s},0,\dots)\}$ and $W(\alpha)=\frac{(|\alpha|-1)!}{\alpha!}$. Hence the generalized Witt formula (2.9) yields

$$\dim L_\alpha=\sum_{d|\alpha}\frac{1}{d}\mu(d)W\left(\frac{\alpha}{d}\right)=\frac{1}{|\alpha|}\sum_{d|\alpha}\mu(d)\frac{|\alpha/d|!}{(\alpha/d)!}$$

(cf. [BMPZ, Chap. 2, Theorem 1.16; Kan]).

On the other hand, consider V as a θ-colored superspace and let $\mathfrak{L}$ be the θ-colored free Lie superalgebra generated by V. The Lie superalgebra $\mathfrak{L}$ has a Γ-gradation $\mathfrak{L}=\bigoplus_{\alpha\in\Gamma}\mathfrak{L}_\alpha$ induced by our definition of the degrees of x_i's. Note that our choice of degrees of x_i's defines a coloring map on Γ. That is, $\alpha=(\alpha_1,\dots,\alpha_{r+s})\in\Gamma$ is even (resp. odd) if and only if $\alpha_{r+1}+\cdots+\alpha_{r+s}$ is even (resp. odd). We denote by Γ_0 (resp. Γ_1) the set of even (resp. odd) elements of Γ. Then, by Proposition 2.5, we obtain

$$\operatorname{Dim}\mathfrak{L}_\alpha=\begin{cases}\dim\mathfrak{L}_\alpha+\dim L_{\alpha/2} & \text{if } \alpha=2\beta \text{ for some } \beta\in\Gamma_1,\\ \psi(\alpha)\dim L_\alpha & \text{otherwise}.\end{cases}$$

If $\alpha = 2\beta$ for some $\beta \in \Gamma_1$, then all α_i are even and $\frac{1}{2}(\alpha_{r+1} + \cdots + \alpha_{r+s})$ is odd. Therefore, for $\alpha = (\alpha_1, \dots, \alpha_{r+s}) \in \Gamma = \mathbf{Z}_{\geq 0}^{r+s} \setminus \{0\}$, we have

$$\operatorname{Dim} \mathfrak{L}_\alpha = \begin{cases} \dim L_\alpha + \dim L_{\alpha/2} & \text{if all } \alpha_i \text{ are even and } \frac{1}{2}\left(\sum_{k=1}^{s} \alpha_{r+k}\right) \\ & \quad \text{is odd,} \\ \psi(\alpha) \dim L_\alpha & \text{otherwise,} \end{cases}$$

which recovers the formula in [BMPZ, Chap. 2, Corollary 2.8]. (There is a minor sign error in their formula.)

2.3. 1-*Dimensional Generalization*

In this subsection, we will consider the 1-*dimensional generalization* of the classical Witt formula for free Lie algebras to free Lie superalgebras. That is, we will discuss the applications of the generalized Witt formula to the $\mathbf{Z}_{>0}$-graded free Lie superalgebras.

Let $T_{\mathbf{Z}_{>0}}(q) = \sum_{i=1}^{\infty} t(i) q^i$ be a formal power series with $t(i) \in \mathbf{Z}$ for all $i \geq 1$. As we have seen in Section 1, the series $T_{\mathbf{Z}_{>0}}(q)$ can be interpreted as the supercharacter of a $\mathbf{Z}_{>0}$-graded superspace $V = \bigoplus_{i=1}^{\infty} V_i$ with $\operatorname{Dim} V_i = t(i) \in \mathbf{Z}$ $(i \geq 1)$. More precisely, take a $\mathbf{Z}$-graded vector space $V = \bigoplus_{i=1}^{\infty} V_i$ with $\dim V_i = |t(i)|$, and let $V_{(0)} = \bigoplus_{t(i)>0} V_i$, $V_{(1)} = \bigoplus_{t(i)<0} V_i$. Then V becomes a superspace with $\operatorname{Dim} V_i = t(i)$ for all $i \geq 1$.

Let $\mathfrak{L}$ be the free Lie superalgebra generated by V. We will apply the generalized Witt formulas (2.5)–(2.7) and (2.9) to this setting. First, we consider the $\mathbf{Z}_{>0}$-gradation on $\mathfrak{L}$. Note that $P(V, \mathbf{Z}_{>0}) = \mathbf{Z}_{>0} = \{1, 2, 3, \dots\}$, and for $n \geq 1$, we have

$$T(n) = \left\{ s = (s_i)_{i \geq 1} \,\middle|\, s_i \in \mathbf{Z}_{\geq 0},\ \sum i s_i = n \right\}, \tag{2.18}$$

the set of all partitions of n into a sum of positive integers. Thus the Witt partition function $W(n)$ is given by

$$W(n) = \sum_{s \in T(n)} \frac{(|s| - 1)!}{s!} \prod t(i)^{s_i}.$$

Therefore, by Proposition 2.3, we obtain the following 1-*dimensional generalization* of the Witt formula:

Proposition 2.7. *Let $V = \bigoplus_{i=1}^{\infty} V_i$ be a $\mathbf{Z}_{>0}$-graded superspace over $\mathbf{C}$ with $\operatorname{Dim} V_i = t(i) \in \mathbf{Z}$ for all $i \geq 1$, and let $\mathfrak{L} = \bigoplus_{n=1}^{\infty} \mathfrak{L}_n$ be the free Lie superalgebra generated by V with $\mathbf{Z}_{>0}$-gradation induced by V.*

Then we have

$$\operatorname{Dim} \mathfrak{L}_n = \sum_{d|n} \frac{1}{d}\mu(d) \sum_{s \in T\left(\frac{n}{d}\right)} \frac{(|s|-1)!}{s!} \prod t(i)^{s_i}. \tag{2.19}$$

Next, we consider the $(\mathbf{Z}_{>0} \times \mathbf{Z}_2)$-gradation on $\mathfrak{L}$. By defining $\varepsilon(i) = 0$ if $t(i) > 0$ and $\varepsilon(i) = 1$ if $t(i) < 0$, we obtain a $(\mathbf{Z}_{>0} \times \mathbf{Z}_2)$-gradation $V = \bigoplus_{(i,k) \in \mathbf{Z}_{>0} \times \mathbf{Z}_2} V_{(i,k)}$ such that $P(V, \mathbf{Z}_{>0} \times \mathbf{Z}_2) = \{(i, \varepsilon(i)) \mid i = 1, 2, 3, \dots\}$ and $\operatorname{Dim} V_{(i,k)} = t(i)$ for all $i \geq 1$, $k \in \mathbf{Z}_2$. Then the free Lie superalgebra $\mathfrak{L}$ on V has a $(\mathbf{Z}_{>0} \times \mathbf{Z}_2)$-gradation induced by V, and for each $n \in \mathbf{Z}_{>0}$, we have

$$\begin{aligned} T(n,0) &= \left\{ s = (s_i)_{i \geq 1} \,\middle|\, s_i \in \mathbf{Z}_{\geq 0},\ \sum s_i(i, \varepsilon(i)) = (n, 0) \right\} \\ &= \left\{ s = (s_i)_{i \geq 1} \,\middle|\, s_i \in \mathbf{Z}_{\geq 0},\ \sum i s_i = n,\ \sum s_i \varepsilon(i) = 0 \bmod 2) \right\} \\ &= \left\{ s \in T(n) \,\middle|\, \sum_{\varepsilon(i)=1} s_i \text{ is even} \right\}. \end{aligned}$$

Similarly, $T(n,1) = \{s \in T(n) \mid \sum_{\varepsilon(i)=1} s_i \text{ is odd}\}$. For $s = (s_i) \in T(n)$, set $|s|_- = \sum_{\varepsilon(i)=1} s_i$, the number of parts in $s = (s_i)$ with $t(i) < 0$. Then the Witt partition functions are given by

$$W^+(n) = W(n,0) = \sum_{\substack{s \in T(n) \\ |s|_-:\ \text{even}}} \frac{(|s|-1)!}{s!} \prod t(i)^{s_i},$$

$$W^-(n) = W(n,1) = \sum_{\substack{s \in T(n) \\ |s|_-:\ \text{odd}}} \frac{(|s|-1)!}{s!} \prod t(i)^{s_i}.$$

Note that $W^+(n) + W^-(n) = W(n)$. By Corollary 2.2, we obtain the following *twisted* 1-*dimensional generalization* of the Witt formula:

Proposition 2.8. *Let* $V = \bigoplus_{(i,k) \in \mathbf{Z}_{>0} \times \mathbf{Z}_2} V_{(i,k)}$ *be a* $(\mathbf{Z}_{>0} \times \mathbf{Z}_2)$-*graded superspace with* $\operatorname{Dim} V_{(i,k)} = t(i) \in \mathbf{Z}$ *for all* $i \geq 1$, *and let* $\mathfrak{L} = \bigoplus_{(n,k) \in \mathbf{Z}_{>0} \times \mathbf{Z}_2} \mathfrak{L}_{(n,k)}$ *be the free Lie superalgebra generated by* V *with* $(\mathbf{Z}_{>0} \times \mathbf{Z}_2)$ *gradation induced by* V.

Then we have

$$\operatorname{Dim} \mathfrak{L}_{(n,0)} = \sum_{d|n} \frac{1}{d}\mu(d)W^{+}\left(\frac{n}{d}\right) + \sum_{\substack{d|n \\ d:\text{ even}}} \frac{1}{d}\mu(d)W^{-}\left(\frac{n}{d}\right)$$

$$= \sum_{d|n} \frac{1}{d}\mu(d) \sum_{\substack{s \in T\left(\frac{n}{d}\right) \\ |s|_{-}:\text{even}}} \frac{(|s|-1)!}{s!} \prod t(i)^{s_i}$$

$$+ \sum_{\substack{d|n \\ d:\text{ even}}} \frac{1}{d}\mu(d) \sum_{\substack{s \in T\left(\frac{n}{d}\right) \\ |s|_{-}:\text{ odd}}} \frac{(|s|-1)!}{s!} \prod t(i)^{s_i}, \quad (2.20)$$

$$\operatorname{Dim} \mathfrak{L}_{(n,1)} = \sum_{\substack{d|n \\ d:\text{ odd}}} \frac{1}{d}\mu(d)W^{-}\left(\frac{n}{d}\right)$$

$$= \sum_{\substack{d|n \\ d:\text{ odd}}} \frac{1}{d}\mu(d) \sum_{\substack{s \in T\left(\frac{n}{d}\right) \\ |s|_{-}:\text{ odd}}} \frac{(|s|-1)!}{s!} \prod t(i)^{s_i}. \quad (2.21)$$

2.4. 2-*Dimensional Generalization*

In this subsection, we will discuss the applications of the generalized Witt formula to the $\mathbf{Z}^2$-graded free Lie superalgebras. Let $T_{\mathbf{Z}^2}(p,q) = \sum_{i,j=1}^{\infty} t(i,j)p^i q^j$ be a formal power series with $t(i,j) \in \mathbf{Z}$ for all $i, j \geq 1$, and let $V = \bigoplus_{i,j=1}^{\infty} V_{(i,j)}$ be a $\mathbf{Z}^2$-graded superspace with $\operatorname{Dim} V_{(i,j)} = t(i,j)$. Thus an element $(i,j) \in \mathbf{Z}^2$ is even (resp. odd) if $t(i,j) > 0$ (resp. $t(i,j) < 0$). Let $\mathfrak{L}$ be the free Lie superalgebra generated by V. We will apply the generalized Witt formulas (2.5)–(2.7) and (2.9) to this setting.

First, we consider the $\mathbf{Z}^2$-gradation on $\mathscr{L}$. Note that

$$P(V, \mathbf{Z}^2) = \mathbf{Z}_{>0} \times \mathbf{Z}_{>0} = \{(i,j) \mid i, j = 1, 2, 3, \ldots\},$$

and for $m, n \geq 1$, we have

$$T(m,n) = \left\{ s = (s_{ij})_{i,j>1} \,\middle|\, s_{i,j} \in \mathbf{Z}_{\geq 0},\ \sum s_{ij}(i,j) = (m,n) \right\}, \quad (2.22)$$

the set of all partitions of (m,n) into a sum of ordered pairs of positive integers. Thus the Witt partition function $W(m,n)$ is equal to

$$W(m,n) = \sum_{s \in T(m,n)} \frac{(|s|-1)!}{s!} \prod t(i,j)^{s_{ij}},$$

where $|s| = \sum s_{ij}$ and $s! = \prod s_{ij}!$. Therefore, by Proposition 2.3, we obtain the following 2-*dimensional generalization* of the Witt formula:

PROPOSITION 2.9. *Let $V = \bigoplus_{i,j=1}^{\infty} V_{(i,j)}$ be a $\mathbf{Z}^2_{>0}$-graded superspace over $\mathbf{C}$ with $\operatorname{Dim} V_{(i,j)} = t(i,j) \in \mathbf{Z}$ for all $i, j \geq 1$, and let $\mathfrak{L} = \bigoplus_{m,n=1}^{\infty} \mathfrak{L}_{(m,n)}$ be the free Lie superalgebra generated by V with $\mathbf{Z}^2$-gradation induced by V with $\mathbf{Z}^2$-gradation induced by .*

Then we have

$$\operatorname{Dim} \mathfrak{L}_{(m,n)} = \sum_{d|(m,n)} \frac{1}{d}\mu(d) \sum_{s \in T\left(\frac{m}{d}, \frac{n}{d}\right)} \frac{(|s|-1)!}{s!} \prod t(i,j)^{s_{ij}}. \quad (2.23)$$

Next, we consider the $(\mathbf{Z}^2_{>0} \times \mathbf{Z}_2)$-gradation on $\mathfrak{L}$. By defining $\varepsilon(i,j) = 0$ if $t(i,j) > 0$ and $\varepsilon(i,j) = 1$ if $t(i,j) < 0$, we obtain a $(\mathbf{Z}^2_{>0} \times \mathbf{Z}_2)$-gradation on $V = \bigoplus_{(i,j,k) \in \mathbf{Z}^2 \times \mathbf{Z}_2} V_{(i,j,k)}$ such that

$$P\left(V, \mathbf{Z}^2_{>0} \times \mathbf{Z}_2\right) = \left\{(i, j, \varepsilon(i,j)) \,\middle|\, i, j = 1, 2, 3, \ldots\right\}$$

and $\operatorname{Dim} V_{(i,j,k)} = t(i,j)$ for all $i, j \geq 1$, $k \in \mathbf{Z}_2$, which induces a $(\mathbf{Z}^2_{>0} \times \mathbf{Z}_2)$-gradation on $\mathfrak{L}$.

For $s = (s_{ij}) \in T(m,n)$, set $|s|_- = \sum_{\varepsilon(i,j)=1} s_{ij}$. Then we have

$$T(m,n,0) = \left\{s \in T(m,n) \,\middle|\, |s|_- \text{ is even}\right\},$$
$$T(m,n,1) = \left\{s \in T(m,n) \,\middle|\, |s|_- \text{ is odd}\right\},$$

and the Witt Partition functions are given by

$$W^+(m,n) = W(m,n,0) = \sum_{\substack{s \in T(m,n) \\ |s|_-:\ \text{even}}} \frac{(|s|-1)!}{s!} \prod t(i,j)^{s_{ij}},$$

$$W^-(m,n) = W(m,n,1) = \sum_{\substack{s \in T(m,n) \\ |s|_-:\ \text{odd}}} \frac{(|s|-1)!}{s!} \prod t(i,j)^{s_{ij}}.$$

Therefore, by Corollary 2.2, we obtain the following *twisted 2-dimensional generalization* of the Witt formula:

PROPOSITION 2.10. *Let $V = \bigoplus_{(i,j,k) \in \mathbf{Z}^2_{>0} \times \mathbf{Z}_2} V_{(i,j,k)}$ be a $(\mathbf{Z}^2_{>0} \times \mathbf{Z}_2)$-graded superspace with $\operatorname{Dim} V_{(i,j,k)} = t(i,j) \in \mathbf{Z}$ for all $i, j \geq 1$, and let $\mathfrak{L}$ be the free Lie superalgebra generated by V with $(\mathbf{Z}^2_{>0} \times \mathbf{Z}_2)$-gradation $\mathfrak{L} = \bigoplus_{(m,n,k) \in \mathbf{Z}^2_{>0} \times \mathbf{Z}_2} \mathfrak{L}_{(m,n,k)}$ induced by V.*

Then we have

$$\operatorname{Dim} \mathfrak{L}_{(m,n,0)} = \sum_{d|(m,n)} \frac{1}{d}\mu(d) W^{+}\left(\frac{m}{d}, \frac{n}{d}\right) + \sum_{\substack{d|(m,n)\\ d:\ \text{even}}} \frac{1}{d}\mu(d) W^{-}\left(\frac{m}{d}, \frac{n}{d}\right)$$

$$= \sum_{d|(m,n)} \frac{1}{d}\mu(d) \sum_{\substack{s \in T\left(\frac{m}{d}, \frac{n}{d}\right)\\ |s|_{-}:\ \text{even}}} \frac{(|s|-1)!}{s!} \prod t(i,j)^{s_{ij}}$$

$$+ \sum_{\substack{d|(m,n)\\ d:\ \text{even}}} \frac{1}{d}\mu(d) \sum_{\substack{s \in T\left(\frac{m}{d}, \frac{n}{d}\right)\\ |s|_{-}:\ \text{odd}}} \frac{(|s|-1)!}{s!} \prod t(i,j)^{s_{ij}}, \tag{2.24}$$

$$\operatorname{Dim} \mathfrak{L}_{(m,n,1)} = \sum_{\substack{d|(m,n)\\ d:\ \text{odd}}} \frac{1}{d}\mu(d) W^{-}\left(\frac{m}{d}, \frac{n}{d}\right)$$

$$= \sum_{\substack{d|(m,n)\\ d:\ \text{odd}}} \frac{1}{d}\mu(d) \sum_{\substack{s \in T\left(\frac{m}{d}, \frac{n}{d}\right)\\ |s|_{-}:\ \text{odd}}} \frac{(|s|-1)!}{s!} \prod t(i,j)^{s_{ij}}. \tag{2.25}$$

Remark. The above discussion can be generalized to the free Lie superalgebras generated by $\mathbf{Z}^n_{>0}$-graded superspaces

$$V = \bigoplus_{(i_1,\ldots,i_n)\in \mathbf{Z}^n_{>0}} V_{(i_1,\ldots,i_n)}$$

such that

$$\operatorname{Dim} V_{(i_1,\ldots,i_n)} = t(i_1,\ldots,i_n) \in \mathbf{Z} \qquad \text{for all } (i_1,\ldots,i_n) \in \mathbf{Z}^n_{>0}.$$

For example, the exchange principle (2.13) can be generalized to the *n-dimensional exchange principle*. More precisely, for $k_1,\ldots,k_n \in \mathbf{Z}_{>0}$, let

$$T(k_1,\ldots,k_n) = \Big\{ s = (s_{i_1,\ldots,i_n}) \Big| s_{i_1,\ldots,i_n} \in \mathbf{Z}_{\geq 0},$$
$$\sum s_{i_1,\ldots,i_n}(i_1,\ldots,i_n) = (k_1,\ldots,k_n) \Big\},$$

the set of all partitions of $(k_1, \ldots, k_n)$ into a sum of ordered n-tuples of positive integers, and let

$$S_r(k_1, \ldots, k_n) = \sum_{s \in T(k_1, \ldots, k_n)} \frac{(|s| - 1)!}{s!} r^{|s|},$$

where $|s| = \sum s_{i_1, \ldots, i_n}$ and $s! = \prod s_{i_1, \ldots, i_n}!$. Then, for any $r, a_1, \ldots, a_n \in \mathbf{Z}$, we have

$$\prod_{k_1, \ldots, k_n = 1}^{\infty} \left(1 - a_1^{k_1} \cdots a_n^{k_n} q_1^{k_1} \cdots q_n^{k_n}\right)^{\sum_{d|(k_1, \ldots, k_n)} \frac{1}{d} \mu(d) S_r\left(\frac{k_1}{d}, \ldots, \frac{k_n}{d}\right)}$$

$$= \prod_{k_1, \ldots, k_n = 1}^{\infty} \left(1 - q_1^{k_1} \cdots q_n^{k_n}\right)^{\sum_{d|(k_1, \ldots, k_n)} \frac{1}{d} \mu(d) a_1^{k_1/d} \cdots a_n^{k_n/d} S_r(\frac{k_1}{d}, \ldots, \frac{k_n}{d})}.$$

2.5. *Product Identities*

The purpose of this subsection is to investigate the relation of graded Lie superalgebras and product identities for normalized formal power series. Let us begin with the binomial expansion

$$(1 - q)^r = \sum_{k=0}^{r} (-1)^k \binom{r}{k} q^k. \tag{2.27}$$

Let $L = \mathbf{C}x_1 \oplus \cdots \oplus \mathbf{C}x_r$ be the r-dimensional abelian Lie algebra with basis $\{x_1, \ldots, x_r\}$. Then we have

$$H_k(L) = \Lambda^k(L) = \operatorname{Span}\left\{x_{i_1} \wedge \cdots \wedge x_{i_k} \middle| 1 \le i_1 < \cdots < i_k \le r\right\},$$

and hence the identity (2.27) can be interpreted as the Euler−Poincaré principle for the abelian Lie algebra L.

Similarly, consider the product expansion

$$(1 - q)^{-r} = \sum_{k=0}^{\infty} \binom{r + k - 1}{k} q^k, \tag{2.28}$$

and let $\mathfrak{L} = \mathbf{C}y_1 \oplus \cdots \oplus \mathbf{C}y_r$ be the abelian Lie superalgebra generated by the *odd* elements $\{y_1, \ldots, y_r\}$. Then we have

$$H_k(\mathfrak{L}) = C_k(\mathfrak{L}) = S^k(\mathfrak{L}) = \operatorname{Span}\left\{y_{i_1} y_{i_2} \cdots y_{i_k} \middle| 1 \le i_1 \le i_2 \le \cdots \le i_k \le r\right\},$$

and therefore, the identity (2.28) can be interpreted as the Euler−Poincaré principle for the abelian Lie superalgebra $\mathfrak{L}$.

In [K2], Jacobi's triple product identity

$$\prod_{n=1}^{\infty}(1-p^nq^n)(1-p^{n-1}q^n)(1-p^nq^{n-1}) = \sum_{k\in\mathbf{Z}}(-1)^k p^{\frac{k(k-1)}{2}} q^{\frac{k(k+1)}{2}} \tag{2.29}$$

was interpreted as the denominator identity for affine Kac–Moody algebra of type $A_1^{(1)}$ (cf. [K4]). In fact, Kac showed that all the Macdonald identities [M] are equivalent to the denominator identities for affine Kac–Moody algebras [K2]. Furthermore, in [KW1, KW2], Kac and Wakimoto investigated the relation of affine Lie superalgebras and many interesting product identities arising from number theory.

Recently, Borcherds completed the proof of the *Moonshine Conjecture* by constructing an infinite dimensional Lie algebra, called the *Monster Lie algebra* [B3]. One of the main ingredients of his proof is the following product identity

$$p^{-1}\prod_{\substack{m>0\\ n\in\mathbf{Z}}}(1-p^mq^n)^{c(mn)} = j(p)-j(q), \tag{2.30}$$

where the $c(n)$ are the coefficients of the elliptic modular function

$$J(q) = j(q) - 744 = \sum_{n=-1}^{\infty} c(n)q^n$$

$$= q^{-1} + 196884q + 21493760q^2 + \cdots . \tag{2.31}$$

It was shown in [B3] that the Monster Lie algebra is a generalized Kac–Moody algebra (cf. [B2]) and the identity (2.30) was interpreted as the denominator identity for the Monster Lie algebra.

More generally, let Γ be a countable abelian semigroup satisfying the finiteness condition (1.5), and consider a normalized formal power series

$$1-T_\Gamma = 1-\sum_{\alpha\in\Gamma} t(\alpha)E^\alpha \qquad \text{with } t(\alpha)\in\mathbf{Z} \text{ for all } \alpha\in\Gamma. \tag{2.32}$$

Suppose we have a product identity for the above formal power series,

$$\prod_{\alpha\in\Gamma}(1-E^\alpha)^{C(\alpha)} = 1-\sum_{\alpha\in\Gamma} t(\alpha)E^\alpha \tag{2.33}$$

with $C(\alpha)\in\mathbf{Z}$ for all $\alpha\in\Gamma$.

If we could construct a "natural" Γ-graded Lie superalgebra $\mathfrak{L} = \bigoplus_{\alpha\in\Gamma}\mathfrak{L}_\alpha$ such that the supercharacter of the homology superspace $H(\mathfrak{L}) = \sum_{k=1}^{\infty}(-1)^{k+1}H_k(\mathfrak{L})$ is given by

$$\operatorname{Ch} H(\mathfrak{L}) = \sum_{\alpha\in\Gamma}\operatorname{Dim} H(\mathfrak{L})_\alpha E^\alpha = \sum_{\alpha\in\Gamma} t(\alpha)E^\alpha, \tag{2.34}$$

then the product identity (2.33) can be interpreted as the Euler–Poincaré principle, i.e., the denominator identity for the graded Lie superalgebra $\mathfrak{L} = \bigoplus_{\alpha \in \Gamma} \mathfrak{L}_\alpha$. In particular, we would have

$$\operatorname{Dim} \mathfrak{L}_\alpha = C(\alpha) \qquad \text{for all } \alpha \in \Gamma. \tag{2.35}$$

In general, it is quite complicated and difficult to construct a "natural" graded Lie superalgebra corresponding to the product identity (2.33). Still, we can always interpret the identity (2.33) as the denominator identity for a suitably defined free Lie superalgebra.

More precisely, consider a Γ-graded vector space $V = \bigoplus_{\alpha \in \Gamma} V_\alpha$ over $\mathbf{C}$ with $\dim V_\alpha = |t(\alpha)|$ for all $\alpha \in \Gamma$. Let $V_{(0)} = \bigoplus_{\alpha:\ t(\alpha) > 0} V_\alpha$ and $V_{(1)} = \bigoplus_{\alpha:\ t(\alpha) < 0} V_\alpha$. Then $V = V_{(0)} \oplus V_{(1)} = \bigoplus_{(\alpha, i) \in \Gamma \times \mathbf{Z}_2} V_{(\alpha, i)}$ becomes a $(\Gamma \times \mathbf{Z}_2)$-graded superspace with $\operatorname{Dim} V_{(\alpha, i)} = t(\alpha)$ for all $\alpha \in \Gamma$, $i \in \mathbf{Z}_2$. Thus $\operatorname{Dim} V_\alpha = t(\alpha)$ for all $\alpha \in \Gamma$ and the right-handed side of (2.33) can be interpreted as $1 - \operatorname{Ch} V$. Let $\mathfrak{L}$ be the free Lie superalgebra generated by V. Then the free Lie superalgebra $\mathfrak{L}$ has a Γ-gradation induced by V, and the denominator identity for $\mathfrak{L} = \bigoplus_{\alpha \in \Gamma} \mathfrak{L}_\alpha$ is the same as the product identity (2.33). Hence we have

$$\operatorname{Dim} \mathfrak{L}_\alpha = C(\alpha) \qquad \text{for all } \alpha \in \Gamma.$$

On the other hand, let $P(V, \Gamma) = \{\alpha \in \Gamma \mid t(\alpha) \neq 0\}$ and let $\{\tau_1, \tau_2, \tau_3, \dots\}$ be an enumeration of $P(V, \Gamma)$. Then, by the generalized Witt formula (2.9), we have

$$\operatorname{Dim} \mathfrak{L}_\alpha = \sum_{\substack{d > 0 \\ \alpha = d\tau}} \frac{1}{d} \mu(d) \sum_{s \in T(\tau)} \frac{(|s| - 1)!}{s!} \prod t(\tau_i)^{s_i},$$

which yields a combinatorial identity

$$C(\alpha) = \sum_{\substack{d > 0 \\ \alpha = d\tau}} \frac{1}{d} \mu(d) \sum_{s \in T(\tau)} \frac{(|s| - 1)!}{s!} \prod t(\tau_i)^{s_i}. \tag{2.36}$$

EXAMPLE 2.11. (a) Consider the generating function for the partition function $p(n)$:

$$\prod_{n=1}^{\infty} (1 - q^n)^{-1} = \sum_{i=0}^{\infty} p(i) q^i = 1 - \sum_{i=1}^{\infty} (-p(i)) q^i. \tag{2.37}$$

Let $V = \bigoplus_{i=1}^{\infty} V_i$ be a $\mathbf{Z}_{>0}$-graded superspace with $\operatorname{Dim} V_i = -p(i)$ for all $i \geq 1$ (thus $V_{(0)} = 0$, $V_{(1)} = V$), and let $\mathfrak{L} = \bigoplus_{n=1}^{\infty} \mathfrak{L}_n$ be the free Lie superalgebra generated by V. Then (2.37) can be interpreted as the denominator identity for the free Lie superalgebra $\mathfrak{L}$, and hence we have

$$\operatorname{Dim} \mathfrak{L}_n = -1 \qquad \text{for all } n \geq 1.$$

Therefore, by (2.36), we obtain

$$\sum_{d|n} \frac{1}{d} \mu(d) \sum_{s \in T\left(\frac{n}{d}\right)} \frac{(|s|-1)!}{s!} (-1)^{|s|} \prod p(i)^{s_i} = -1. \tag{2.38}$$

(b) Recall the definition of the *Ramanujan tau-function*:

$$\begin{aligned} \Delta(q) &= q \prod_{n=1}^{\infty} (1-q^n)^{24} = \sum_{i=1}^{\infty} \tau(i) q^i \\ &= q - 24q^2 + 252q^3 - 1472q^4 - \cdots . \end{aligned} \tag{2.39}$$

We can rewrite it as

$$\prod_{n=1}^{\infty} (1-q^n)^{24} = 1 - \sum_{i=1}^{\infty} (-\tau(i+1)) q^i. \tag{2.40}$$

Let $V = \bigoplus_{i=1}^{\infty} V_i$ be a $\mathbf{Z}_{>0}$-graded superspace with $\operatorname{Dim} V_i = -\tau(i+1)$ for all $i \geq 1$ (thus $V_{(0)} = \bigoplus_{i:\ \tau(i+1)<0} V_i$, $V_{(1)} = \bigoplus_{i:\ \tau(i+1)>0} V_i$), and let $\mathfrak{L} = \bigoplus_{n=1}^{\infty} \mathfrak{L}_n$ be the free Lie superalgebra generated by V. Then (2.40) can be interpreted as the denominator identity for the free Lie superalgebra $\mathfrak{L}$, and hence we have

$$\operatorname{Dim} \mathfrak{L}_n = 24 \qquad \text{for all } n \geq 1.$$

Therefore, by (2.36), we obtain

$$\sum_{d|n} \frac{1}{d} \mu(d) \sum_{s \in T\left(\frac{n}{d}\right)} \frac{(|s|-1)!}{s!} (-1)^{|s|} \prod \tau(i+1)^{s_i} = 24. \tag{2.41}$$

The relation (2.41) allows us to determine the values of the Ramanujan tau-function $\tau(n)$ recursively.

(c) Recall the product identity (2.30) for the elliptic modular function $J(q) = j(q) - 744$. Observing that $c(0) = 0$ and $c(-k) = 0$ for $k > 1$, it can be written as

$$\begin{aligned} \prod_{m,n=1}^{\infty} (1-p^m q^n)^{c(mn)} &= \frac{j(p) - j(q)}{p^{-1} - q^{-1}} \\ &= 1 - \sum_{i,j=1}^{\infty} c(i+j-1) p^i q^j. \end{aligned} \tag{2.42}$$

Let $V = \bigoplus_{i,j=1}^{\infty} V_{(i,j)}$ be a $\mathbf{Z}_{>0}^2$-graded vector space over $\mathbf{C}$ with $\dim V_{(i,j)} = c(i+j-1)$ for $i, j \geq 1$, and let $L = \bigoplus_{m,n=1}^{\infty} L_{(m,n)}$ be the free Lie algebra generated by V. Then the product identity (2.42) for the elliptic modular function J is the denominator identity for the free Lie algebra L, and we have $\dim L_{(m,n)} = c(mn)$.

(d) Let $V = \bigoplus_{i,j=1}^{\infty} V_{(i,j)}$ be a $\mathbf{Z}_{>0}^2$-graded superspace over $\mathbf{C}$ with $\operatorname{Dim} V_{(i,j)} = (-1)^{i+j} c(i+j-1)$, and let $\mathfrak{L} = \bigoplus_{m,n=1}^{\infty} \mathfrak{L}_{(m,n)}$ be the free Lie superalgebra generated by V. Replacing p and q by $-p$ and $-q$ in (2.42) yields

$$\prod_{m,n=1}^{\infty} \left(1 - (-1)^{m+n} p^m q^n\right)^{c(mn)} = 1 - \sum_{i,j=1}^{\infty} (-1)^{i+j} c(i+j-1) p^i q^j,$$

which can be regarded as the denominator identity for the free Lie superalgebra $\mathfrak{L}$. Observe that

$$\prod_{m,n=1}^{\infty} \left(1 - (-1)^{m+n} p^m q^n\right)^{c(mn)} = \prod_{m,n=1}^{\infty} (1 - p^m q^n)^{(-1)^{m+n} c(mn)} \prod_{\substack{m,n:\ \text{even} \\ m+n \equiv 2 \pmod 4}} (1 - p^m q^n)^{c(mn/4)}.$$

Hence the denominator identity for the free Lie superalgebra $\mathfrak{L} = \bigoplus_{m,n=1}^{\infty} \mathfrak{L}_{(m,n)}$ is the same as

$$\prod_{m,n=1}^{\infty} (1 - p^m q^n)^{(-1)^{m+n} c(mn)} \prod_{\substack{m,n:\ \text{even} \\ m+n \equiv 2 \pmod 4}} (1 - p^m q^n)^{c(mn/4)} = 1 - \sum_{i,j=1}^{\infty} (-1)^{i+j} c(i+j-1) p^i q^j, \tag{2.43}$$

which implies

$$\operatorname{Dim} \mathfrak{L}_{(m,n)} = \begin{cases} c(mn) = c\left(\dfrac{mn}{4}\right) & \text{if } m, n \text{ are even, } m+n \equiv 2 \pmod 4, \\ (-1)^{m+n} c(mn) & \text{otherwise.} \end{cases}$$

Note that this is a special case of Proposition 2.5.

In [B4], Borcherds gave a very important method for constructing automorphic forms on $O_{s+2,2}(\mathbf{R})^+$ with infinite product expansions. In particular, let

$$F(\tau) = \sum_{n\in\mathbf{Z}_{>0},\ \mathrm{odd}} \sigma_1(n)q^n = q + 4q^3 + 6q^5 + \cdots,$$

$$\theta(\tau) = \sum_{n\in\mathbf{Z}} q^{n^2} = 1 + 2q + 2q^4 + \cdots,$$

and define

$$\begin{aligned} f_0(\tau) &= F(\tau)\theta(\tau)\frac{\big(\theta(\tau)^4 - 2F(\tau)\big)\big(\theta(\tau)^4 - 16F(\tau)\big)E_6(4\tau)}{\Delta(4\tau)} + 56\theta(\tau) \\ &= \sum_{n\in\mathbf{Z}} c_0(n)q^n = q^{-3} - 248q + 26752q^4 - \cdots, \\ g_0(\tau) &= (j(4\tau) - 876)\theta(\tau) \\ &\quad - 2F(\tau)\theta(\tau)\frac{\big(\theta(\tau)^4 - 2F(\tau)\big)\big(\theta(\tau)^4 - 16F(\tau)\big)E_6(4\tau)}{\Delta(4\tau)} \\ &= \sum_{n\in\mathbf{Z}} b_0(n)q^n = q^{-4} + 6 + 504q + 143388q^4 - \cdots. \end{aligned}$$

Then he proved the following product identities for the elliptic modular function j and the Eisenstein series E_4, E_6, E_8, E_{10}, and E_{14}:

$$j(\tau) = q^{-1} + 744 + 196884q + \cdots = q^{-1}\prod_{n=1}^{\infty}(1-q^n)^{3c_0(n^2)},$$

$$E_4(\tau) = 1 + 240\sum_{n=1}^{\infty}\sigma_3(n)q^n = \prod_{n=1}^{\infty}(1-q^n)^{c_0(n^2)+8},$$

$$E_6(\tau) = 1 - 504\sum_{n=1}^{\infty}\sigma_5(n)q^n = \prod_{n=1}^{\infty}(1-q^n)^{b_0(n^2)},$$

$$E_8(\tau) = 1 + 480\sum_{n=1}^{\infty}\sigma_7(n)q^n = \prod_{n=1}^{\infty}(1-q^n)^{2c_0(n^2)+16},$$

$$E_{10}(\tau) = 1 - 264\sum_{n=1}^{\infty}\sigma_9(n)q^n = \prod_{n=1}^{\infty}(1-q^n)^{b_0(n^2)+c_0(n^2)+8},$$

$$E_{14}(\tau) = 1 - 24\sum_{n=1}^{\infty}\sigma_{13}(n)q^n = \prod_{n=1}^{\infty}(1-q^n)^{b_0(n^2)+2c_0(n^2)+16}.$$

The product identity for the modular function j can be written as

$$\prod_{n=1}^{\infty}(1-q^n)^{3c_0(n^2)} = 1 + 744q + 196884q^2 + \cdots$$
$$= 1 - \sum_{i=1}^{\infty}\left(-c_1(i-1)\right)q^i, \qquad (2.44)$$

where the $c_1(i)$ are the coefficients of j. (Note that $c_1(0) = 744$ and $c_1(n) = c(n)$ for all $n \neq 0$.) Let $V = \bigoplus_{i=1}^{\infty} V_i$ be the $\mathbf{Z}_{>0}$-graded superspace with $\operatorname{Dim} V_i = -c_1(i-1)$ for all $i \geq 1$ (thus $V_{(0)} = 0$, $V_{(1)} = V$), and let $\mathfrak{L} = \bigoplus_{n=1}^{\infty} \mathfrak{L}_n$ be the free Lie superalgebra genreated by V. Then (2.44) can be interpreted as the denominator identity for the free Lie superalgebra $\mathfrak{L}$, and hence

$$\operatorname{Dim} \mathfrak{L}_n = 3c_0(n^2) \qquad \text{for all } n \geq 1.$$

Therefore, we obtain a combinatorial identity:

$$3c_0(n^2) = \sum_{d|n}\frac{1}{d}\mu(d)\sum_{s\in T\left(\frac{n}{d}\right)}\frac{(|s|-1)!}{s!}(-1)^{|s|}\prod c_1(i-1)^{s_i}. \quad (2.45)$$

Similarly, the product identities for the Eisenstein series can be interpreted as the denominator identities for the free Lie superalgebras $\mathfrak{L} = \bigoplus_{n=1}^{\infty} \mathfrak{L}_n$ generated by the $\mathbf{Z}_{>0}$-graded superspaces $V = \bigoplus_{i=1}^{\infty} V_i$ with $\operatorname{Dim} V_i = -240\sigma_3(i)$, $504\sigma_5(n)$, $-480\sigma_7(i)$, $264\sigma_9(n)$, and $24\sigma_{13}(n)$, respectively. Therefore, the superdimensions of the homogeneous subspaces $\mathfrak{L}_n$ are given by

$$\operatorname{Dim} \mathfrak{L}_n = c_0(n^2) + 8 \qquad (\text{corresponding to } E_4(\tau)),$$
$$\operatorname{Dim} \mathfrak{L}_n = b_0(n^2) \qquad (\text{corresponding to } E_6(\tau)),$$
$$\operatorname{Dim} \mathfrak{L}_n = 2c_0(n^2) + 16 \qquad (\text{corresponding to } E_8(\tau)),$$
$$\operatorname{Dim} \mathfrak{L}_n = b_0(n^2) + c_0(n^2) + 8 \qquad (\text{corresponding to } E_{10}(\tau)),$$
$$\operatorname{Dim} \mathfrak{L}_n = b_0(n^2) + 2c_0(n^2) + 16 \qquad (\text{corresponding to } E_{14}(\tau)),$$

and (2.36) yields the following combinatorial identities:

$$
\begin{aligned}
c_0(n^2) + 8 &= \sum_{d|n} \frac{1}{d}\mu(d) \sum_{s \in T\left(\frac{n}{d}\right)} \frac{(|s|-1)!}{s!}(-240)^{|s|} \prod \sigma_3(i)^{s_i},\\
b_0(n^2) &= \sum_{d|n} \frac{1}{d}\mu(d) \sum_{s \in T\left(\frac{n}{d}\right)} \frac{(|s|-1)!}{s!} 504^{|s|} \prod \sigma_5(i)^{s_i},\\
2c_0(n^2) + 16 &= \sum_{d|n} \frac{1}{d}\mu(d) \sum_{s \in T\left(\frac{n}{d}\right)} \frac{(|s|-1)!}{s!}(-480)^{|s|} \prod \sigma_7(i)^{s_i},\\
b_0(n^2) + c_0(n^2) + 8 &= \sum_{d|n} \frac{1}{d}\mu(d) \sum_{s \in T\left(\frac{n}{d}\right)} \frac{(|s|-1)!}{s!} 264^{|s|} \prod \sigma_9(i)^{s_i},\\
b_0(n^2) + 2c_0(n^2) + 16 &= \sum_{d|n} \frac{1}{d}\mu(d) \sum_{s \in T\left(\frac{n}{d}\right)} \frac{(|s|-1)!}{s!} 24^{|s|} \prod \sigma_{13}(i)^{s_i}.
\end{aligned}
\tag{2.46}
$$

3. GENERALIZED KAC–MOODY SUPERALGEBRAS

3.1. *Weyl–Kac–Borcherds Character Formula*

Let I be a countable (possibly infinite) index set. A real square matrix $A = (a_{ij})_{i,j \in I}$ is called a *Borcherds–Cartan matrix* if it satisfies (i) $a_{ii} = 2$ or $a_{ii} \leq 0$ for all $i \in I$; (ii) $a_{ij} \leq 0$ if $i \neq j$, and $a_{ij} \in \mathbf{Z}$ if $a_{ii} = 2$; (iii) $a_{ij} = 0$ implies $a_{ji} = 0$. We say that an index i is *real* if $a_{ii} = 2$ and *imaginary* if $a_{ii} \leq 0$. We denote by $I^{re} = \{i \in \mid a_{ii} = 2\}$, $I^{im} = \{i \in \mid a_{ii} \leq 0\}$. Let $\underline{m} = (m_i \in \mathbf{Z}_{>0} \mid i \in I)$ be a collection of positive integers such that $m_i = 1$ for all $i \in I^{re}$. We call $\underline{m}$ the *charge* of A. In this paper, we assume that the Borcherds–Cartan matrix A is *symmetrizable*, i.e., there is a diagonal matrix $D = \operatorname{diag}(s_i \mid i \in I)$ with $s_i > 0$ $(i \in I)$ such that DA is symmetric.

Let $C = (\theta_{ij})_{i,j \in I}$ be a complex matrix satisfying $\theta_{ij}\theta_{ji} = 1$ for all $i, j \in I$. Thus we have $\theta_{ii} = \pm 1$ for all $i \in I$. We call $i \in I$ an *even index* if $\theta_{ii} = 1$ and an *odd index* if $\theta_{ii} = -1$. We denote by I^{even} (resp. I^{odd}) the set of all even (resp. odd) indices. We say that a Borcherds–Cartan matrix $A = (a_{ij})_{i,j \in I}$ is *restricted* with respect to C if it satisfies the following condition:

$$\text{if } a_{ii} = 2 \text{ and } \theta_{ii} = -1, \text{ then the } a_{ij} \text{ are even integers for all } j \in I. \tag{3.1}$$

In this case, the matrix C is called a *coloring matrix* of A.

Let $\mathfrak{h} = (\bigoplus_{i \in I} \mathbf{C} h_i) \oplus (\bigoplus_{i \in I} \mathbf{C} d_i)$ be a complex vector space with a basis $\{h_i, d_i \mid i \in I\}$, and for each $i \in I$ define a linear functional $\alpha_i \in \mathfrak{h}^*$ by

$$\alpha_i(h_j) = a_{ji}, \qquad \alpha_i(d_j) = \delta_{ij} \qquad \text{for all } j \in I. \tag{3.2}$$

The free abelian group $Q = \bigoplus_{i \in I} \mathbf{Z} \alpha_i$ generated by α_i's $(i \in I)$ is called the *root lattice* associated with A. Since A is assumed to be symmetrizable, there is a symmetric bilinear form $(\,|\,)$ on Q given by $(\alpha_i \mid \alpha_j) = s_i a_{ij} = s_j a_{ji}$ for all $i, j \in I$. Let $Q_+ = \sum_{i \in I} \mathbf{Z}_{\geq 0} \alpha_i$ and $Q_- = -Q_+$. There is a partial ordering $\geq$ on Q given by $\lambda \geq \mu$ if and only if $\lambda - \mu \in Q_+$. The coloring matrix $C = (\theta_{ij})_{i,j \in I}$ defines a bimultiplicative map $\theta \colon Q \times Q \to \mathbf{C}^\times$ on Q by

$$\begin{aligned} \theta(\alpha_i, \alpha_j) &= \theta_{ij} \qquad \text{for all } i, j \in I, \\ \theta(\alpha + \beta, \gamma) &= \theta(\alpha, \gamma)\theta(\beta, \gamma), \\ \theta(\alpha, \beta + \gamma) &= \theta(\alpha, \beta)\theta(\alpha, \gamma) \end{aligned} \tag{3.3}$$

for all $\alpha, \beta, \gamma \in Q$. Note that, since $\theta_{ij}\theta_{ji} = 1$ for all $i, j \in I$, θ satisfies

$$\theta(\alpha, \beta)\theta(\beta, \alpha) = 1 \qquad \text{for all } \alpha, \beta \in Q. \tag{3.4}$$

That is, θ is a coloring map on Q. In particular, $\theta(\alpha, \alpha) = \pm 1$ for all $\alpha \in Q$. We say $\alpha \in Q$ is *even* if $\psi(\alpha) = \theta(\alpha, \alpha) = 1$ and *odd* if $\psi(\alpha) = \theta(\alpha, \alpha) = -1$.

DEFINITION 3.1. The *generalized Kac–Moody superalgebra* $\mathfrak{g} = \mathfrak{g}(A, \underline{m}, C)$ associated with a symmetrizable Borcherds–Cartan matrix $A = (a_{ij})_{i,j \in I}$ of charge $\underline{m} = (m_i \mid i \in I)$ with a coloring matrix $C = (\theta_{ij})_{i,j \in I}$ is the θ-colored Lie superalgebra generated by the elements

h_i, d_i $(i \in I)$, e_{ik}, f_{ik} $(i \in I,\ k = 1, 2, \ldots, m_i)$ with the defining relations,

$$\begin{gathered}[h_i, h_j] = [h_i, d_j] = [d_i, d_j] = 0,\\ [h_i, e_{jl}] = a_{ij} e_{jl}, \qquad [h_i, f_{jl}] = -a_{ij} f_{jl},\\ [d_i, e_{jl}] = \delta_{ij} e_{jl}, \qquad [d_i, f_{jl}] = -\delta_{ij} f_{jl},\\ [e_{ik}, f_{jl}] = \delta_{ij}\delta_{kl} h_i,\\ (ad e_{ik})^{1-a_{ij}}(e_{jl}) = (ad f_{ik})^{1-a_{ij}}(f_{jl}) = 0 \qquad \text{if } a_{ii} = 2 \text{ and } i \neq j,\\ [e_{ik}, e_{jl}] = [f_{ik}, f_{jl}] = 0 \qquad \text{if } a_{ij} = 0\end{gathered} \tag{3.5}$$

for $i, j \in I$, $k = 1, \ldots, m_i$, $l = 1, \ldots, m_j$.

The abelian subalgebra $\mathfrak{h} = (\oplus_{i \in I} \mathbf{C} h_i) \oplus (\oplus_{i \in I} \mathbf{C} d_i)$ is called the *Cartan subalgebra* of $\mathfrak{g}$, and the linear functionals $\alpha_i \in \mathfrak{h}^*$ $(i \in I)$ defined by (3.2) are called the *simple roots* of $\mathfrak{g}$. For each $i \in I^{re}$, let $r_i \in GL(\mathfrak{h}^*)$ be the *simple reflection on* $\mathfrak{h}^*$ *defined by*

$$r_i(\lambda) = \lambda - \lambda(h_i)\alpha_i \qquad (\lambda \in \mathfrak{h}^*). \tag{3.6}$$

The subgroup W of $GL(\mathfrak{h}^*)$ generated by the r_i's $(i \in I^{re})$ is called the *Weyl group* of the generalized Kac–Moody superalgebra $\mathfrak{g}$.

The generalized Kac–Moody superalgebra $\mathfrak{g} = \mathfrak{g}(A, \underline{m}, C)$ has the *root space decomposition* $\mathfrak{g} = \oplus_{\alpha \in Q} \mathfrak{g}_\alpha$, where

$$\mathfrak{g}_\alpha = \{x \in \mathfrak{g} \mid [h, x] = \alpha(h) x \text{ for all } h \in \mathfrak{h}\}. \tag{3.7}$$

Note that $\mathfrak{g}_{\alpha_i} = \mathbf{C} e_{i,1} \oplus \cdots \oplus \mathbf{C} e_{i, m_i}$, and $\mathfrak{g}_{-\alpha_i} = \mathbf{C} f_{i,1} \oplus \cdots \oplus \mathbf{C} f_{i, m_i}$. We say that $\alpha \in Q^\times$ is a *root* of $\mathfrak{g}$ if $\mathfrak{g}_\alpha \neq 0$. The subspace $\mathfrak{g}_\alpha$ is called the *root space* of $\mathfrak{g}$ attached to α. A root α is called *real* if $(\alpha|\alpha) > 0$ and *imaginary* if $(\alpha|\alpha) \leq 0$. In particular, a simple root α_i is real if $a_{ii} = 2$ (i.e., $i \in I^{re}$) and imaginary if $a_{ii} \leq 0$ (i.e., $i \in I^{im}$). Note that the imaginary simple roots may have multiplicity > 1. A root $\alpha > 0$ (resp. $\alpha < 0$) is called *positive* (resp. *negative*). One can show that all the roots are either positive or negative. We denote by Φ, Φ^+, and Φ^- the set of all roots, positive roots, and negative roots, respectively. We also denote by Φ_0 (resp. Φ_1) the set of all even (resp. odd) roots of $\mathfrak{g}$. Hence, for example, Φ_0^+ will denote the set of all positive even roots of $\mathfrak{g}$. Define the subalgebras $\mathfrak{g}^\pm = \oplus_{\alpha \in \Phi^\pm} \mathfrak{g}_\alpha$. Then we have the *triangular decomposition* of $\mathfrak{g}$:

$$\mathfrak{g} = \mathfrak{g}^- \oplus \mathfrak{h} \oplus \mathfrak{g}^+.$$

A $\mathfrak{g}$-module V is called $\mathfrak{h}$-*diagonalizable* if it admits a *weight space decomposition* $V = \oplus_{\mu \in \mathfrak{h}^*} V_\mu$, where

$$V_\mu = \{v \in V \mid h \cdot v = \mu(h) v \text{ for all } h \in \mathfrak{h}\}. \tag{3.8}$$

If $V_\mu \neq 0$, then μ is called a *weight* of V, and V_μ is called the *μ-weight space*. We denote by $P(V)$ the set of all weights of V. When all the weight spaces are finite dimensional, we define the *character* of V to be

$$\operatorname{ch} V = \sum_{\mu \in \mathfrak{h}^*} \left(\dim V_\mu\right) e^{\mu}, \tag{3.9}$$

where the e^μ are the basis elements of the group algebra $\mathbf{C}[\mathfrak{h}^*]$ with the multiplication given by $e^\mu e^\nu = e^{\mu+\nu}$ for $\mu, \nu \in \mathfrak{h}^*$.

An $\mathfrak{h}$-diagonalizable $\mathfrak{g}$-module V is called a *highest weight module* with highest weight $\lambda \in \mathfrak{h}^*$ if there is a nonzero vector $v_\lambda \in V$ such that (i) $e_{ik} \cdot v_\lambda = 0$ for all $i \in I$, $k = 1, \dots, m_i$; (ii) $h \cdot v_\lambda = \lambda(h) v_\lambda$ for all $h \in \mathfrak{h}$; (iii) $V = U(\mathfrak{g}) \cdot v_\lambda$. The vector v_λ is called a *highest weight vector*. For a highest weight module V with highest weight λ, we have (i) $V = U(\mathfrak{g}^-) \cdot v_\lambda$; (ii) $V = \bigoplus_{\mu \leq \lambda} V_\mu$, $V_\lambda = \mathbf{C} v_\lambda$; and (iii) $\dim V_\mu < \infty$ for all $\mu \leq \lambda$.

Let $\mathfrak{b}^+ = \mathfrak{h} \oplus \mathfrak{g}^+$, and let $\mathbf{C}_\lambda$ be the 1-dimensional $\mathfrak{b}^+$-module defined by $h \cdot 1 = \lambda(h) 1$ for all $h \in \mathfrak{h}$ and $\mathfrak{g}^+ \cdot 1 = 0$. The induced module $M(\lambda) = U(\mathfrak{g}) \otimes_{U(\mathfrak{b}^+)} \mathbf{C}_\lambda$ is called the *Verma module* over $\mathfrak{g}$ with highest weight λ. Every highest weight $\mathfrak{g}$-module with highest weight λ is a homomorphic image of $M(\lambda)$ and the Verma module $M(\lambda)$ contains a unique maximal submodule $J(\lambda)$. Hence the quotient $V(\lambda) = M(\lambda)/J(\lambda)$ is irreducible.

Let $P = \{\lambda \in \mathfrak{h}^* \mid \lambda(h_i) \in \mathbf{Z} \text{ for all } i \in I\}$ be the *weight lattice* of $\mathfrak{g}$, and let P^+ be the set of all linear functionals $\lambda \in \mathfrak{h}^*$ satisfying

$$\begin{cases} \lambda(h_i) \in \mathbf{Z}_{\geq 0} & \text{for all } i \in I^{re}, \\ \lambda(h_i) \in 2\mathbf{Z}_{\geq 0} & \text{for all } i \in I^{re} \cap I^{odd}, \\ \lambda(h_i) \geq 0 & \text{for all } i \in I^{im}. \end{cases} \tag{3.10}$$

The elements of P^+ are called the *dominant integral weights*.

Take a $\mathbf{C}$-linear functional $\rho \in \mathfrak{h}^*$ satisfying $\rho(h_i) = \frac{1}{2} a_{ii}$ for all $i \in I$, and let R be the set of all imaginary simple roots counted with multiplicities. For $\lambda \in P^+$, let R_λ be the set of all elements $\beta \in P$ of the form

$$\beta = \alpha_{i_1} + \cdots + \alpha_{i_r} + p_{j_1} \beta_{j_1} + \cdots + p_{j_s} \beta_{j_s} \qquad (r = s = 0 \text{ if } \beta = 0),$$

where α_{i_k} (resp. β_{j_l}) are distinct even (resp. odd) imaginary simple roots of R satisfying $(\alpha_{i_k} \mid \alpha_{i_l}) = (\beta_{j_k} \mid \beta_{j_l}) = 0$ if $k \neq l$, $(\alpha_{i_k} \mid \alpha_{j_l}) = 0$ for all k, l, $(\beta_{j_k} \mid \beta_{j_k}) = 0$ if $p_{j_k} \geq 2$, and $\lambda(h_{i_k}) = \lambda(h_{j_l}) = 0$ for all k, l. For such $\beta \in R_\lambda$, we denote $|\beta| = r + \sum_{k=1}^{s} p_{j_k}$. Then the character of the irreducible highest weight module $V(\lambda)$ with highest weight $\lambda \in P^+$ is determined by the *Weyl–Kac–Borcherds formula*:

PROPOSITION 3.2 [Mi, R].

$$\operatorname{ch} V(\lambda) = \frac{\prod_{\alpha \in \Phi_1^-}(1+e^{\alpha})^{\dim \mathfrak{g}_\alpha}}{\prod_{\alpha \in \Phi_0^-}(1-e^{\alpha})^{\dim \mathfrak{g}_\alpha}} \sum_{w \in W,\ \beta \in R_\lambda} (-1)^{l(w)+|\beta|} e^{w(\lambda+\rho-\beta)-\rho}. \tag{3.11}$$

Letting $\lambda = 0$, we obtain the *denominator identity*:

$$\frac{\prod_{\alpha \in \Phi_0^-}(1-e^{\alpha})^{\dim \mathfrak{g}_\alpha}}{\prod_{\alpha \in \Phi_1^-}(1+e^{\alpha})^{\dim \mathfrak{g}_\alpha}} = \sum_{w \in W,\ \beta \in R_0} (-1)^{l(w)+|\beta|} e^{w(\rho-\beta)-\rho}. \tag{3.12}$$

3.2. *Root Multiplicity Formula*

Let J be a finite subset of I^{re}, and we denote by $\Phi_J = \Phi \cap (\sum_{j \in J} \mathbf{Z}\alpha_j)$, $\Phi_J^{\pm} = \Phi^{\pm} \cap \Phi_J$, and $\Phi^{\pm}(J) = \Phi^{\pm} \setminus \Phi_J^{\pm}$. We also denote by $Q_J = Q \cap (\sum_{j \in J} \mathbf{Z}\alpha_j)$, $Q_J^{\pm} = Q^{\pm} \cap Q_J$, and $Q^{\pm}(J) = Q^{\pm} \setminus Q_J^{\pm}$. Let $\mathfrak{g}_0^{(J)} = \mathfrak{h} \oplus (\bigoplus_{\alpha \in \Phi_J} \mathfrak{g}_\alpha)$, and $\mathfrak{g}_{\pm}^{(J)} = \bigoplus_{\alpha \in \Phi^{\pm}(J)} \mathfrak{g}_\alpha$. Then we have the *triangular decomposition*,

$$\mathfrak{g} = \mathfrak{g}_-^{(J)} \oplus \mathfrak{g}_0^{(J)} \oplus \mathfrak{g}_+^{(J)}, \tag{3.13}$$

where $\mathfrak{g}_0^{(J)}$ is the Kac–Moody superalgebra (with an extended Cartan subalgebra) associated with the generalized Cartan matrix $A_J = (a_{ij})_{i,j \in J}$ and the set of odd indices $J^{odd} = J \cap I^{odd} = \{j \in J \mid \theta_{jj} = -1\}$, and $\mathfrak{g}_-^{(J)}$ (resp. $\mathfrak{g}_+^{(J)}$) is a direct sum of irreducible highest weight (resp. lowest weight) modules over $\mathfrak{g}_0^{(J)}$ (cf. [K4]).

Let W_J be the subgroup of W generated by the simple reflections r_j with $j \in J$, and let $W(J) = \{w \in W \mid \Phi_w \subset \Phi^+(J)\}$, where $\Phi_w = \{\alpha \in \Phi^+ \mid w^{-1}\alpha < 0\}$. Thus W_J is the Weyl group of the Kac–Moody superalgebra $\mathfrak{g}_0^{(J)}$ and $W(J)$ is the set of right coset representatives of W_J in W. That is, $W = W_J W(J)$. Let us denote by $\Phi_{J,i}^{\pm} = \Phi_J \cap \Phi_i^{\pm}$ $(i = 0, 1)$ and $\Phi_i^{\pm}(J) = \Phi_i^{\pm} \setminus \Phi_{J,i}^{\pm}$ $(i = 0, 1)$. Then we have the following generalization of the denominator identity for generalized Kac–Moody superalgebras:

PROPOSITION 3.3. *Let J be a finite subset of the set of real indices I^{re}. Then we have*

$$\frac{\prod_{\alpha \in \Phi_0^-(J)}(1-e^{\alpha})^{\dim \mathfrak{g}_\alpha}}{\prod_{\alpha \in \Phi_1^-(J)}(1+e^{\alpha})^{\dim \mathfrak{g}_\alpha}} = \sum_{\substack{w \in W(J) \\ \beta \in R_0}} (-1)^{l(w)+|\beta|} \operatorname{ch} V_J(w(\rho-\beta)-\rho), \tag{3.14}$$

where $V_J(\mu)$ denotes the irreducible highest weight module over the Kac–Moody superalgebra $\mathfrak{g}_0^{(J)}$ with highest weight μ.

Proof. Let $P_J^+ \subset P^+$ be the set of all $\lambda \in \mathfrak{h}^*$ such that $\lambda(h_j) \in \mathbf{Z}_{\geq 0}$ for all $j \in J$ and $\lambda(h_j) \in 2\mathbf{Z}_{\geq 0}$ for all $j \in J^{odd}$. We first show that $w(\rho - \beta) - \rho \in P_J^+$ for all $w \in W(J)$. Since $w(\rho - \beta) - \rho \in Q_-$, it is clear that $(w(\rho - \beta) - \rho)(h_j) \in \mathbf{Z}$ for all $j \in J$, and since the Borcherds–Cartan matrix A is restricted with respect to the coloring matrix C, we have $(w(\rho - \beta) - \rho)(h_j) \in 2\mathbf{Z}$ if $j \in J^{odd}$. Thus it remains to show that $(w(\rho - \beta) - \rho)(h_j) \geq 0$ for all $j \in J$. For each $j \in J$, since $w \in W(J)$, we have $w^{-1}(\alpha_j) > 0$. Hence $(w\rho|\alpha_j) = (\rho|w^{-1}(\alpha_j)) > 0$, which implies $(w\rho)(h_j) = 2(w\rho|\alpha_j)/(\alpha_j|\alpha_j) > 0$. Therefore, $(w\rho - \rho)(h_j) = (w\rho)(h_j) - 1 \geq 0$. Moreover, since $J \subset I^{re}$, we have $\beta(h_j) \leq 0$ for all $j \in J$. Hence $(w(\rho - \beta) - \rho)(h_j) \geq 0$ for all $j \in J$.

By the Weyl–Kac character formula for the irreducible highest weight modules over the Kac–Moody superalgebra $\mathfrak{g}_0^{(J)}$ with dominant integral highest weights [K4], we have

$$\begin{aligned}
&\operatorname{ch} V_J(w(\rho - \beta) - \rho) \\
&\quad = \frac{\prod_{\alpha \in \Phi_{J,1}^-}(1 + e^{\alpha})^{\dim \mathfrak{g}_\alpha}}{\prod_{\alpha \in \Phi_{J,0}^-}(1 - e^{\alpha})^{\dim \mathfrak{g}_\alpha}} \sum_{w' \in W_J} (-1)^{l(w')} e^{w'w(\rho - \beta) - \rho}.
\end{aligned}$$

Therefore, the right-hand side of (3.14) yields

$$\begin{aligned}
&\sum_{\substack{w \in W(J) \\ \beta \in R_0}} (-1)^{l(w) + |\beta|} \operatorname{ch} V_J(w(\rho - \beta) - \rho) \\
&\quad = \sum_{\substack{w \in W(J) \\ \beta \in R_0}} (-1)^{l(w) + |\beta|} \frac{\prod_{\alpha \in \Phi_{J,1}^-}(1 + e^{\alpha})^{\dim \mathfrak{g}_\alpha}}{\prod_{\alpha \in \Phi_{J,0}^-}(1 - e^{\alpha})^{\dim \mathfrak{g}_\alpha}} \\
&\qquad \times \sum_{w' \in W_J} (-1)^{l(w')} e^{w'w(\rho - \beta) - \rho} \\
&\quad = \frac{\prod_{\alpha \in \Phi_{J,1}^-}(1 + e^{\alpha})^{\dim \mathfrak{g}_\alpha}}{\prod_{\alpha \in \Phi_{J,0}^-}(1 - e^{\alpha})^{\dim \mathfrak{g}_\alpha}} \sum_{\substack{w \in W(J) \\ w' \in W_J \\ \beta \in R_0}} (-1)^{l(w) + l(w') + |\beta|} e^{w'w(\rho - \beta) - \rho} \\
&\quad = \frac{\prod_{\alpha \in \Phi_{J,1}^-}(1 + e^{\alpha})^{\dim \mathfrak{g}_\alpha}}{\prod_{\alpha \in \Phi_{J,0}^-}(1 - e^{\alpha})^{\dim \mathfrak{g}_\alpha}} \sum_{\substack{w \in W \\ \beta \in R_0}} (-1)^{l(w) + |\beta|} e^{w(\rho - \beta) - \rho}.
\end{aligned}$$

By the denominator identity (3.12), this is equal to

$$\frac{\prod_{\alpha\in\Phi_{J,1}^-}(1+e^{\alpha})^{\dim\mathfrak{g}_\alpha}}{\prod_{\alpha\in\Phi_{J,0}^-}(1-e^{\alpha})^{\dim\mathfrak{g}_\alpha}}\,\frac{\prod_{\alpha\in\Phi_0^-}(1-e^{\alpha})^{\dim\mathfrak{g}_\alpha}}{\prod_{\alpha\in\Phi_1^-}(1+e^{\alpha})^{\dim\mathfrak{g}_\alpha}}=\frac{\prod_{\alpha\in\Phi_0^-(J)}(1-e^{\alpha})^{\dim\mathfrak{g}_\alpha}}{\prod_{\alpha\in\Phi_1^-(J)}(1+e^{\alpha})^{\dim\mathfrak{g}_\alpha}},$$

which proves the identity (3.14). ∎

Recall that $\operatorname{Dim}\mathfrak{g}_\alpha=\psi(\alpha)\dim\mathfrak{g}_\alpha$. Since $w(\rho-\beta)-\rho$ is an element of Q_-, by setting $E^{\alpha}=\psi(\alpha)e^{\alpha}$, we can define the supercharacter of the irreducible highest weight $\mathfrak{g}_0^{(J)}$-module $V_J(w(\rho-\beta)-\rho)$. Thus, the identity (3.14) can be written as

$$\prod_{\alpha\in\Phi^-(J)}(1-E^{\alpha})^{\operatorname{Dim}\mathfrak{g}_\alpha}=\sum_{\substack{w\in W(J)\\ \beta\in R_0}}(-1)^{l(w)+|\beta|}\operatorname{Ch}V_J(w(\rho-\beta)-\rho),\tag{3.15}$$

which will be called the *denominator identity* for the θ-colored Lie superalgebra $\mathfrak{g}_-^{(J)}$.

For each $k\geq 1$, let

$$H_k^{(J)}=\sum_{\substack{w\in W(J)\\ \beta\in R_0\\ l(w)+|\beta|=k}}V_J(w(\rho-\beta)-\rho),\tag{3.16}$$

and define the *homology superspace*

$$H^{(J)}=\sum_{k=1}^{\infty}(-1)^{k+1}H_k^{(J)}=H_1^{(J)}\ominus H_2^{(J)}\oplus H_3^{(J)}\ominus\cdots,\tag{3.17}$$

an alternating direct sum of superspaces. Then the denominator identity (3.15) can be written as

$$\prod_{\alpha\in\Phi^-(J)}(1-E^{\alpha})^{\operatorname{Dim}\mathfrak{g}_\alpha}=1-\operatorname{Ch}H^{(J)}.\tag{3.18}$$

Let $P(H^{(J)})=\{\tau\in Q_-\mid \operatorname{Dim}H_\tau^{(J)}\neq 0\}=\{\tau_1,\tau_2,\tau_3,\dots\}$, and $t(i)=\operatorname{Dim}H_{\tau_i}^{(J)}$. For $\tau\in Q_-$, define the set $T^{(J)}(\tau)$ of all partitions of τ into a sum of τ_i's as in (1.21) and define the Witt partition function $W^{(J)}(\tau)$ as in (1.22). Then, our superdimension formula (1.23) yields a closed form root multiplicity formula for all symmetrizable generalized Kac–Moody superalgebras:

THEOREM 3.4. *Let J be a finite subset of I^{re}. Then, for all $\alpha \in \Phi^-(J)$, we have*

$$\operatorname{Dim} \mathfrak{g}_\alpha = \sum_{d|\alpha} \frac{1}{d} \mu(d) W^{(J)}\left(\frac{\alpha}{d}\right)$$

$$= \sum_{d|\alpha} \frac{1}{d} \mu(d) \sum_{s \in T^{(J)}(\alpha/d)} \frac{(|s|-1)!}{s!} \prod t(i)^{s_i}, \qquad (3.19)$$

where μ is the classical Möbius function, and, for a positive integer d, $d|\alpha$ denotes $\alpha = d\tau$ for some $\tau \in Q_-$, in which case $\alpha/d = \tau$.

3.3. *Maximal and Minimal Graded Lie Superalgebras*

In this subsection, we consider the realization of generalized Kac–Moody superalgebras as the minimal graded Lie superalgebras. We first recall the basic definition introduced by Kac [K1]. Let $\mathfrak{L} = \bigoplus_{n \in \mathbf{Z}} \mathfrak{L}_n$ be a $\mathbf{Z}$-graded Lie superalgebra and let $\mathfrak{L}_{loc} = \mathfrak{L}_{-1} \oplus \mathfrak{L}_0 \oplus \mathfrak{L}_1$. The subspace $\mathfrak{L}_{loc}$ is called the *local part* of $\mathfrak{L}$ if it generates the Lie superalgebra $\mathfrak{L}$. More generally, a direct sum of superspaces $\mathfrak{L}_{loc} = \mathfrak{L}_{-1} \oplus \mathfrak{L}_0 \oplus \mathfrak{L}_1$ is called a *local Lie superalgebra* if there exists a bilinear map $[\,,]$: $\mathfrak{L}_i \times \mathfrak{L}_j \to \mathfrak{L}_{i+j}$, called the *superbracket*, defined for $|i+j| \le 1$ satisfying the condition (1.3) whenever the superbrackets are defined.

A $\mathbf{Z}$-graded Lie superalgebra $\mathfrak{L} = \bigoplus_{n \in \mathbf{Z}} \mathfrak{L}_n$ with local part $\mathfrak{L}_{loc} = \mathfrak{L}_{-1} \oplus \mathfrak{L}_0 \oplus \mathfrak{L}_1$ is called the *maximal graded Lie superalgebra* (resp. *minimal graded Lie superalgebra*) if for any $\mathbf{Z}$-graded Lie superalgebra $\mathfrak{L}' = \bigoplus_{n \in \mathbf{Z}} \mathfrak{L}'_n$ with local part $\mathfrak{L}'_{loc} = \mathfrak{L}'_{-1} \oplus \mathfrak{L}'_0 \oplus \mathfrak{L}'_1$, every isomorphism between the local parts $\mathfrak{L}_{loc}$ and $\mathfrak{L}'_{loc}$ can be extended to an epimorphism of $\mathfrak{L}$ onto $\mathfrak{L}'$ (resp. $\mathfrak{L}'$ onto $\mathfrak{L}$). In particular, a $\mathbf{Z}$-graded Lie superalgebra $\mathfrak{L} = \bigoplus_{n \in \mathbf{Z}} \mathfrak{L}_n$ is the minimal graded Lie superalgebra with local part $\mathfrak{L}_{loc} = \mathfrak{L}_{-1} \oplus \mathfrak{L}_0 \oplus \mathfrak{L}_1$ if and only if there is no nonzero $\mathbf{Z}$-graded ideal of $\mathfrak{L}$ that intersects the local part $\mathfrak{L}_{loc}$ trivially. In [K3], using the same argument in [K1], Kac proved:

PROPOSITION 3.5 [K3]. *For any local Lie superalgebra $\mathfrak{L}_{loc} = \mathfrak{L}_{-1} \oplus \mathfrak{L}_0 \oplus \mathfrak{L}_1$, there exist unique (up to isomorphism) maximal and minimal $\mathbf{Z}$-graded Lie superalgebras whose local parts are isomorphic to $\mathfrak{L}_{loc} = \mathfrak{L}_{-1} \oplus \mathfrak{L}_0 \oplus \mathfrak{L}_1$.*

We return to the generalized Kac–Moody superalgebras. Let $\mathfrak{g} = \mathfrak{g}(A, \underline{m}, C)$ be a generalized Kac–Moody superalgebra associated with a Borcherds–Cartan matrix $A = (a_{ij})_{i,j \in I}$ of charge $\underline{m} = (m_i \in \mathbf{Z}_{>0} \mid$

$i \in I$) with a coloring matrix $C = (\theta_{ij})_{i,j \in I}$. Let J be a finite subset of I^{re}, and consider the corresponding triangular decomposition $\mathfrak{g} = \mathfrak{g}_-^{(J)} \oplus \mathfrak{g}_0^{(J)} \oplus \mathfrak{g}_+^{(J)}$, where the subalgebra $\mathfrak{g}_0^{(J)} = \mathfrak{h} \oplus (\bigoplus_{\alpha \in \Phi_J} \mathfrak{g}_\alpha)$ is the Kac–Moody superalgebra (with an extended Cartan subalgebra) associated with the generalized Cartan matrix $A_J = (a_{ij})_{i,j \in J}$ and the set of odd indices $J^{odd} = \{j \in J \mid \theta_{jj} = -1\}$. For each root $\alpha = \sum_{i \in I} k_i \alpha_i \in \Phi$, we define the *generalized height* of α to be $\mathrm{ht}^{(J)}(\alpha) = \sum_{i \in I \setminus J} k_i$. Note that if $J = \phi$, $\mathrm{ht}^{(J)}$ is the usual height function.

For all $n \geq 1$, let

$$\mathfrak{g}_n^{(J)} = \bigoplus_{\substack{\alpha \in \Phi^+(J) \\ \mathrm{ht}^{(J)}(\alpha) = n}} \mathfrak{g}_\alpha \qquad \text{and} \qquad \mathfrak{g}_{-n}^{(J)} = \bigoplus_{\substack{\alpha \in \Phi^-(J) \\ \mathrm{ht}^{(J)}(\alpha) = -n}} \mathfrak{g}_\alpha. \tag{3.20}$$

Then the generalized Kac–Moody superalgebra $\mathfrak{g}$ becomes a **Z**-graded Lie superalgebra $\mathfrak{g} = \bigoplus_{n \in \mathbf{Z}} \mathfrak{g}_{\pm n}^{(J)}$ generated by the local part $\mathfrak{g}_{loc} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$, and, by the same argument given in [BKM, Ju2, K1, Ka1], one can show that there is no nonzero **Z**-graded ideal of $\mathfrak{g}$ that intersects the local part $\mathfrak{g}_{loc} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$ trivially. Moreover, as $\mathfrak{g}_0^{(J)}$-modules, we have

$$\mathfrak{g}_{-1}^{(J)} \cong \bigoplus_{i \in I \setminus J} V_J(-\alpha_i)^{\oplus m_i}, \qquad \mathfrak{g}_1^{(J)} \cong \bigoplus_{i \in I \setminus J} V_J^*(-\alpha_i)^{\oplus m_i}, \tag{3.21}$$

where $V_J(\mu)$ (resp. $V_J^*(\mu)$) denotes the irreducible highest weight (resp. lowest weight) module over the Kac–Moody superalgebra $\mathfrak{g}_0^{(J)}$ with highest weight μ (resp. lowest weight $-\mu$), and the m_i's are the multiplicities of the imaginary simple roots α_i's. Therefore, the generalized Kac–Moody superalgebra is the minimal $\mathbb{Z}$-graded Lie superalgebra with local part

$$\mathfrak{g}_{loc} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \cong V \oplus \mathfrak{g}_0^{(J)} \oplus V^*,$$

where $V = \bigoplus_{i \in I \setminus J} V_J(-\alpha_i)^{\oplus m_i}$ and $V^* = \bigoplus_{i \in I \setminus J} V_J^*(-\alpha_i)^{\oplus m_i}$.

Suppose that the Borcherds–Cartan matrix A satisfies (i) the set I^{re} is finite, (ii) $a_{ij} \neq 0$ for all $i, j \in I^{im}$. If we take $J = I^{re}$, then we have $W(J) = \{1\}$ and every element $\beta \in R_0$ has the form $\beta = 0$ or $\beta = \alpha_i$ for $i \in I^{im}$. Hence the denominator identity for the Lie superalgebra $\mathfrak{g}_-^{(J)}$ is the same as

$$\begin{aligned} \prod_{\alpha \in Q^-(J)} (1 - E^\alpha)^{\mathrm{Dim}(\mathfrak{g}_-^{(J)})_\alpha} &= 1 - \sum_{\substack{w \in W(J),\ \beta \in R_0 \\ l(w) + |\beta| \geq 1}} \mathrm{Ch}\, V_J(w(\rho - \beta) - \rho) \\ &= 1 - \sum_{i \in I^{im}} m_i \mathrm{Ch}\, V_J(-\alpha_i). \end{aligned} \tag{3.22}$$

But, since $\mathfrak{g} = \mathfrak{g}_{-}^{(J)} \oplus \mathfrak{g}_0^{(J)} \oplus \mathfrak{g}_{+}^{(J)}$ is the minimal **Z**-graded Lie superalgebra with local part $\mathfrak{g}_{loc} \cong V \oplus \mathfrak{g}_0^{(J)} \oplus V^*$, the Lie superalgebra $\mathfrak{g}_{-}^{(J)}$ is a homomorphic image of the free Lie superalgebra $\mathfrak{F} = \bigoplus_{\alpha \in Q^-(J)} \mathfrak{F}_\alpha$ generated by the superspace $V = \bigoplus_{i \in I^{im}} V_J(-\alpha_i)^{\oplus m_i}$. By (2.8), the denominator identity for the free Lie superalgebra $\mathfrak{F}$ is equal to

$$\prod_{\alpha \in Q^-(J)} (1 - E^\alpha)^{\mathrm{Dim}\, \mathfrak{F}_\alpha} = 1 - \sum_{i \in I^{im}} m_i \mathrm{Ch}\, V_J(-\alpha_i).$$

In particular, $\mathrm{Dim}\, \mathfrak{F}_\alpha = \mathrm{Dim}(\mathfrak{g}_{-}^{(J)})_\alpha$ for all $\alpha \in Q^-(J)$. Hence the Lie superalgebra $\mathfrak{g}_{-}^{(J)}$ is isomorphic to the free Lie superalgebra generated by the superspace $V = \bigoplus_{i \in I^{im}} V_J(-\alpha_i)^{\oplus m_i}$. Therefore, we obtain:

PROPOSITION 3.6. *Let $A = (a_{ij})_{i,j \in I}$ be a Borcherds–Cartan matrix of charge $\underline{m} = (m_i \in \mathbf{Z}_{>0} \mid i \in I)$ with a coloring matrix $C = (\theta_{ij})_{i,j \in I}$. Suppose A satisfies* (i) *the set I^{re} is finite,* (ii) *$a_{ij} \neq 0$ for all $i, j \in I^{im}$. Let $J = I^{re}$, and consider the corresponding triangular decomposition of the generalized Kac–Moody superalgebra*

$$\mathfrak{g} = \mathfrak{g}(A, \underline{m}, C) = \mathfrak{g}_{-}^{(J)} \oplus \mathfrak{g}_0^{(J)} \oplus \mathfrak{g}_{+}^{(J)}.$$

Then the Lie superalgebra $\mathfrak{g}_{-}^{(J)} = \bigoplus_{\alpha \in \Phi^-(J)} \mathfrak{g}_\alpha$ (resp. $\mathfrak{g}_{+}^{(J)} = \bigoplus_{\alpha \in \Phi^+(J)} \mathfrak{g}_\alpha$) is isomorphic to the free Lie superalgebra generated by $V = \bigoplus_{i \in I^{im}} V_J(-\alpha_i)^{\oplus m_i}$ (resp. $V^* = \bigoplus_{i \in I^{im}} V_J^*(-\alpha_i)^{\oplus m_i}$), where $V_J(\mu)$ (resp. $V_J^*(\mu)$) denotes the irreducible highest weight (resp. lowest weight) module over the Kac–Moody superalgebra $\mathfrak{g}_0^{(J)}$ with highest weight μ (resp. lowest weight $-\mu$).

Remark. Proposition 3.6 shows that, under the above assumptions, the generalized Kac–Moody superalgebra $\mathfrak{g} = \mathfrak{g}_{-}^{(J)} \oplus \mathfrak{g}_0^{(J)} \oplus \mathfrak{g}_{+}^{(J)}$ is isomorphic to the *maximal graded Lie superalgebra* with local part $\mathfrak{g}_{loc} \cong V \oplus \mathfrak{g}_0^{(J)} \oplus V^*$.

4. MONSTROUS LIE SUPERALGEBRAS

4.1. *Monstrous Moonshine*

The classification theorem of finite simple groups tells that there are exactly 26 sporadic simple groups besides the family of alternating groups on n letters ($n \geq 5$) and the families of simple groups of Lie type (see, for example, [GLS]). The largest among the sporadic simple groups has order

$$2^{46} \cdot 3^{20} \cdot 5^9 \cdot 7^6 \cdot 11^2 \cdot 13^3 \cdot 17 \cdot 19 \cdot 23 \cdot 29 \cdot 31 \cdot 41 \cdot 47 \cdot 59 \cdot 71,$$

and it is called the *Monster* due to its enormous size.

The trivial character degree of the Monster simple group G is, by definition, one, and the smallest nontrivial irreducible character degree of G is 196883 [FLT]. It was noticed by McKay that $1 + 196883 = 196884$, which is the first nontrivial coefficient of the elliptic modular function

$$\begin{aligned} J(q) &= j(q) - 744 \\ &= \sum_{n \geq -1} c(n) q^n = q^{-1} + 196884q + 21493760q^2 + \cdots . \end{aligned} \tag{4.1}$$

Later, Thompson found that the first few coefficients of the modular function $j(q) - 744$ are simple linear combinations of the irreducible character degrees of G [T]. Motivated by these observations, Conway and Norton conjectured that there exists an infinite dimensional graded representation $V^\natural = \oplus_{n \geq -1} V_n^\natural$ of the Monster simple group G with $\dim V_n^\natural = c(n)$ such that the Thompson series

$$T_g(q) = \sum_{n \geq -1} Tr\left(g | V_n^\natural\right) q^n = \sum_{n \geq -1} c_g(n) q^n \tag{4.2}$$

are the normalized generators of the genus zero function fields arising from certain discrete subgroups of $PSL(2, \mathbf{R})$ [CN]. Moreover, they also noticed that the Thompson series seem to satisfy certain functional equation which they call the *replication formulae*. Their conjecture is referred to as the *Moonshine conjecture*.

The natural graded representation $V^\natural = \oplus_{n \geq -1} V_n^\natural$ of the Monster simple group G in the Moonshine conjecture, called the *Moonshine module*, was constructed by Frenkel, Lepowsky, and Meurman using the theory of vertex (operator) algebras [FLM]. They also calculated the Thompson series for some conjugacy classes of the Monster, and verified the Moonshine conjecture for these Thompson series.

In [B3], Borcherds completed the proof of the Moonshine conjecture by constructing a $II_{1,1}$-graded Lie algebra $L = \oplus_{(m,n) \in II_{1,1}} L_{(m,n)}$, called the *Monster Lie algebra*, where $II_{1,1}$ is the 2-dimensional even Lorentzian lattice associated with the matrix $\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$. The Monster Lie algebra L is a $II_{1,1}$-graded representation of the Monster simple group G such that $L_{(m,n)} \cong V_{mn}^\natural$ as G-modules for $(m, n) \neq (0, 0)$. In particular, we have

$$\dim L_{(m,n)} = \dim V_{mn}^\natural = c(mn) \qquad \text{for all } (m, n) \neq (0, 0).$$

On the other hand, the Monster Lie algebra can be regarded as a generalized Kac–Moody algebra. We take $I = \{-1\} \cup \{1, 2, 3, \ldots\}$ as the index set, and consider the Borcherds–Cartan matrix $A = (-(i + j))_{i, j \in I}$ of charge $\underline{m} = (c(i) \,|\, i \in I)$, where $c(i)$ are the coefficients of the elliptic modular function $J(q) = j(q) - 744 = \sum_{n=-1}^{\infty} c(n)q^n$. Then, by using Borcherds' product identity (2.30) for the elliptic modular function $J(q) = j(q) - 744$, it can be shown that the Monster Lie algebra is isomorphic to the generalized Kac–Moody algebra $\mathfrak{g} = \mathfrak{g}(A, \underline{m})$.

In [Ka3], we derived a root multiplicity formula for all symmetrizable generalized Kac–Moody algebras, which turns out to be a special case of our formula (3.19). Applying that formula to the Monster Lie algebra, we obtained the following interesting relations for the coefficients $c(n)$ of the elliptic modular function $J(q) = j(q) - 744 = \sum_{n=-1}^{\infty} c(n)q^n$,

$$c(mn) = \sum_{d|(m,n)} \frac{1}{d}\mu(d) \sum_{s \in T\left(\frac{m}{d}, \frac{n}{d}\right)} \frac{(|s|-1)!}{s!} \prod c(i+j-1)^{s_{ij}}, \quad (4.3)$$

where $T(k, l)$ denotes the set of all partitions of (k, l) as a sum of ordered pairs of positive integers (see also [Ju1]).

In [JLW, KK], the relation (4.3) was generalized to the relation of the coefficients $c_g(n)$ of the Thompson series:

$$c_g(mn) = \sum_{d|(m,n)} \frac{1}{d}\mu(d) \sum_{s \in T\left(\frac{m}{d}, \frac{n}{d}\right)} \frac{(|s|-1)!}{s!} \prod c_{g^d}(i+j-1)^{s_{ij}}. \quad (4.4)$$

It was pointed out in [JLW] that these relations completely determine all the coefficients $c_g(n)$ if the values of $c_h(1)$, $c_h(2)$, $c_h(3)$, and $c_h(5)$ are known for all $h \in G$. In particular, the relation (4.3) is a complete recursive relation determining the coefficients $c(n)$ of the elliptic modular function $j(q) - 744$. Moreover in [KK], we have noticed the ghost functions introduced in [CN] also seem to satisfy the same relations as the Thompson series, which leads us to consider a more general class of modular functions—*replicable functions*.

4.2. *Replicable Functions*

We recall the definition of replicable functions following the exposition given in [H] (see also [ACMS, CN, F2, N]). Let $F(q) = \sum_{n=-1}^{\infty} f(n)q^n = q^{-1} + f(1)q + f(2)q^2 + \cdots$ be a normalized q-series such that $f(-1) = 1$, $f(0) = 0$, and $f(n) \in \mathbf{Z}$ for all $n \geq 1$. Observe that, for each $m \geq 1$, there exists a unique polynomial $P_m(t) \in \mathbf{Z}[t]$ such that

$$P_m(F) \equiv q^{-m} \mod q\mathbf{Z}[[q]].$$

For example, $P_1(t) = t$, $P_2(t) = t^2 - 2f(1)$, etc. The polynomials $P_m(t)$ are uniquely determined by the recursive relation [ACMS, H]

$$P_{m+1}(t) + \sum_{i=0}^{m-1} f(i)P_{m-i}(t) + (m+1)f(m) = tP_m(t). \quad (4.5)$$

DEFINITION 4.1. A normalized q-series $F(q) = \sum_{n=-1}^{\infty} f(n)q^n = q^{-1} + f(1)q + f(2)q^2 + \cdots$ is called *replicable* if for all $m > 0$ and for all $a|m$, there exist normalized q-series $F^{(a)}(q) = \sum_{n=-1}^{\infty} f^{(a)}(n)q^n = q^{-1} + f^{(a)}(1)q + f^{(a)}(2)q^2 + \cdots$ with $f^{(a)}(-1) = 1$, $f^{(a)}(0) = 0$, and $f^{(a)}(n) \in \mathbf{Z}$ for all $n \geq 1$ such that

$$F^{(1)} = F, \qquad \text{and} \qquad \sum_{\substack{ad=m \\ 0 \leq b < d}} F^{(a)}\left(\frac{az+b}{d}\right) = P_m(F), \tag{4.6}$$

where $q = e^{2\pi i z}$, Im $z > 0$.

The normalized q-series $F^{(a)}$ is called the *ath replicate* of F. If all the replicates $F^{(a)}$ are also replicable, then F is called *completely replicable*.

EXAMPLE 4.2. (a) A finite series $F(q) = q^{-1} + rq$ $(r \in \mathbf{Z})$ is completely replicable with $F^{(a)}(q) = q^{-1} + r^a q$ for all $a \geq 1$. Norton showed that a finite normalized q-series $F(q) = \sum_{n=-1}^{N} f(n)q^n$ is replicable only if $f(n) = 0$ for all $n \geq 2$ [N].

(b) The elliptic modular function $J(q) = j(q) - 744$ is completely replicable with $J^{(a)} = J$ for all $a \geq 1$ [CN, Se2].

(c) The Thompson series $T_g(q) = \sum_{n=-1}^{\infty} c_g(n)q^n$ are completely replicable with $T_g^{(a)} = T_{g^a}$ for all $a \geq 1$ [Ko, F2].

(d) In [F1], Ferenbaugh classified all genus 0 modular groups between $\Gamma_0(N)$ and its normalizer in $PSL(2, \mathbf{R})$. The corresponding Hauptmoduls are called *$n \mid h$-type Hauptmoduls*, and it was proved that they are all completely replicable [F2, Ko]. This fact played a crucial role in Borcherds' proof of the Moonshine Conjecture.

In [ACMS], it was proposed that we can define the notion of *generalized Hecke operators* on the replicable functions, and in [F2], Ferenbaugh verified that for the $n|h$-type modular functions, it is a reasonable definition with the Galois action incorporated.

More generally, for each $m \geq 1$, we define the *formal Hecke operators* T_m on the set of normalized q-series with values in the set of formal Laurent q-series by

$$T_m(F) = \frac{1}{m} P_m(F). \tag{4.7}$$

Hence if F is replicable with replicates $F^{(a)}$ for $a \geq 1$, we have

$$T_m(F) = \frac{1}{m} \sum_{\substack{ad=m \\ 0 \leq b < d}} F^{(a)}\left(\frac{az+b}{d}\right). \tag{4.8}$$

Furthermore, the same argument in [Se2, Chap. VII, Proposition 12] yields

$$\frac{1}{m}\sum_{\substack{ad=m\\0\le b<d}} F^{(a)}\left(\frac{az+b}{d}\right) = \sum_{n\in\mathbf{Z}}\left(\sum_{\substack{a>0\\a|(m,n)}}\frac{1}{a}f^{(a)}\left(\frac{mn}{a^2}\right)\right)q^n.$$

Hence we obtain:

PROPOSITION 4.3. *A normalized q-series $F(q)=\sum_{n=-1}^{\infty} f(n)q^n = q^{-1} + f(1)q + f(2)q^2 + \cdots$ is replicable if and only if there exist normalized q-series $F^{(a)}(q)=\sum_{n=-1}^{\infty} f^{(a)}(n)q^n = q^{-1} + f^{(a)}(1)q + f^{(a)}(2)q^2 + \cdots$ for all $a\ge 1$ satisfying*

$$T_m(F) = \frac{1}{m}P_m(F) = \sum_{n\in\mathbf{Z}}\left(\sum_{\substack{a>0\\a|(m,n)}}\frac{1}{a}f^{(a)}\left(\frac{mn}{a^2}\right)\right)q^n \tag{4.9}$$

for all $m\ge 1$.

Now, we will characterize the replicable functions in terms of product identities:

THEOREM 4.4. *The following conditions on a normalized q-series $F(q) = q^{-1} + f(1)q + f(2)q^2 + \cdots = \sum_{n=-1}^{\infty} f(n)q^n$ are equivalent.*

(a) *F is replicable.*

(b) *For all $k\ge 1$, there exist normalized q-series $F^{(k)}(q) = \sum_{n=-1}^{\infty} f^{(k)}(n)q^n$ satisfying the product identity*

$$p^{-1}\prod_{\substack{m>0\\n\in\mathbf{Z}}}\exp\left(-\sum_{k=1}^{\infty}\frac{1}{k}f^{(k)}(mn)p^{km}q^{kn}\right) = F(p)-F(q). \tag{4.10}$$

(c) *For all $k\ge 1$, there exist normalized q-series $F^{(k)}(q) = \sum_{n=-1}^{\infty} f^{(k)}(n)q^n$ satisfying the product identity*

$$\prod_{m,n=1}^{\infty}\exp\left(-\sum_{k=1}^{\infty}\frac{1}{k}f^{(k)}(mn)p^{km}q^{kn}\right) = 1-\sum_{i,j=1}^{\infty}f(i+j-1)p^iq^j. \tag{4.11}$$

Proof. It is easy to prove the equivalence of (b) and (c). Observing that $f(0)=0$ and $f(-k)=0$ for $k>1$, the identity (4.10) can be rewritten as

$$\prod_{m,n=1}^{\infty}\exp\left(-\sum_{k=1}^{\infty}\frac{1}{k}f^{(k)}(mn)p^{km}q^{kn}\right) = \frac{F(p)-F(q)}{p^{-1}-q^{-1}}$$
$$= 1-\sum_{i,j=1}^{\infty}f(i+j-1)p^iq^j,$$

which is the identity (4.11).

We will prove the equivalence of (a) and (b). Suppose F is replicable with replicates $F^{(k)}(q) = \sum_{n=-1}^{\infty} f^{(k)}(n)q^n$ for $k \geq 1$. By Proposition 4.3, we have

$$T_m(F) = \frac{1}{m}P_m(F) = \sum_{n \in \mathbf{Z}} \left(\sum_{\substack{k>0 \\ k|(m,n)}} \frac{1}{k} f^{(k)}\left(\frac{mn}{k^2}\right)\right) q^n.$$

Observe that

$$\begin{aligned}
&\log\left(\prod_{\substack{m>0 \\ n\in \mathbf{Z}}} \exp\left(-\sum_{k=1}^{\infty} \frac{1}{k} f^{(k)}(mn) p^{km} q^{kn}\right)\right) \\
&\quad = -\sum_{m>0} \sum_{n \in \mathbf{Z}} \sum_{k=1}^{\infty} \frac{1}{k} f^{(k)}(mn) p^{km} q^{kn} \\
&\quad = -\sum_{m>0} \sum_{n \in \mathbf{Z}} \sum_{\substack{k>0 \\ k|(m,n)}} \frac{1}{k} f^{(k)}\left(\frac{mn}{k^2}\right) p^m q^n \\
&\quad = -\sum_{m=1}^{\infty} \frac{1}{m} P_m(F) p^m.
\end{aligned}$$

So the left-hand side of (4.10) is equal to

$$\begin{aligned}
&p^{-1} \exp\left(-\sum_{m=1}^{\infty} \frac{1}{m} P_m(F) p^m \right) \\
&\quad = p^{-1}\left[1 + \left(-\sum_{m=1}^{\infty} \frac{1}{m} P_m(F) p^m \right) + \frac{1}{2!}\left(-\sum_{m=1}^{\infty} \frac{1}{m} P_m(F) p^m \right)^2 + \cdots \right] \\
&\quad = \sum_{m=-1}^{\infty} G_m(q) p^m,
\end{aligned}$$

where $G_{-1}(q) = 1$, $G_0(q) = -F(q)$, and

$$G_m(q) = \sum_{s \in T(m+1)} \frac{(-1)^{|s|}}{s!} \prod \left(\frac{P_i(F)}{i} \right)^{s_i} \qquad (m \geq 1).$$

Here, $T(m+1)$ is the set of all partitions of $m+1$ into a sum of positive integers. In particular, $G_m(q)$ is a polynomial in F.

On the other hand, the right-hand side of (4.10) can be written as

$$F(p) - F(q) = p^{-1} - F(q) + \sum_{m=1}^{\infty} f(m)p^m = \sum_{m=-1}^{\infty} H_m(q)p^m,$$

and each $H_m(q)$ is obviously a polynomial in F. Thus, in order to prove the identity (4.10), we have only to check that the coefficients of q^n of $G_m(q)$ and $H_m(q)$ are the same for all $n \leq 0$.

Write $G_m(q) = \sum_{n\in\mathbf{Z}} g_m(n)q^n$ and $H_m(q) = \sum_{n\in\mathbf{Z}} h_m(n)q^n$. Then the left-hand side of (4.10) is equal to

$$\sum_{m=-1}^{\infty} G_m(q)p^m = \sum_{m=-1}^{\infty}\sum_{n\in\mathbf{Z}} g_m(n)q^n p^m = \sum_{n\in\mathbf{Z}}\left(\sum_{m=-1}^{\infty} g_m(n)p^m\right)q^n,$$

and, similarly, the right-hand side of (4.10) is equal to

$$\sum_{m=-1}^{\infty} H_m(q)p^m = \sum_{n\in\mathbf{Z}}\left(\sum_{m=-1}^{\infty} h_m(n)p^m\right)q^n.$$

Suppose $\sum_{m=-1}^{\infty} g_m(n)p^m = \sum_{m=-1}^{\infty} h_m(n)p^m$ for all $n \leq 0$. Then $g_m(n) = h_m(n)$ for all $n \leq 0$, which implies $G_m(q) = H_m(q)$, since they are polynomials in F. Hence, to prove (4.10), it suffices to check that the coefficients of q^n in the left-hand side and the right-hand side of (4.10) are the same for all $n \leq 0$, which is straightforward. (They are -1 if $n = -1$, $F(p)$ if $n = 0$, and 0 otherwise.)

Conversely, suppose there exist normalized q-series $F^{(k)}(q) = \sum_{n=-1}^{\infty} f^{(k)}(n)q^n$ for all $k \geq 1$ satisfying the identity (4.10). As we have seen in the previous argument, we have

$$\begin{aligned}\log\left(\prod_{\substack{m>0\\ n\in\mathbf{Z}}} \exp\left(-\sum_{k=1}^{\infty}\frac{1}{k}f^{(k)}(mn)p^{km}q^{kn}\right)\right)\\ = -\sum_{m>0}\sum_{n\in\mathbf{Z}}\sum_{\substack{k>0\\ k\mid(m,n)}}\frac{1}{k}f^{(k)}\left(\frac{mn}{k^2}\right)p^m q^n.\end{aligned}$$

Let

$$\hat{T}_m(q) = \sum_{n\in\mathbf{Z}}\left(\sum_{\substack{k>0\\ k\mid(m,n)}}\frac{1}{k}f^{(k)}\left(\frac{mn}{k^2}\right)\right)q^n.$$

We would like to show that $\hat{T}_m(q) = \dfrac{1}{m}P_m(F)$ for all $m \geq 1$. The left-hand side of (4.10) is equal to

$$\begin{aligned} & p^{-1}\exp\left(-\sum_{m=1}^{\infty}\hat{T}_m p^m\right) \\ &= p^{-1}\left[1 + \left(-\sum_{m=1}^{\infty}\hat{T}_m p^m\right) + \frac{1}{2!}\left(-\sum_{m=1}^{\infty}\hat{T}_m p^m\right)^2 + \cdots\right] \\ &= \sum_{m=-1}^{\infty}\hat{G}_m(q)p^m, \end{aligned}$$

where $\hat{G}_{-1}(q) = 1$, $\hat{G}_0(q) = -\hat{T}_1(q)$, and

$$\hat{G}_m(q) = \sum_{s \in T(m+1)} \frac{(-1)^{|s|}}{s!}\prod \hat{T}_i(q)^{s_i} \qquad (m \geq 1).$$

Since the right-hand side of (4.10) is equal to $p^{-1} - F(q) + \sum_{m=1}^{\infty} f(m)p^m$, we obtain

$$\hat{T}_1(q) = F(q), \qquad \text{and} \qquad \sum_{s \in T(m+1)} \frac{(-1)^{|s|}}{s!}\prod \hat{T}_i(q)^{s_i} = f(m) \qquad (m \geq 1). \tag{4.12}$$

By induction, it follows from (4.12) that $\hat{T}_m(q)$ is a polynomial in F for all $m \geq 1$. It is easy to check that the coefficients of q^n in $\hat{T}_m(q)$ and $\frac{1}{m}P_m(F)$ are the same for all $n \leq 0$. (They are $\frac{1}{m}$ if $n = -m$ and 0 otherwise.) Hence, $\hat{T}_m(q) = \frac{1}{m}P_m(F)$ for all $m \geq 1$, and by Proposition 4.3, F is replicable. ∎

COROLLARY 4.5. *If a normalized q-series $F(q) = \sum_{n=-1}^{\infty} f(n)q^n$ is completely replicable with replicates $F^{(a)}(q) = \sum_{n=-1}^{\infty} f^{(a)}(n)q^n$ for $a \geq 1$, then we have*

$$f(mn) = \sum_{d|(m,n)} \frac{1}{d}\mu(d) \sum_{s \in T\left(\frac{m}{d}, \frac{n}{d}\right)} \frac{(|s|-1)!}{s!}\prod f^{(d)}(i+j-1)^{s_{ij}}. \tag{4.13}$$

In particular, the coefficients of a completely replicable function F are determined recursively by the coefficients of its replicates $F^{(a)}$.

Proof. If F is completely replicable, then all of its replicates $F^{(a)}$ are also replicable with $(F^{(a)})^{(k)} = F^{(ak)}$ for all $a, k \geq 1$ (cf. [N]). Hence, by (4.11), we have

$$\prod_{m,n=1}^{\infty} \exp\left(-\sum_{k=1}^{\infty} \frac{1}{k} f^{(ak)}(mn)\, p^{km} q^{kn}\right) = 1 - \sum_{i,j=1}^{\infty} f^{(a)}(i+j-1)\, p^i q^j$$

for all $a \geq 1$. By taking the logarithm, we obtain

$$\log \prod_{m,n=1}^{\infty} \exp\left(-\sum_{k=1}^{\infty} \frac{1}{k} f^{(ak)}(mn)\, p^{km} q^{kn}\right) = -\sum_{m,n=1}^{\infty} \sum_{k=1}^{\infty} \frac{1}{k} f^{(ak)}(mn)\, p^{km} q^{kn},$$

and

$$\begin{aligned}
\log&\left(1 - \sum_{i,j=1}^{\infty} f^{(a)}(i+j-1)\, p^i q^j\right) \\
&= -\sum_{k=1}^{\infty} \frac{1}{k} \left(\sum_{i,j=1}^{\infty} f^{(a)}(i+j-1)\, p^i q^j\right)^k \\
&= -\sum_{k=1}^{\infty} \frac{1}{k} \sum_{\substack{s=(s_{ij}) \\ s_{ij} \in \mathbf{Z}_{\geq 0} \\ \Sigma s_{ij} = k}} \frac{(\Sigma s_{ij})!}{\Pi(s_{ij}!)} \prod f^{(a)}(i+j-1)^{s_{ij}}\, p^{\Sigma i s_{ij}} q^{\Sigma j s_{ij}} \\
&= -\sum_{l,t=1}^{\infty} \left(\sum_{s \in T(l,t)} \frac{(|s|-1)!}{s!} \prod f^{(a)}(i+j-1)^{s_{ij}}\right) p^l q^t.
\end{aligned}$$

Therefore, we have

$$\sum_{k(m,n)=(l,t)} \frac{1}{k} f^{(ak)}(mn) = \sum_{s \in T(l,t)} \frac{(|s|-1)!}{s!} \prod f^{(a)}(i+j-1)^{s_{ij}}.$$

Hence, by Möbius inversion, we obtain

$$f^{(a)}(mn) = \sum_{d|(m,n)} \frac{1}{d}\mu(d) \sum_{s\in T\left(\frac{m}{d},\frac{n}{d}\right)} \frac{(|s|-1)!}{s!} \prod f^{(ad)}(i+j-1)^{s_{ij}}$$

for all $a \geq 1$. In particular, if $a = 1$, we get (4.13). ∎

Remark. Note that the relations (4.3) and (4.4) are special cases of (4.13).

4.3. *Monstrous Lie Superalgebras*

We take $I = \{-1\} \cup \{1, 2, 3, \ldots\}$ to be the index set, and let $A = (-(i+j))_{i,j\in I}$ be the Borcherds–Cartan matrix of the Monster Lie algebra. Consider a normalized q-series $F(q) = \sum_{n=-1}^{\infty} f(n)q^n$ such that $f(-1) = 1$, $f(0) = 0$, and $f(n) \in \mathbf{Z}$ for all $n \geq 1$. We define the charge of the matrix A to be $\underline{m} = (|f(i)|\colon i \in I)$, and choose a coloring matrix $C = (\theta_{ij})_{i,j\in I}$ such that $\theta_{ii} = 1$ if $f(i) > 0$ and $\theta_{ii} = -1$ if $f(i) < 0$. That is, an index $i \in I$ is even if $f(i) > 0$ and is odd if $f(i) < 0$.

DEFINITION 4.6. The generalized Kac–Moody superalgebra $\mathfrak{L}(F) = \mathfrak{g}(A, \underline{m}, C)$ associated with the above data is called the *Monstrous Lie superalgebra* associated with the normalized q-series $F(q) = \sum_{n=-1}^{\infty} f(n)q^n$.

For example, the Monstrous Lie superalgebra associated with the elliptic modular function $J(q) = j(q) - 744$ is the Monster Lie algebra, and the Monstrous Lie superalgebras associated with the Thompson series $T_g(q) = \sum_{n=-1}^{\infty} c_g(n)q^n$ are the Monstrous Lie superalgebras given in [B3].

By our choice of the charge and the coloring matrix, we see that α_{-1} is the only real even simple root, α_i $(i \geq 1)$ is an imaginary even simple root with multiplicity $f(i)$ if $f(i) > 0$, and α_i $(i \geq 1)$ is an imaginary odd simple root with multiplicity $-f(i)$ if $f(i) < 0$. We just neglect those α_i's for which $f(i) = 0$. In short, we have $\operatorname{Dim} \mathfrak{L}(F)_{\alpha_i} = f(i)$ for all $i \in I$. Thus $W = \{1, r_{-1}\}$ and $R = \{\alpha_i, \ldots, \alpha_i \mid i \geq 1\}$, where each α_i is counted $|f(i)|$ times. Since $(\alpha_i|\alpha_i) = -2i \neq 0$ for $i \geq 1$, $\beta \in R_0$ if and only if $\beta = 0$ or $\beta = \alpha_i$ for some $i \geq 1$ ($|f(i)|$ choices). Hence, if we take $J = \phi$, by (3.16), we obtain

$$H^{(\phi)} = \mathbf{C}_{-\alpha_{-1}} \oplus \left(\bigoplus_{i=1}^{\infty} \mathbf{C}_{-\alpha_i}^{\oplus|f(i)|} \right) \ominus \left(\bigoplus_{i=1}^{\infty} \mathbf{C}_{-i\alpha_{-1}-\alpha_i}^{\oplus|f(i)|} \right). \tag{4.14}$$

Therefore, the denominator identity (3.12) for the Lie superalgebra $\mathfrak{L}(F)_-$ yields

$$\frac{\prod_{\alpha \in \Phi_0^-}(1 - e^{\alpha})^{\dim \mathfrak{L}(F)_\alpha}}{\prod_{\alpha \in \Phi_1^-}(1 + e^{\alpha})^{\dim \mathfrak{L}(F)_\alpha}} = 1 - e^{-\alpha_{-1}} - \sum_{i=1}^{\infty} |f(i)| e^{-\alpha_i} + \sum_{i=1}^{\infty} |f(i)| e^{-i\alpha_{-1} - \alpha_i},$$

which is equivalent to

$$\prod_{\alpha \in Q^-} (1 - E^{\alpha})^{\operatorname{Dim} \mathfrak{L}(F)_\alpha} = 1 - E^{-\alpha_{-1}} - \sum_{i=1}^{\infty} f(i) E^{-\alpha_i} + \sum_{i=1}^{\infty} f(i) E^{-i\alpha_{-1} - \alpha_i}.$$

We identify the simple roots α_{-1} with $(1, -1) \in II_{1,1}$ and α_i with $(1, i) \in II_{1,1}$ $(i \geq 1)$. Then the Monstrous Lie superalgebra $\mathfrak{L}(F)$ becomes a $II_{1,1}$-graded Lie superalgebra, and we have

$$\mathfrak{L}(F)_+ = \bigoplus_{\substack{m>0 \\ n \in \mathbf{Z}}} \mathfrak{L}(F)_{(m,n)}, \qquad \mathfrak{L}(F)_- = \bigoplus_{\substack{m>0 \\ n \in \mathbf{Z}}} \mathfrak{L}(F)_{(-m,n)}.$$

By letting $p = E^{-(1,0)}$ and $q = E^{-(0,1)}$, the denominator identity for the Lie superalgebra $\mathfrak{L}(F)_-$ can be written as

$$\begin{aligned}\prod_{\substack{m>0 \\ n \in \mathbf{Z}}} (1 - p^m q^n)^{\operatorname{Dim} \mathfrak{L}(F)_{(m,n)}} &= 1 - pq^{-1} - \sum_{i=1}^{\infty} f(i) pq^i + \sum_{i=1}^{\infty} f(i) p^{i+1} \\ &= p\left(p^{-1} + \sum_{i=1}^{\infty} f(i) p^i\right) - p\left(q^{-1} + \sum_{i=1}^{\infty} f(i) q^i\right) \\ &= p(F(p) - F(q)),\end{aligned}$$

which yields:

PROPOSITION 4.7. *Let* $\mathfrak{L}(F) = \bigoplus_{(m,n) \in II_{1,1}} \mathfrak{L}(F)_{(m,n)}$ *be the Monstrous Lie superalgebra associated with a normalized q-series* $F(q) = \sum_{n=-1}^{\infty} f(n) q^n$ *such that* $f(-1) = 1$, $f(0) = 0$, *and* $f(n) \in \mathbf{Z}$ *for all* $n \geq 1$.

Then the denominator identity for the Lie superalgebra $\mathfrak{L}(E)_-$ *is given by*

$$p^{-1} \prod_{\substack{m>0 \\ n \in \mathbf{Z}}} (1 - p^m q^n)^{\operatorname{Dim} \mathfrak{L}(F)_{(m,n)}} = F(p) - F(q). \tag{4.15}$$

We will apply our root multiplicity formula (3.19) to the Monstrous Lie superalgebra $\mathfrak{L}(F) = \bigoplus_{(m,n) \in II_{1,1}} \mathfrak{L}(F)_{(m,n)}$. Take $J = \{-1\}$. Then $\mathfrak{L}(F)_0^{(J)} = \langle e_{-1}, f_{-1}, h_{-1} \rangle + \mathfrak{h} \cong sl(2, \mathbf{C}) + \mathfrak{h}$, and $W(J) = \{1\}$. By (3.16), we obtain

$$H^{(J)} = H_1^{(J)} = \bigoplus_{i=1}^{\infty} V_J(-\alpha_i)^{\oplus |f(i)|},$$

where $V_J(-\alpha_i)$ is the i-dimensional irreducible representation of $sl(2, \mathbf{C})$ (since $-\alpha_i(h_{-1}) = i - 1$). Hence the weights of $V_J(-\alpha_i)$ are $-\alpha_i = (-1, -i)$, $-\alpha_i - \alpha_{-1} = (-2, -i+1), \ldots, -\alpha_i - (i-1)\alpha_{-1} = (-i, -1)$ with the superdimensions $f(i)$ $(i \geq 1)$. It follows that

$$P(H^{(J)}) = \left\{(-i, -j) \,\middle|\, i, j \in \mathbf{Z}_{>0}\right\}$$

and

$$\operatorname{Dim} H^{(J)}_{(-i,-j)} = f(i+j-1) \qquad \text{for all } i, j \geq 1.$$

Hence the denominator identity for the Lie superalgebra $\mathfrak{L}(F)_-^{(J)}$ is equal to

$$\prod_{m,n=1}^{\infty} (1 - p^m q^n)^{\operatorname{Dim} \mathfrak{L}(F)_{(m,n)}} = 1 - \sum_{i,j=1}^{\infty} f(i+j-1) p^i q^j. \quad (4.16)$$

For $k, l > 0$, we have

$$T^{(J)}(k,l) = T(k,l) = \left\{ s = (s_{ij})_{i,j \geq 1} \,\middle|\, s_{ij} \in \mathbf{Z}_{\geq 0}, \sum s_{ij}(i,j) = (k,l) \right\},$$

the set of all partitions of (k, l) into a sum of ordered pairs of positive integers, and the Witt partition function $W^{(J)}(k, l)$ is given by

$$W^{(J)}(k,l) = \sum_{s \in T(k,l)} \frac{(|s|-1)!}{s!} \prod f(i+j-1)^{s_{ij}}.$$

Therefore, our root multiplicity formula (3.19) yields:

THEOREM 4.8. *Let $\mathfrak{L}(F) = \bigoplus_{(m,n) \in II_{1,1}} \mathfrak{L}(F)_{(m,n)}$ be the Monstrous Lie superalgebra associated with a normalized q-series $F(q) = \sum_{n=-1}^{\infty} f(n) q^n$ such that $f(-1) = 1$, $f(0) = 0$, and $f(n) \in \mathbf{Z}$ for all $n \geq 1$.*

Then, for all $m, n \in \mathbf{Z}_{>0}$, we have

$$\operatorname{Dim} \mathfrak{L}(F)_{(m,n)} = \sum_{d|(m,n)} \frac{1}{d}\mu(d) \sum_{s \in T\left(\frac{m}{d}, \frac{n}{d}\right)} \frac{(|s|-1)!}{s!} \prod f(i+j-1)^{s_{ij}}. \tag{4.17}$$

Remark. It follows from Proposition 3.6 that the Lie superalgebra $\mathfrak{L}(F)_{-}^{(J)}$ is isomorphic to the free Lie superalgebra generated by

$$H^{(J)} = \bigoplus_{i=1}^{\infty} V_J(-\alpha_i)^{\oplus |f(i)|},$$

where $H^{(J)}$ is regarded as a $\mathbf{Z}_{>0}^2$-graded superspace

$$H^{(J)} = \bigoplus_{i,j=1}^{\infty} H_{(-i,-j)}^{(J)} \qquad \text{with } \operatorname{Dim} H_{(-i,-j)}^{(J)} = f(i+j-1).$$

Hence the Monstrous Lie superalgebra $\mathfrak{L}(F)$ is isomorphic to the maximal graded Lie superalgebra with local part $\mathfrak{L}(F)_{loc} = H^{(J)} \oplus (sl(2, \mathbf{C}) + \mathfrak{h}) \oplus H^{(J)*}$.

Since we have identified the roots of the Monstrous Lie superalgebra $\mathfrak{L}(F)$ with the elements of the lattice $II_{1,1}$, the superdimension $\operatorname{Dim} \mathfrak{L}(F)_{(m,n)}$ is the difference of the *even dimension* and the *odd dimension*. Still, thanks to Corollary 1.3 and Proposition 2.10, we can recover its even dimension and odd dimension explicitly. We consider the $(II_{1,1} \times \mathbf{Z}_2)$-gradation on the superspace $H^{(J)}$ and the Monstrous Lie superalgebra $\mathfrak{L}(F)$ as follows. For each $(i,j) \in \mathbf{Z}_{>0} \times \mathbf{Z}_{>0}$, define $\varepsilon(i,j) = 0$ if $f(i+j-1) > 0$ and $\varepsilon(i,j) = 1$ if $f(i+j-1) < 0$. Then we obtain a $(II_{1,1} \times \mathbf{Z}_2)$-gradation on $H^{(J)} = \bigoplus_{(i,j,k) \in II_{1,1} \times \mathbf{Z}_2} H_{(-i,-j,k)}^{(J)}$ such that

$$P\big(H^{(J)}, II_{1,1} \times \mathbf{Z}_2\big) = \{(-i, -j, \varepsilon(i,j)) \,|\, i, j = 1, 2, 3, \cdots.\}$$

and $\operatorname{Dim} H_{(i,j,k)}^{(J)} = f(i+j-1)$ for all $i, j \geq 1$, $k \in \mathbf{Z}_2$, which induces a $(II_{1,1} \times \mathbf{Z}_2)$-gradation on $\mathfrak{L}(F)$.

For $s = (s_{ij}) \in T(m,n)$, set $|s|_{-} = \sum_{f(i+j-1)<0} s_{ij}$. Then, as we have seen in Subsection 2.4, the Witt partition functions are given by

$$W^{+}(m,n) + W(m,n,0) = \sum_{\substack{s \in T(m,n) \\ |s|_{-}:\text{ even}}} \frac{(|s|-1)!}{s!} \prod f(i+j-1)^{s_{ij}},$$

$$W^{-}(m,n) = W(m,n,1) = \sum_{\substack{s \in T(m,n) \\ |s|_{-}:\text{ odd}}} \frac{(|s|-1)!}{s!} \prod f(i+j-1)^{s_{ij}}.$$

Therefore, by Corollary 1.3 and Proposition 2.10, we recover the even dimensions and the odd dimensions of homogeneous subspaces of the Monstrous Lie superalgebras:

PROPOSITION 4.9. *Let* $\mathfrak{L}(F) = \bigoplus_{(m,n)\in II_{1,1}} \mathfrak{L}(F)_{(m,n)}$ *be the Monstrous Lie superalgebra associated with a normalized q-series* $F(q) = \sum_{n=-1}^{\infty} f(n)q^n$ *such that* $f(-1) = 1$, $f(0) = 0$, *and* $f(n) \in \mathbf{Z}$ *for all* $n \geq 1$.

Then, for each $m, n \in \mathbf{Z}_{>0}$, *the even dimension and the odd dimension of the homogeneous subspace* $\mathfrak{L}(F)_{(m,n)}$ *are given by*

$$\begin{aligned}
\operatorname{Dim} \mathfrak{L}(F)_{(m,n,0)} &= \sum_{d|(m,n)} \frac{1}{d}\mu(d) W^{+}\left(\frac{m}{d}, \frac{n}{d}\right) + \sum_{\substack{d|(m,n)\\ d:\ \text{even}}} \frac{1}{d}\mu(d) W^{-}\left(\frac{m}{d}, \frac{n}{d}\right) \\
&= \sum_{d|(m,n)} \frac{1}{d}\mu(d) \sum_{\substack{s\in T\left(\frac{m}{d},\frac{n}{d}\right)\\ |s|_{-}:\ \text{even}}} \frac{(|s|-1)!}{s!} \prod f(i+j-1)^{s_{ij}} \\
&\quad + \sum_{\substack{d|(m,n)\\ d:\ \text{even}}} \frac{1}{d}\mu(d) \sum_{\substack{s\in T\left(\frac{m}{d},\frac{n}{d}\right)\\ |s|_{-}:\ \text{odd}}} \frac{(|s|-1)!}{s!} \prod f(i+j-1)^{s_{ij}}, \qquad (4.18)
\end{aligned}$$

$$\begin{aligned}
\operatorname{Dim} \mathfrak{L}_{(m,n,1)} &= \sum_{\substack{d|(m,n)\\ d:\ \text{odd}}} \frac{1}{d}\mu(d) W^{-}\left(\frac{m}{d}, \frac{n}{d}\right) \\
&= \sum_{d|(m,n)}^{d:\ \text{odd}} \frac{1}{d}\mu(d) \sum_{\substack{s\in T\left(\frac{m}{d},\frac{n}{d}\right)\\ |s|_{-}:\ \text{odd}}} \frac{(|s|-1)!}{s!} \prod f(i+j-1)^{s_{ij}}. \qquad (4.19)
\end{aligned}$$

EXAMPLE 4.10. In this example, we will apply Theorem 4.4, Corollary 4.5, and Theorem 4.8 to the Monstrous Lie superalgebras associated with replicable functions.

(a) If $F(q) = q^{-1}$, the trivial case, then

$$\mathfrak{L}(F) \cong sl(2, \mathbf{C}) \oplus \mathbf{C}d \cong gl(2, \mathbf{C}).$$

(b) If $F(q) = q^{-1} + rq$ ($r \in \mathbf{Z}$), then

$$P(H^{(J)}) = \{(-1, -1)\} \qquad \text{with } \operatorname{Dim} H^{(J)}_{(-1,-1)} = r.$$

Hence the denominator identity for the Lie superalgebra $\mathfrak{L}(F)^{(J)}_{-}$ is equal to

$$\prod_{m,n=1}^{\infty} (1 - p^m q^n)^{\operatorname{Dim} \mathfrak{L}(F)_{(m,n)}} = 1 - \sum_{i,j=1}^{\infty} f(i+j-1) p^i q^j = 1 - rpq.$$

Thus, we have

$$W^{(k,l)} = \begin{cases} \frac{1}{k} r^k & \text{if } k = l, \\ 0 & \text{otherwise,} \end{cases}$$

which yields

$$\operatorname{Dim} \mathfrak{L}(F)_{(m,n)} = \begin{cases} \frac{1}{n} \sum_{d|n} \mu(d) r^{\frac{n}{d}} & \text{if } m = n, \\ 0 & \text{otherwise.} \end{cases}$$

Therefore, the Lie superalgebra $\mathfrak{L}(F)^{(J)}_{-} = \bigoplus_{n=1}^{\infty} \mathfrak{L}(F)_{(-n,-n)}$ is isomorphic to the free Lie superalgebra generated by a superspace of superdimension $r \in \mathbf{Z}$.

(c) If $F^{(a)} = F$ for all $a \geq 1$ (e.g., $F(q) = j(q) - 744$), then by Corollary 4.5 and Theorem 4.8, we have

$$\begin{aligned} \operatorname{Dim} \mathfrak{L}(F)_{(m,n)} &= \sum_{d|(m,n)} \frac{1}{d} \mu(d) \sum_{s \in T\left(\frac{m}{d}, \frac{n}{d}\right)} \frac{(|s|-1)!}{s!} \prod f(i+j-1)^{s_{ij}} \\ &= \sum_{d|(m,n)} \frac{1}{d} \mu(d) \sum_{s \in T\left(\frac{m}{d}, \frac{n}{d}\right)} \frac{(|s|-1)!}{s!} \prod f^{(d)}(i+j-1)^{s_{ij}} \\ &= f(mn). \end{aligned}$$

(d) More generally, if F is replicable, then by Theorem 4.4 and the identity (4.16), the denominator identity for the Lie superalgebra $\mathfrak{L}(F)^{(J)}_{-}$ is equal to

$$\begin{aligned} \prod_{m,n=1}^{\infty} (1 - p^m q^n)^{\operatorname{Dim} \mathfrak{L}(F)_{(m,n)}} &= 1 - \sum_{i,j=1}^{\infty} f(i+j-1) p^i q^j \\ &= \prod_{m,n=1}^{\infty} \exp\left(- \sum_{a=1}^{\infty} \frac{1}{a} f^{(a)}(mn) p^{am} q^{an} \right). \end{aligned}$$

Taking the logarithm, we have

$$\begin{aligned}\log \prod_{m,n=1}^{\infty} (1-p^m q^n)^{\operatorname{Dim} \mathfrak{L}(F)_{(m,n)}} &= -\sum_{m,n=1}^{\infty} \sum_{k=1}^{\infty} \frac{1}{k} \operatorname{Dim} \mathfrak{L}(F)_{(m,n)} p^{km} q^{kn} \\ &= -\sum_{m,n=1}^{\infty} \sum_{\substack{k>0 \\ k|(m,n)}} \frac{1}{k} \operatorname{Dim} \mathfrak{L}(F)_{(m/k,n/k)} p^m q^n,\end{aligned}$$

and

$$\begin{aligned}\log \prod_{m,n=1}^{\infty} \exp\left(-\sum_{a=1}^{\infty} \frac{1}{a} f^{(a)}(mn) p^{am} q^{an}\right) &= -\sum_{m,n=1}^{\infty} \sum_{a=1}^{\infty} \frac{1}{a} f^{(a)}(mn) p^{am} q^{an} \\ &= -\sum_{m,n=1}^{\infty} \sum_{\substack{a>0 \\ a|(m,n)}} \frac{1}{a} f^{(a)}\left(\frac{mn}{a^2}\right) p^m q^n.\end{aligned}$$

It follows that

$$\sum_{\substack{k>0 \\ k|(m,n)}} \frac{1}{k} \operatorname{Dim} \mathfrak{L}(F)_{(m/k,n/k)} = \sum_{\substack{a>0 \\ a|(m,n)}} \frac{1}{a} f^{(a)}\left(\frac{mn}{a^2}\right).$$

Therefore, by Möbius inversion, we obtain

$$\operatorname{Dim} \mathfrak{L}(F)_{(m,n)} = \sum_{ad|(m,n)} \frac{1}{ad} \mu(d) f^{(a)}\left(\frac{mn}{a^2 d^2}\right).$$

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